International Journal of Differential Equations and Applications
Volume 16 No. 1 2017, 45-52
ISSN: 1311-2872
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijdea.v16i1.932

acadpubl.eu

A CONTROLLABILITY RESULT FOR AFFINE CONTROL SYSTEMS ON SOME NON-COMPACT LIE GROUP

I. Akdenizci Demirtas[1], A. Kara Hansen[2 §]

[1,2]Department of Mathematics
Bilgi University
Istanbul, TURKEY
[1,2]Institut for Matematik
Aarhus Universitet
Ny Munkegade 118, 8000 Aarhus C, DENMARK

Abstract: In this paper we deal with affne control systems on a non-compact Lie group $ax+b$ group. First, we study topological properties of the state space $Af(1)$ and the automorphism orbit of $Af(1)$. Then, we establish controllability of affne systems on $Af(1)$ by considering controllability of associated bilinear parts on the automorphism orbit of $Af(1)$.

AMS Subject Classification: 93B05, 93C10
Key Words: affne control systems, non-compact Lie group, state space $Af(1)$, controllability of affne systems, automorphism orbit of $Af(1)$

1. Introduction and Notation

The purpose of this paper is to study controllability property of affine control systems on some specific Lie group called "ax+b" group by relating to their associated bilinear parts.

First, Jurdjevic and Sallet have studied the controllability of affine systems on Euclidean spaces, [3]. Later, in [4], Kara and San Martin extended this ap-

Received: December 21, 2016

[§]Correspondence author

proach to the controllability of affine control systems on generalized Heisenberg Lie groups. There is also a work with similar technique on Carnot groups by Kara and Kule in [5].

Related to the controllability of affine control systems on Lie groups, in [1], the authors Ayala and San Martin have studied the controllability problem through the subalgebra of the Lie algebra $\mathrm{Af}(G)$ generated by the vector fields of a linear control system, where the drift vector field X is an infinitesimal automorphism, i.e., $(X_t)_{t\in\mathbb{R}}$ is a one-parameter subgroup of $\mathrm{Aut}(G)$; have lifted the system itself to a right-invariant control system on Lie group $\mathrm{Af}(G)$ and have given the controllability results for compact connected and noncompact semi-simple Lie groups cases.

In this paper, the affine control systems on a non-compact Lie group $ax+b$-group have been investigated and given a controllability characterization.

2. Affine Control Systems on Lie Groups

If G is a connected Lie group with its Lie algebra $L(G)$, then the affine group $\mathrm{Af}(G)$ of G is the semi-direct product of $\mathrm{Aut}(G)$ with G itself, [4], i.e., $\mathrm{Af}(G) = \mathrm{Aut}(G) \times_s G$. The group operation of $\mathrm{Af}(G)$ is given by

$$(\phi, g_1) \cdot (\psi, g_2) = (\phi \circ \psi, g_1\phi(g_2)).$$

And, if 1 denotes the identity element of $\mathrm{Aut}(G)$ and e denotes the neutral element of G, then the group identity of $\mathrm{Af}(G)$ is $(1, e)$ and $(\phi^{-1}, \phi^{-1}(g^{-1}))$ is the inverse of $(\phi, g) \in \mathrm{Af}(G)$. Hence, $g \to (1, g)$ and $\phi \to (\phi, e)$ embed G into $\mathrm{Af}(G)$ and $\mathrm{Aut}(G)$ into $\mathrm{Af}(G)$, respectively. Therefore, G and $\mathrm{Aut}(G)$ are subgroups of $\mathrm{Af}(G)$. Then, there is a natural transitive action

$$\mathrm{Af}(G) \times G \to G$$

defined by

$$(\phi, g_1) \cdot g_2 \to g_1\phi(g_2),$$

where $(\phi, g_1) \in \mathrm{Af}(G)$ and $g_2 \in G$.

The groups G and $\mathrm{Aut}(G)$ are closed subgroups of $\mathrm{Af}(G)$. Denote by $\mathrm{Aut}(L(G))$ the automorphism group of $L(G)$ which is a Lie group and its Lie algebra is $\mathrm{Der}(L(G))$, the Lie algebra of derivations of $L(G)$. If G is simply connected, then $\mathrm{Aut}(L(G))$ and $\mathrm{Aut}(G)$ are isomorphic. In fact, there is an isomorphism Φ between $\mathrm{Aut}(L(G))$ and $\mathrm{Aut}(G)$ which assigns to each authomorphism ϕ of G its differential $d\phi\,|_1$ at the identity. And, any authomorphism of $L(G)$ extends to an authomorphism of G, therefore Φ is indeed an isomorphism between $\mathrm{Aut}(L(G))$ and $\mathrm{Aut}(G)$. Thus, in this case, the Lie algebra of $\mathrm{Aut}(G)$ is $\mathrm{Der}(L(G))$.

The Lie algebra $\mathrm{af}(G)$ of $\mathrm{Af}(G)$ is the semi-direct product $\mathrm{Der}(L(G)) \times_s L(G)$. Its Lie bracket is given by

$$[(D_1, X_1), (D_2, X_2)] = ([D_1, D_2], D_1 X_2 - D_2 X_1 + [X_1, X_2])$$

An affine control system $\Sigma = (G, \mathcal{D})$ on a Lie group $G \subset \mathrm{Af}(G)$ is determined by the specification of the following data :

$$\dot{x} = (D + X)(x) + \sum_{j=1}^{d} u_j(t)(D^j + Y^j)(x)$$

parametrized by U, family of piecewise constant real valued funtions, where $x \in G$; $D, D^1, \ldots, D^d \in \mathrm{Der}(L(G))$ and $X, Y^1, \ldots, Y^d \in L(G)$. And, the dynamic is given by

$$\mathrm{D} = \{D + X + \sum_{j=1}^{d} u_j(D^j + Y^j) \mid u \in \mathbb{R}^d\}.$$

If an affine control system is considered on an Abelian Lie group, then it becomes a linear control system since any bracket between the elements of an Abelian Lie algebra is null. Therefore, in this case, the affine system turns to the form of the linear control system.

If it is considered $X = 0$ and $Y^1 = Y^2 = \cdots = Y^d = 0$ for affine control system on a Lie group, then it becomes a bilinear control system. In the general case, affine control systems define much richer class of control systems than the bilinear class, and their controllability properties on Carnot groups are essentially governed by their bilinear parts $D, D^1, \ldots, D^d$.

If we denote the space of the all endomorphisms of $\mathbb{R}^n$ by $End(\mathbb{R}^n)$ and denote the set of all linear automorphisms by $Aut(\mathbb{R}^n)$, then the semi direct product of $End(\mathbb{R}^n)$ and $Aut(\mathbb{R}^n)$,, i.e.; $\mathrm{Af}(\mathbb{R}^n) = \mathrm{Aut}(\mathbb{R}^n) \times_s \mathbb{R}^n$ defines the affine group of $\mathbb{R}^n$ and Lie algebra of this group is of the following from $\mathrm{af}(\mathbb{R}^n) = \mathrm{End}(\mathbb{R}^n) \times_s \mathbb{R}^n$ Therefore, for $A \in \mathrm{End}(\mathbb{R}^n)$ and $a \in \mathbb{R}^n$, the vector field $X(p) = Ap + a$ is called affine vector field. $exptX$ on $\mathbb{R}^n$ creats 1-parameter group of diffeomorphisms on $\mathbb{R}^n$ and these infinitesimal generators, $X(p) = Ap + a$ are affine vector fields. Here, any finite number of affine vector fields $X_1, X_2, \ldots, X_m$ such that $X_i(p) = A_i p + a$, $i = 1, 2, 3, \ldots,$ define a control system :

$$\frac{dp}{dt} = X_0(p) + \sum_{i=1}^{m} u_i(t)X_i(p) = (A_0p + a_0) + \sum_{i=1}^{m} u_i(t)(A_ip + a_i)$$

Such control-parameter differential equations on $\mathbb{R}^n$ form the dynamics of affine systems. Here, $u_i \in U \subset \mathbb{R}^n$ functions are of arbitrary functions of controls, $A_1, A_2, \ldots, A_m$ are $n \times n$ matrices and $a_1, a_2, \ldots, a_m$ are the column vectors of $\mathbb{R}^n$. Let G be a connected Lie group and $L(G)$ be its Lie algebra, and then the family of differential equations

$$p(t) = (D + X)(p) + \sum_{i=1}^{d} u_j(t)(D^j + X^j)(p)$$

determine the affine control system on G.

Here, $p \in G$, $D^1, D^2, \ldots, D^d \in \mathrm{aut}(G)$, $X^1, X^2, \ldots, X^d \in L(G)$ and u_i denotes the controls. The dynamics of the system is in the following form

$$\mathrm{D} = \{D + X + \sum_{i=1}^{d} u_j(t)(D^j + X^j) | u \in \mathbb{R}^d\}.$$

3. "$ax + b$" Groups

A translation is called an affine translation, if it is defined by

$$f_{a,b}(x) = ax + b; \quad \forall a \in \mathbb{R}^+, \ \forall b \in \mathbb{R}.$$

Product of any two affine translation is an affine translation and inverse of an affine translation is also an affine translation. Therefore, the set of all affine tranlations forms a group and it is denoted by $Af(1)$. Any $f_{a,b}(x)$ function is associated to the matrix $F_{a,b} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. In fact,

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} ax + b \\ 1 \end{pmatrix}, \quad \forall a \in \mathbb{R}^+, \ \forall b, x \in \mathbb{R}.$$

Since Af(1) is a group of real matrices, it is a geometric object in $\mathbb{R}^4$ and on the other hand it is an infinite subset of $\mathbb{R}^4$. Because of the definition of Af(1), b is any number and a is any positive number. In addition, Af(1) is a non-compact Lie group. Af(1), with all of these properties, is the only non-abelian 2-dimensional connected Lie group. Af(1) is not a simple group. Its group homomorphism has the following form

$$\varphi : \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

Moreover, normal subgroup of matrices in kernel of φ is their matrix Lie group. Kernel of Af(1) is a group homomorphism that sends all of matrices to unit matrix. So, for $a \to 1$ and $b = 0$,

$$\varphi \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Geometrically, this subgroup is a line and

$$\begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b_1 + b_2 \\ 0 & 1 \end{pmatrix}.$$

Because of that, the group operation corresponds to the addition on the line. Since Af(1) is the half of a plane in the four-dimensional Euclidean space $\mathbb{R}^4$, it is geometrically clear that it is of the form of the tangent space at the identity element. However, to find explicit matrices for the elements of the tangent space we look at the vectors from the identity element of Af(1). Here, Lie directions in the plane are stretched by the vectors $J = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $K = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Lie brackets of these base vectors have the following form;

$$[J, K] = JK - KJ = K$$

Since, af(1) has a Lie algebra structure, provides the following properties;

$$[J, K] = -[K, J], \quad \forall J, K \in \text{af}(1)$$

$$[I, [J, K]] + [J, [K, I]] + [K, [I, J]] = 0, \quad \forall I, J, K \in \text{af}(1).$$

The center of the Lie group Af(1) which is the Lie group of af(1) consists of the identity element. Moreover, "$ax + b$" group is not nilpotent but solvable Lie group.

Theorem 3.1. *Let* $\Sigma = (\text{Af}(1), \text{D})$ *be an affine control system. Then, the state space* Af(1) *is a locally compact Hausdorff space.*

Proof. Af(1) is a Hausdorff space since it is a Lie group. To show that its locally compactness it is enough to prove, for a given $x \in Af(1)$ and a given neighborhood V of x, the existence of some neighborhood U of x such that $\overline{U}$ is compact and $\overline{U} \subset V$. The topology on Af(1) half plane is homeomorphic to the standard topology of $\mathbb{R}^2$. Therefore, $\forall x \in \mathrm{Af}(1)$, the neighborhood V of x is homeomorphic to an open ball. For each neighborhood V of x, there is a neighborhood U of x such that $x \in U \subset \overline{U} \subset V$. Since U is also homeomorphic to an open ball, the closure of U is a closed ball. Since in $\mathbb{R}^2$ finite closed balls are compact, their homeomorphic images are compact, too.

Lemma 3.2. *The automorphism orbit of the state space* Af(1) *is dense.*

Proof. The set

$$Y := exp(\mathrm{af}(1) - [\mathrm{af}(1), \mathrm{af}(1)])$$

is an Aut(Af(1)) - orbit of Af(1). In fact, the classical Hadamard theorem asserts that at each point of the surface, the exponential mapping from the tangent plane to the surface defines a global diffeomorphism, [2]. Moreover, given any two elements $g_1, g_2 \in Y$ such that the line segment $\overline{g_1 g_2}$ which is parallel to $[\mathrm{Af}(1), \mathrm{Af}(1)]$,

$$\Phi : Y \to Y$$

defined by

$$g_1 \to t_1 g_1 + t_2 = Y, t_1, t_2 \in \mathbb{R}$$

is an authomorphism. Also, it is possible to connect those segments with the perpendicular segments to each other in the same way. Aut(Af(1)) orbit is open, since the center $[\mathrm{Af}(1), \mathrm{Af}(1)]$ forms a line for any element $x \in [\mathrm{Af}(1), \mathrm{Af}(1)]$ and every neighborhood $B(x, \delta)$ of x have some element of Af(1) different than x . Therefore,

$$\overline{\mathrm{Af}(1) - [\mathrm{Af}(1), \mathrm{Af}(1)]} = \mathrm{Af}(1).$$

Theorem 3.3. *The affine control system Σ_a on the state space* Af(1) *is controllable if it does not have any equilibrium point and the associated bilinear system $\Sigma_b = (\mathrm{Af}(1), \mathrm{D_b})$ is controllable on the* Aut(Af(1)) *orbit.*

Proof. For the controllability not having any equilibrium point is necessary. Now, consider the associated bilinear system $\Sigma_b = (\mathrm{Af}(1), \mathrm{D_b})$ is controllable on the Aut(Af(1)) orbit. And, define the following authomorphism at the algebra level

$$\Psi_\lambda : \partial L(G) \times L(G) \to \partial L(G) \times L(G)$$

such that $\Psi_\lambda = Id \times \frac{1}{\lambda}$. $\forall D + X \in \mathrm{af}(1) = \partial L(G) \times L(G)$, we have

$$\Psi_\lambda(D + X) = D\frac{1}{\lambda}X.$$

Since $\Psi_\lambda(D+X) = D$ as $\lambda \to \infty$, $\Psi_\lambda(\mathrm{D}_a) \to \mathrm{D}_b$ as $\lambda \to \infty$. Therefore, $\Psi_\lambda(\Sigma_a) \to \Sigma_b$ as $\lambda \to \infty$. By the hypothesis, the bilineer system Σ_b is controllable on the authomorphism $\mathrm{Aut}(\mathrm{Af}(1))$ orbit and according to the Lemma 3.2, $\mathrm{Aut}(\mathrm{Af}(1))$ orbit is dense in the state space $\mathrm{Af}(1)$, therefore, bilineer system Σ_b is controllable on every orbit. Orbits of state spaces have differentiable manifold structure, [7]. Hence, the system can be analyzed on the orbit of the system by keeping the all information of the dynamic of the system. Controllability on every orbit of Σ_b reflects to the state space. Let us consider the unit sphere $S(1_e, 1)$ which is the boundary of the unit ball $B(1_e, 1)$ centered at 1_e. Since complete controllability is preserved under small perturbations, [6], for sufficiently large λ, $\Psi_\lambda(\Sigma_a)$ is controllable on $S(1_e, 1) - [\mathrm{Af}(1), \mathrm{Af}(1)]$. Therefore, since normally controllable finite systems are open on $S(1_e, 1)$, [6], the system $\Psi_\lambda(\Sigma_a)$ is also controllable on $B(1_e, 1) - [\mathrm{Af}(1), \mathrm{Af}(1)]$. Hence, Σ_a is controllable on $B(\Psi_\lambda^{-1}(1_e), 1) - [\mathrm{Af}(1), \mathrm{Af}(1)]$, where $\Psi_\lambda^{-1}(1_e) = Id\ e \times \lambda\ Id\ e$. In this case, the positive orbit of the affine system through $\Psi_\lambda^{-1}(1_e) = Id\ e \times \lambda\ Id\ e$ is open and its interior is nonempty since it contains $B(\Psi_\lambda^{-1}(1_e), 1) - [\mathrm{Af}(1), \mathrm{Af}(1)]$. Thus, the system Σ_a is normally accessible from $\Psi_\lambda^{-1}(1_e)$. Since the state space is connected, the affine system Σ_a is controllable on $\mathrm{Af}(1)$.

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