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MONOTONIC INTEGRABLE SOLUTION FOR A MIXED TYPE INTEGRAL AND DIFFERENTIAL INCLUSIONS OF FRACTIONAL ORDERS

A.M.A. EL-SAYED[1] AND SH.M. AL-ISSA[2]

[1]Faculty of Science
Alexandria University Alexandria, EGYPT
[2]Faculty of Science
Lebanese International University
Saida, LEBANON

ABSTRACT: In this paper, we present a global existence theorem of positive monotonic integrable solution for the mixed type integral inclusion of fractional order

$$x(t) \in p(t) + \int_0^1 k(t,s)\ F_1(s, I^\beta f_2(s, x(s))ds, \quad t \in [0,1],\ \beta > 0.$$

The initial value problem of arbitrary (fractional) orders differential inclusion will be considered as an application.

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1. INTRODUCTION

Let $\alpha \in (0,1)$, and consider the fractional order functional integral inclusion

$$x(t) \in p(t) + F_1(t, I^\alpha f_2(t, x(\varphi(t))), \quad t \in [0,1]. \tag{1}$$

In [8], the authors proved the existence of global integrable solution for the nonlinear functional integral inclusion (1), where the set-valued map $F_1 : (0,1) \times R^+ \to 2^{R^+}$ has nonempty closed values which is satisfy Caratheodory and growth conditions.

The existence of positive monotonic continuous solution of the mixed type integral inclusion

$$x(t) \in p(t) + \int_0^1 k(t,s)\ F_1(s, I^\beta f_2(s,x(s))ds, \quad t \in [0,1],\ \beta > 0. \tag{2}$$

was proved in [9] by using Arzela -Ascoli Theorem and applying Schauder's fixed-point Theorem, also, in [7] the local existence of monotonic integrable solution of the mixed type nonlinear integral equation of fractional order

$$x(t) = p(t) + \int_0^1 k(t,s)\ f_1(s, I^\beta f_2(s,x(s))ds, \quad t \in [0,1],\ \beta > 0 \tag{3}$$

has been proved in when the two functions f_1 and f_2 are monotonic nondecreasing (in both variables).

Here, we omit this conditions and prove a global existence theorem for a positive nondecreasing integrable solution of (3).

As a generalization of our results we study the global existence of positive monotonic integrable solution for the integral inclusion (2), and the initial value problem of arbitrary (fractional) orders differential inclusion

$$\frac{dx}{dt} \in p(t) + \int_0^1 k(t,s)\ F_1(s, D^\alpha x(s))ds, \text{ a.e. } t \in [0,1],\ \beta > 0 \tag{4}$$

$$x(0) = x_\circ \tag{5}$$

will be studied.

2. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let $L^1 = L^1(I)$ be the class of Lebesgue integrable function on the interval $I = [a,b]$, where $0 \leq a < b < \infty$ and let $\Gamma(.)$ be the gamma function.

Definition 1. The fractional integral of the function $f(.) \in L^1(I)$.of order $\alpha \in R^+$ is defined by (cf. [11] [12] and [14] [16])

$$I_a^\alpha\ f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ f(s)\ d(s).$$

For the properties of the fractional order integral see [11], [10].

Definition 2. The (Caputo) fractional derivative D^{α} of order $\alpha \in (a,b]$ of the absolutely continuous function g is defined as (see [2] [12] [13] and [16])

$$D_a^{\alpha}\, g(t) = I_a^{1-\alpha}\, \frac{d}{dt}\, g(t) \quad , \qquad t \in [a,b]$$

Recall that the operator T is compact if it is continuous and maps bounded sets into relatively compact sets from a subspace $U \subset X$ into the Banach space X is denoted by $C(U,X)$.

Moreover, let us define the operator T as

$$Tx(t) = p(t) + \int_0^1 k(t,s) f_1(s, I^{\beta} f_2(s, x(s)))ds, \tag{6}$$

The following two theorems will be needed in the proof of our main result.

Theorem 3. *(Nonlinear Alternative of Leray-Shauder type) [4]. Let U be an open subset of a convex set D in a Banach space X. Assume $0 \in U$ and $T \in C(\bar{U}, D)$. Then either:*

(A_1) T has a fixed point in $\bar{U}$, or

(A_2) there exists $\gamma \in (0,1)$ and $x \in \partial U$ such that $x = \gamma T x$.

Theorem 4. *(Kolmogorov Compactness Criterion) [5]. Let $\Omega \subseteq L^p(0,1)$, $1 \leq P \leq \infty$. If:*

(i) Ω is bounded in $L^p(0,1)$, and

(ii) $x_h \to x$ as $h \to 0$ uniformly with respect to $x \in \Omega$,

then Ω is relatively compact in $L^p(0,1)$, where

$$x_h(t) = \frac{1}{h} \int_0^{t+h} x(s)\ ds.$$

3. MAIN RESULTS

To facilitate our discussion, let us first state the following assumptions:

(i) $p \in L^1$;

(ii) $f_i : [0,1] \times R^+ \to R^+$, $i = 1, 2$ satisfies caratheodory condition i.e f_i are measurabel in t for any $x \in R^+$ and continuous in x for almost all $t \in [0,1]$.

There exists two functions $t \to a(t)$, $t \to b(t)$ such that

$$|f_i(t,x)| \le \; a_i(t) + \; b_i(t)|x|, \; i = 1,2 \; \forall \; t \; \in [0,1] \text{ and } x \in R,$$

where $a_i(.) \in L^1$ and $b_i(.)$ are measurable and bounded;

(iii) $k : [0,1] \times R^+ \to R^+$ is measurable and the integral operator K defined by

$$(Kx)(t) = \int_0^1 k(t,s)x(s)ds$$

map L^1 into itself and such that

$$\int_0^1 |k(t,s)|dt \; < \; k;$$

(iv) Assume that every solution $x(.) \in L^1$ to the equation

$$x(t) = \gamma(p(t) + \int_0^1 k(t,s) f_1(s, I^\beta f_2(s, x(s))ds), \; t \in [0,1], \; 0 < \beta < 1, \; \gamma \in (0,1),$$

satisfies $\|x\| \neq r$ ($r > 0$ is arbitrary but fixed).

Now, we are in position to state and prove our main result

Theorem 5. *Let the assumptions (i)-(iv) be satisfied. Then equation (3) has at least one solution in* L^1.

Proof. let x be an arbitrary element in the open set $B_r = \{x : \; \|x\| < r \; , \; r > 0\}$. Then from assumption (i) and (ii) we have,

$$\begin{aligned}
\|Tx\| &= \int_0^1 |(Tx)(t)| \; dt \\
&\le \int_0^1 |p(t)|dt + \int_0^1 |k(t,s)| \; |f_1(s, \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} \; f_2(\tau, x(\tau))d\tau|ds \\
&\le \|p\| + k \int_0^1 [a_1(s) + b_1(s)| \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} \; f_2(\tau, x(\tau))d\tau|]ds \\
&\le \|p\| + k \int_0^1 |a_1(s)|ds + k \int_0^1 |b_1(s) \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} \; |f_2(\tau, x(\tau))|d\tau ds \\
&\le \|p\| + k\|a_1\| + kb_1 \int_0^1 \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} [a_2(\tau) + b_2(\tau)|x(\tau)|]d\tau ds \\
&\le \|p\| + k\|a_1\|
\end{aligned}$$

$$
\begin{aligned}
&+ kb_1[\int_0^1\int_\tau^1 \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} a_2(\tau) ds d\tau + \int_0^1\int_\tau^1 \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} |b_2(\tau)||x(\tau)| ds d\tau] \\
&\leq \|p\| + k\|a_1\| + kb_1 \int_0^1 a_2(\tau) \int_\tau^1 \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} ds d\tau + kb_1 b_2 \int_0^1 |x(\tau)| \frac{(1-\tau)^\beta}{\Gamma(\beta+1)} d\tau \\
&\leq \|p\| + k\|a_1\| + kb_1 \int_0^1 a_2(\tau) \frac{(1-\tau)^\beta}{\Gamma(\beta+1)} d\tau + \frac{kb_1 b_2}{\Gamma(\beta+1)} \int_0^1 |x(\tau)| d\tau \\
&\leq \|p\| + k\|a_1\| + \frac{kb_1}{\Gamma(\beta+1)} \int_0^1 a_2(\tau) d\tau + \frac{kb_1 b_2 \|x\|}{\Gamma(\beta+1)} \\
&\leq \|p\| + k\|a_1\| + \frac{kb_1\|a_2\|}{\Gamma(\beta+1)} + \frac{kb_1 b_2 \|x\|}{\Gamma(\beta+1)} \\
&\leq \|p\| + k\|a_1\| + \frac{kb_1\|a_2\| + kb_1 b_2 \|x\|}{\Gamma(\beta+1)}.
\end{aligned}
$$

Hence, the above inequality means that the operator T maps B_r into L^1.

Now, we will show that T is compact. To achieve this goal we will apply Theorem 3. So let Ω be a bounded subset of B_r. Then $T(\Omega)$ is bounded in L^1 i.e condition (i) of Theorem 4 satisfied.

It remains to show that $(Tx)_h \to Tx$ in L^1 as $h \to 0$ uniformly with respect to $Tx \in \Omega$. We have the following :

$$
\begin{aligned}
\|(Tx)_h - (Tx)\| &= \int_0^1 |(Tx)_h(t) - (Tx)(t)| dt, \\
&= \int_0^1 |\frac{1}{h}\int_t^{t+h} (Tx)_h(\tau) d\tau - (Tx)(t)| dt, \\
&= \int_0^1 |\frac{1}{h}\int_t^{t+h} ((Tx)_h(\tau) - (Tx)(t)) d\tau| dt \\
&\leq \int_0^1 (\frac{1}{h}\int_t^{t+h} |p(\tau) - p(t)| d\tau dt \\
&\quad + \int_0^1 \frac{1}{h}\int_t^{t+h} | \int_0^1 k(\tau,s) f_1(s, I^\beta f_2(s, x(s)) \\
&\quad - \int_0^1 k(t,s) f_1(s, I^\beta f_2(s, x(s)) ds | d\tau dt \\
&\leq \int_0^1 (\frac{1}{h}\int_t^{t+h} |p(\tau) - p(t)| d\tau dt \\
&\quad + \int_0^1 \frac{1}{h}\int_t^{t+h} | \int_0^1 (k(\tau,s) - k(t,s)) f_1(s, I^\beta f_2(s, x(s)) ds | d\tau dt, \\
&\leq \int_0^1 (\frac{1}{h}\int_t^{t+h} |p(\tau) - p(t)| d\tau dt \\
&\quad + \int_0^1 \frac{1}{h}\int_t^{t+h} \int_0^1 |k(\tau,s) - k(t,s)| f_1(s, I^\beta f_2(s, x(s)) ds d\tau dt
\end{aligned}
$$

since $f_1, f_2 \in L^1$, we get $I^\beta f_2 \in L^1$ and $K f_1 f_2 \in L^1$, then

$$\frac{1}{h}\int_t^{t+h}\int_0^1 |k(\tau,s)-k(t,s)| f_1(s, I^\beta f_2(s,x(s)) ds d\tau \to 0$$

Moreover, $p(.) \in L^1$. So, we have

$$\frac{1}{h}\int_t^{t+h} |p(\tau)-p(t)| d\tau \to 0$$

for a.e $t \in L^1$. Therefore, by Theorem 4 we have that $T(\Omega)$ is relatively compact, that is, T is compact operator.

Set $U = B_r$ and $D = X = L^1[0,1]$. Then in the view of assumption (iv) condition (A_2) of Theorem 3 does not hold. Theorem 1, implies that T has a fixed point. This completes the proof. □

Corollary 6. *Let the assumptions of Theorem 5 be satisfied. If the function p is nondecreasing on $[0,1]$ and the kernel k is nondecreasing with respect to $t \in [0,1]$ then the solution of equation (3) is nondecreasing*

Proof. For $t_1, t_2 \in [0,1]$ and $t_1 < t_2$, we have

$$\begin{aligned} x(t_1) =& p(t_1) + \int_0^1 k(t_1,s)\ f_1(s, I^\beta f_2(s,x(s)))\ ds, \\ \leq & p(t_2) + \int_0^1 k(t_2,s)\ f_1(s, I^\beta f_2(s,x(s)))\ ds = x(t_2). \end{aligned}$$

□

4. MIXED TYPE INTEGRAL INCLUSION

Consider now inclusion (2), where $F_1 : [0,1] \times R^+ \to 2^{R^+}$ has nonempty closed values.

As an important consequence of the Theorem 5 we can present the following:

Theorem 7. *Let the functions f_1, p and k be satisfied all Assumptions of Theorem 5.*

Let the set-valued function F_1 be satisfied the following :

(i) $F_1(t,x)$ are non empty, closed and convex for all $(t,x) \in [0,1] \times R^+$,

(ii) $F_1(t,.)$ is lower semicontinuous from R^+ into R^+,

(iii) $F_1(.,.)$ is measurable,

(iv) There exists a function $a \in L_1$ and a positive number b such that

$$|F_1(t,x)| \leq a_1(t) + b_1(t)|x| \;\forall\; t \in [0,1].$$

Then there exists at least one positive non decreasing solution $x \in L^1$ of the integral inclusion (2).

Proof. By conditions $(i) - (iv)$ (see [1], [3], [6] [15]), we can find a selection function(Caratheodory selection) $f_1 : [0,1] \times R^+ \to R^+$ such that $f_1(t,x) \in F_1(t,x)$ for all $(t,x) \in [0,1] \times R^+$. which satisfies the assumption (2) of Theorem 5.

Clearly all assumptions of Theorem 5 are hold, then there exists at least one positive solution $x \in L^1$ such that

$$x(t) - p(t) = \int_0^1 K(t,s)\; f_1(s, I^\beta\; f_2(s,x(s)) \;\in\; \int_0^1 K(t,s)\; F_1(s, I^\beta\; f_2(s,x(s)). \quad \square$$

Now, we can easily proved the following corollary

Corollary 8. *Let the assumptions of Theorem 7 and Corollary 6 be satisfied, then the solution of integral inclusion (2) is nondecreasing.*

5. DIFFERENTIAL INCLUSION

Consider now the initial value problem of the differential inclusion (4) with the initial data (5).

Theorem 9. *Let the assumptions of Theorem 5 be satisfied, then the initial value problem (4)-(5) has at least one positive nondecreasing solution $x \in L^1$.*

Proof. Let $y(t) = \frac{dx(t)}{dt}$, then equation (4) transformed to the integral inclusion

$$y(t) \in P(t) + \int_0^1 k(t,s) F_1(s, I^{1-\alpha} y(s)) ds$$

which by Theorem 7 has at least one positive solution $y \in L^1$.

This implies that the existence of absolutely continuous solution

$$x(t) = x_\circ + \int_0^t y(s) ds$$

is a positive non decreasing solution of the initial-valued problem (4)-(5). $\square$

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