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A NOTE ON THE WILSON–HILFERTY DISTRIBUTION

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ABSTRACT: P. Ramos, M. Almeida, V. Tomazella and F. Louzada (*Anais da Academia Brasileira de Ciencias*, **91** (2019), 28 pp.) presented some mathematical properties of the Wilson–Hilferty distribution. The aim of this note is to study "saturation" of the cumulative sigmoid

$$M(t) = 1 - \frac{1}{\Gamma(\alpha)}\Gamma(\alpha, \frac{\alpha}{\lambda}t^3)$$

to the horizontal asymptote with respect to Hausdorff distance. We prove upper and lower estimates for the one–sided Hausdorff approximation of the Heaviside step–function $h_{t_1}(t)$ by means of this function. Numerical examples using *CAS Mathematica*, illustrating our results are given.

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1. INTRODUCTION AND PRELIMINARIES

In [1], the authors presented some mathematical properties of the Wilson–Hilferty distribution with PDF (for more details see [1] and the bibliography therein)

$$f(t) = \frac{3}{\Gamma(\alpha)}\left(\frac{\alpha}{\lambda}\right)^{\alpha} t^{3\alpha-1}e^{-\frac{\alpha}{\lambda}t^3}, \tag{1}$$

where $\alpha > 0$ and $\lambda > 0$ are shape and scale parameters, respectively and showed that

this distribution is appropriate for modeling of some specific data.

When studying the intrinsic properties of this distribution, it is also appropriate to study the following cumulative sigmoid:

$$M(t) = 1 - \frac{1}{\Gamma(\alpha)}\Gamma(\alpha, \frac{\alpha}{\lambda}t^3), \tag{2}$$

where $\Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}dt$ is the upper incomplete gamma function, and in particular "saturation" to the horizontal asymptote.

Definition 1. The shifted Heaviside step function is defined by

$$h_{t_1}(t) = \begin{cases} 0, & \text{if} \quad t < t_1, \\ [0,1], & \text{if} \quad t = t_1, \\ 1, & \text{if} \quad t > t_1. \end{cases} \tag{3}$$

Definition 2. [2] The Hausdorff distance (the H–distance) $\rho(f,g)$ between two interval functions f, g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,

$$\rho(f,g) = \max\{\sup_{A\in F(f)} \inf_{B\in F(g)} ||A-B||, \sup_{B\in F(g)} \inf_{A\in F(f)} ||A-B||\}, \tag{4}$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t,x)|| = \max\{|t|,|x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in $\mathbb{R}^2$ is $||A-B|| = \max(|t_A - t_B|, |x_A - x_B|)$.

2. MAIN RESULTS

In this Section we prove upper and lower estimates for the one–sided Hausdorff approximation of the Heaviside step–function $h_{t_1}(t)$ by means of families (2).

Let t_1 is the unique positive root of the nonlinear equation $M(t_1) - \frac{1}{2} = 0$.

The one–sided Hausdorff distance d satisfies the relation

$$M(t_1 + d) = 1 - d. \tag{5}$$

The following theorem gives upper and lower bounds for d

Theorem. *Let*

$$\begin{aligned} p &= -\frac{\Gamma\left(\alpha, \frac{\alpha}{\lambda}t_1^3\right)}{\Gamma(\alpha)}, \\ q &= 1 + \frac{3\alpha e^{-\frac{\alpha}{\lambda}t_1^3} t_1^2 \left(\frac{\alpha}{\lambda}t_1^3\right)^{\alpha-1}}{\lambda\Gamma(\alpha)} \\ s &= 2.1q. \end{aligned} \tag{6}$$

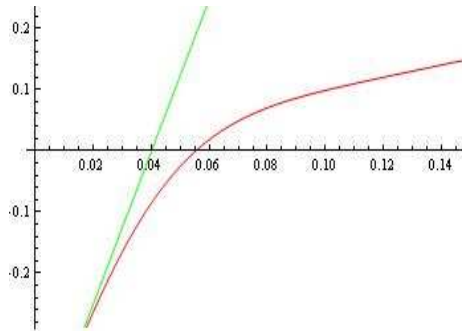

Figure 1: The functions $F(d)$ and $G(d)$ for $\lambda = 0.001;\ \alpha = 0.95$.

Let $s > e^{1.05}$. For the one–sided Hausdorff distance d between $h_{t_1}(t)$ and the sigmoid (2) the following inequalities hold:

$$d_l = \frac{1}{s} < d < \frac{\ln s}{s} = d_r. \tag{7}$$

Proof. Let us examine the function:

$$F(d) = M(t_1 + d) - 1 + d. \tag{8}$$

With some constraints imposed on the parameters α and λ which we will not stop here (for example, see [1]), it can be shown that $F'(d) > 0$ and the function F is increasing.

Consider the function

$$G(d) = p + qd. \tag{9}$$

From Taylor expansion we obtain $G(d) - F(d) = O(d^2)$.

Hence $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see Figure 1).

In addition $G'(d) > 0$ and the function G is also increasing.

Further, for $s > e^{1.05}$ we have

$$G(d_l) < 0; \quad G(d_r) > 0.$$

This completes the proof of the theorem.

Approximations of the $h_{t_1}(t)$ by function (2) for various λ and α are visualized on Figure 2–Figure 3.

Some computational examples using relations (5) and (7) are presented in Table 1.

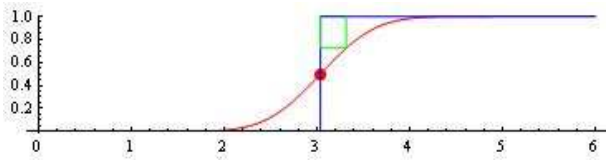

Figure 2: The sigmoid (2) for $\lambda = 30$; $\alpha = 5$; $t_1 = 3.03751$; Hausdorff distance $d = 0.276387$; $d_l = 0.256466$; $d_r = 0.348988$.

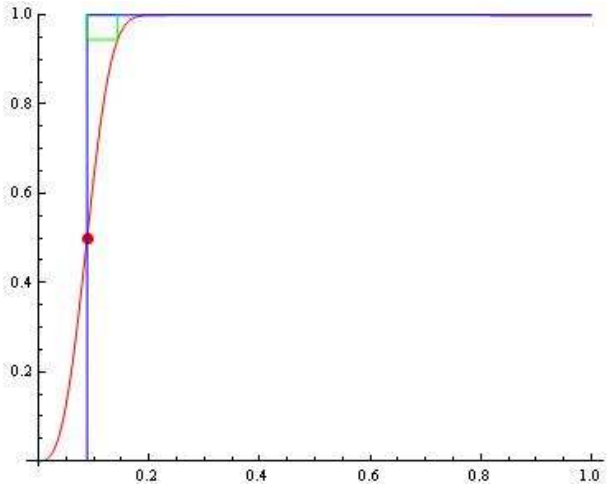

Figure 3: The sigmoid (2) for $\lambda = 0.001$; $\alpha = 0.95$; $t_1 = 0.0878839$; Hausdorff distance $d = 0.0554314$; $d_l = 0.0382575$; $d_r = 0.12485$.

We will illustrate the advances of the model $M^*(t) = \omega M(t)$ for approximating and modelling of the following "growth data (mean height) of sunflower plants"- (DSP) [6]

$$data_DSP$$
$$:= \{\{14, 36.4\}, \{28, 98.1\}, \{49, 205.5\}, \{56, 228.3\}, \{70, 250.5\},$$
$$\{84, 254.5\}\}.$$

For $\alpha = 0.545$, $\lambda = 71000$ and $\omega = 254.5$ we obtain the fitted model (see Figure 4).

3. CONCLUSIONS

In this paper we prove upper and lower estimates for the one–sided Hausdorff approximation of the shifted Heaviside function $h_{t_1}(t)$ by means of the sigmoid $M(t)$. The

λ	α	t_1	d_l	$H-distance$	d_r
0.1	0.6	0.374722	0.160712	0.18493	0.293804
0.001	0.9	0.187866	0.0771781	0.0993676	0.197702
0.001	0.95	0.0878839	0.0382575	0.0554314	0.12485
30	5	3.03751	0.2564665	0.276387	0.348988

Table 1: Bounds for d computed by (5) and (7) for various λ and α

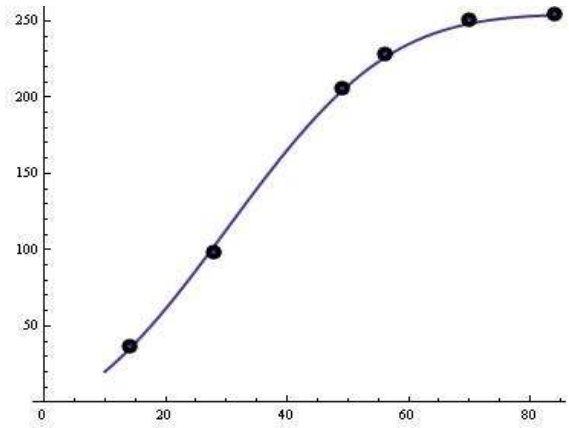

Figure 4: The fitted model $M^*(t)$.

estimates can be used in practice as one possible additional criterion in "saturation" study. The results are relevant for applied insurance mathematics. For example, the estimates obtained give more insight on the parameters in the strategy "Insurance responsibility". For some results, see [3]–[5].

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