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DICHOTOMY AND WELL-CONDITIONING OF DISCRETE MATRIX SYLVESTER SYSTEMS

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ABSTRACT: Dichotomy and conditioning of boundary value problems associated with difference equations is an interesting area of research and several papers appeared in that direction. These results can be used in the analysis of algorithms, in devising numerical schemes for solutions. We propose to study the dichotomy and conditioning for two-point boundary value problems associated with Sylvester discrete dynamical systems, with the help of Kronecker product of matrices. We establish a relationship between the stability bounds of the problem and the growth behavior of the fundamental matrix solution.

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1. INTRODUCTION

The analogy of Dichotomy between differential equations and difference equations goes back to Perron's paper on stability [18] and paper by Tali [6]. The dichotomy for system of differential equations by Daleckii and Krein [3] is the starting point of extensive later developments by Massera and Schaffer [7] and followed by Coppel [2]. The conditioning of Linear boundary Value problems was studied by Mattheji R M M [8] in 1982. In 1995, Murthy, K N, Prasad, K R and Anand P V S [11] constructed the solution of the Two point Boundary value problem associated with Lyapunov Type Matrix Difference System. The dichotomy and well conditioning of two point boundary value problems associated with matrix differential equations was studied by Murthy K N and Lakshmi P.V.S [10] in 1990 and Matrix Lyapunov Systems by Murthy M S N and Suresh Kumar G [16] in 2008. In 2009, Osvaldo Mendez and Nada al Hanna [17] presented elementary functional analysis proof of the roughness of exponential dichotomy of ordinary differential equations on an arbitrary Banach Space. K.N. Murty, Yan Wu and Viswanadh Kanuri [15] generalized results on time scale dynamical systems on Dichotomy and well conditions of two-point boundary value problems by unifying both continuous and discrete systems in 2011 and presented empirical validation of a set of theoretical-grounded metrics on object-oriented design. In this paper we consider a two point boundary value problem associated with a Discrete Matrix Sylvester Systems

$$LT = \Delta T(n) - (A(n)T(n) + T(n)B(n) + A(n)T(n)B(n) = F(n) \tag{1}$$

satisfying the boundary conditions

$$M_0T(n_0) + M_1T(n_1) = W \tag{2}$$

where $A(n), B(n), F(n) \in [L_p(n_0, n_1)]^{s\times s}$ for some p satisfying condition $1 \leq p \leq \infty$, M_0, N_0, M_1, N_1 and W are constant quare matrices of order s. This paper is organized as follows: Section 2 presents basic definitions and results required to understand the paper. Section 3 is concerned with the Dichotomy and Strong Dichotomy of Discrete Matrix Sylvester Systems. Conditioning of Boundary Value Problems associated with of Discrete Matrix Sylvester Systems are presented in Section 4.

2. PRELIMINARIES

This section present the definitions of kronecker product of matrices, Vectorization, basic properties of kronecker product and Vectorization and also results on Greens matrix of the Boundary value problem associated with Discrete Matrix Sylvester System.

Definition 1. Let $A \in C^{r\times s}(R^{r\times s})$ and $AB \in C^{p\times q}(R^{p\times q})$. Then kronecker product of of A and B is written as $A \otimes B$ is defined as A partitioned matrix

$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & ... & a_{1s}B \\ a_{21}B & a_{22}B & ... & a_{2s}B \\ ... & ... & ... & ... \\ a_{r1}B & a_{r2}B & ... & a_{rs}B \end{bmatrix}$ matrix and is in is an $rp \times sq$ and is in

$C^{rp\times sq}(R^{rp\times sq})$.

Definition 2. Let $A=[a_{ij}]\in C^{r\times s}(R^{r\times s})$, we denote $\widehat{A}$=Vec $A=[A_{.1},A_{.2},...,A_{.s}]^T$, where $A_{.j}$=$[a_{1j},a_{2j},...,a_{rj}]^T$, $(1\le j\le s)$.

The kronecker product has the following properties.
1.$(A\otimes B)^*=A^*\otimes B^*$
2.$(A\otimes B)^{-1}=A^{-1}\otimes B^{-1}$
3..$(A\otimes B)(C\otimes D)=(AC\otimes BD)$ This rule holds, provided the dimensions of the matrices are such that expressions are defined
4.$\|(A\otimes B)\|=\|A\|\|B\|$ (where $\|A\|=\underset{i,j}{Max}|a_{ij}|$)
5.$(A+B)\otimes C=(A\otimes C)+(B\otimes C)$.
6. If $A(n)$ and $B(n)$ are matrices and Δ is the forward difference operator then $\Delta(A\otimes B)=(\Delta A)\otimes B+A\otimes(\Delta B)+(\Delta A)\otimes(\Delta B)$
7. $Vec(ATB)=(B^*\otimes A)VecT$
8. If A and B are square matrices of order "s", then
(i) $Vec(AT)=((I_s\otimes A)VecT$
(ii)$Vec(TB)=((B^*\otimes I_s)VecT$.
Now by applying vector operation to the Discrete Matrix Sylvester System [1] satisfying the boundary conditions [2] we get

$$\Delta\widehat{X}(n)=H(n)\widehat{X}(n)+\widehat{F}(n) \tag{3}$$

satisfying

$$(N_0\otimes M_0)\widehat{X}(n_0)+(N_1\otimes M_1)\widehat{X}(n_1)=\widehat{W} \tag{4}$$

where $H(n)=[(I_s\otimes A)+(B^*\otimes I_s)+(B^*\otimes A)]$, $\widehat{X}=VecX$, $\widehat{F}=VecF$ and $\widehat{W}=VecW$. The corresponding homogeneous system of [3] is

$$L\widehat{T}(n)=\Delta\widehat{T}(n)-H(n)\widehat{T}(n)=0 \tag{5}$$

Lemma 3. *Let $\Phi_1(n,n_0)$ and $\Phi_2(n,n_0)$ be the fundamental matrix solutions for the systems*

$$\Delta T(n)=A(n)T(n) \tag{6}$$

$$\Delta T(n)=B^*(n)T(n) \tag{7}$$

respectively. Then the matrix $\Phi_1(n,n_0)\otimes\Phi_2^*(n,n_0)$ is a fundamental matrix of [5] and every solution of (5) is of the form $\widehat{T}(n)=(\Phi_2(n,n_0)\otimes\Phi_1^*(n,n_0))c$, where c is a s^2-column vectors.

Proof. Consider

$$\Delta(\Phi_2(n,n_0)\otimes\Phi_1^*(n,n_0)) = \Delta\Phi_2(n,n_0))\otimes\Phi_1^*(n,n_0) + (\Phi_2(n,n_0)\otimes$$
$$\Delta\Phi_1^*(n,n_0)) + \Delta\Phi_2(n,n_0)\otimes\Delta\Phi_1^*(n,n_0) = B^*(n)\Phi_2(n,n_0))\otimes\Phi_1^*(n,n_0)$$
$$+\Phi_2(n,n_0))\otimes A(n)\Phi_1^*(n,n_0) + B^*(n)\Phi_2(n,n_0))\otimes A(n)\Phi_1^*(n,n_0) =$$
$$[(B^*(n)\otimes I_s) + (I_s\otimes A(n)) + (B^*(n)\otimes A(n))](\Phi_2(n,n_0)\otimes\Phi_1^*(n,n_0))$$
$$= H(n)(\Phi_2(n,n_0)\otimes\Phi_1^*(n,n_0))$$

Hence $(\Phi_2(n,n_0)\otimes\Phi_1^*(n,n_0))$ is a fundamental matrix of (5). Clearly $\widehat{T}(n) = (\Phi_2(n,n_0)\otimes \Phi_1^*(n,n_0))c$ is a solution of (5) and every solution is of this form. □

The two-point Boundary Value Problem (3) – (4) has a unique solution $\widehat{T}$ if and only if the charecteristic matrix D defined by

$$D = (N_0\otimes M_0)(\Phi_2(n_0,n_0))\otimes\Phi_1^*(n_0,n_0)) + (N_1\otimes M_1)(\Phi_2(n_1,n_0))\otimes\Phi_1^*(n_1,n_0)) \tag{8}$$

is non singular. In this case the formal solution of the form

$$\widehat{T}(n) = (\Phi_2(n,n_0)\otimes\Phi_1^*(n,n_0))D^{-1}\widehat{W} + \sum_{j=n_0}^{n_1-1} G(n,j+1)\widehat{F}(j) \tag{9}$$

where G is the Greens matrix for the homogeneous Boundary value Problem (3) – (4) given by

$$G(n,j+1) = \begin{cases} (\Phi_2(n,n_0)\otimes\Phi_1^*(n,n_0))D^{-1}(N_0\otimes M_0)(\Phi_2(n_0,n_0) \\ \otimes\Phi_1^*(n_0,n_0))(\Phi_2(n_0,j+1))\otimes\Phi_1^*(n_0,j+1)), \\ n_0\le j\le n\le n_1-1 \\ \\ -(\Phi_2(n,n_0)\otimes\Phi_1^*(n,n_0))D^{-1}(N_1\otimes M_1)(\Phi_2(n_1,n_0) \\ \otimes\Phi_1^*(n_1,n_0))(\Phi_2(n_0,j+1))\otimes\Phi_1^*(n_0,j+1)), \\ n_0\le n\le j\le n_1-1 \end{cases} \tag{10}$$

Let $\|\nu\|_p = \left[\sum_{j=n_0}^{n_1-1}|\nu(j)|^p\right]^{1/p}, 1\le p\le\infty$ and $\|\nu\|_\infty = \underset{n\in[n_0,n_1]}{Sup}|\nu(n)|$

$$\|\widehat{T}\| = \|\widehat{T}\|_\infty \leq \eta|\widehat{W}| + \gamma_q\|\widehat{F}\|_p, \frac{1}{p} + \frac{1}{q} = 1, \tag{11}$$

where

$$\eta = \|(\Phi_2(n, n_0)) \otimes \Phi_1^*(n, n_0))D^{-1}\| \tag{12}$$

$$\gamma_q = \underset{n\in[n_0,n_1]}{Sup} \left[\sum_{j=n_0}^{n_1-1} |G(n, j)|^q\right]^{1/q} \tag{13}$$

Depending upon the problem of consideration suitable norm is chosen in (11). When $p = 1$ equations $(11), (12), (13)$ become

$$\|\widehat{T}\| \leq \eta|\widehat{W}| + \gamma\|\widehat{F}\|, \tag{14}$$

$$\eta = \|(\Phi_2(n, n_0)) \otimes \Phi_1^*(n, n_0))D^{-1}\| \tag{15}$$

$$\gamma = \underset{n\in[n_0,n_1]}{Sup} \left[|G(n, j)|\right] \tag{16}$$

Consider

$$\|(\Phi_2(n, n_0)) \otimes \Phi_1^*(n, n_0))D^{-1}\|^2$$
$$= \|G(n, n_0)G^*(n, n_0) + G(n, n_1)G^*(n, n_1)\|$$
$$\{\text{By assuming that}(N_0^* N_0 \otimes M_0 M_0^*) + (N_1^* N_1 \otimes M_1 M_1^*) = I_{s^2}\}$$
i.e., $\eta^2 \leq \gamma^2 + \gamma^2$ and hence $\eta \leq \sqrt{2}\gamma$.

To simplify the algebra, we investigate the fundamental matrix $(\Phi_2(n, n_0)) \otimes \Phi_1^*(n, n_0))$ whose characteristic matrix is Identity. Thus $(\Phi_2(n, n_0)) \otimes \Phi_1^*(n, n_0))$ is the fundamental matrix for $L\widehat{T} = 0$ for which

$$\begin{aligned} D =&(N_0^* \otimes M_0)(\Phi_2(n_0, n_0)) \otimes \Phi_1^*(n_0, n_0))+ \\ &(N_1^* \otimes M_1)(\Phi_2(n_1, n_0)) \otimes \Phi_1^*(n_1, n_0)) = I_{s^2}. \end{aligned} \tag{17}$$

Then the Greens matrix is given by

$$G(n,j+1)=\begin{cases}(\Phi_2(n,n_0)\otimes\Phi_1^*(n,n_0))(N_0\otimes M_0)(\Phi_2(n_0,n_0)\\ \otimes\Phi_1^*(n_0,n_0))(\Phi_2(n_0,j+1))\otimes\Phi_1^*(n_0,j+1)),\\ n_0\leq j\leq n\leq n_1-1\\ \\ -(\Phi_2(n,n_0)\otimes\Phi_1^*(n,n_0))(N_1\otimes M_1)(\Phi_2(n_1,n_0)\\ \otimes\Phi_1^*(n_1,n_0))(\Phi_2(n_0,j+1))\otimes\Phi_1^*(n_0,j+1)),\\ n_0\leq n\leq j\leq n_1-1\end{cases}\tag{18}$$

Result 4. *The fundamental matrix* $(\Phi_2(n,n_0)\otimes\Phi_1^*(n,n_0))$ *of* $L\widehat{T}=0$ *satisfies the following relations*

$$\begin{aligned}(i)(\Phi_2(n,n_0)\otimes\Phi_1(n,n_0)) =&G(n,j+1)(\Phi_2(j+1,n_0)\otimes\Phi_1(j+1,n_0))\\ &-G(n,k+1)(\Phi_2(k+1,n_0)\otimes\Phi_1(k+1,n_0))\\ &n_0\leq j+1<n\leq u\leq n_1-1\end{aligned}$$

and

$$\begin{aligned}(ii)(\Phi_2(n_0,n)\otimes\Phi_1(n_0,n)) =&(\Phi_2(n_0,u+1)\otimes\Phi_1(n_0,u+1))G(u+1,n)\\ &-(\Phi_2(n_0,j+1)\otimes\Phi_1(n_0,j+1))G(j+1,n)\\ &n_0\leq j<n\leq u\leq n_1-1\end{aligned}$$

Proof. From equation(17) we have

$$\begin{aligned}&((\Phi_2(n,n_0)\otimes\Phi_1(n,n_0))=\\ &((\Phi_2(n,n_0)\otimes\Phi_1(n,n_0))[(N_0^*\otimes M_0)(\Phi_2(n_0,n_0))\otimes\Phi_1^*(n_0,n_0))+\\ &(N_1^*\otimes M_1)(\Phi_2(n_1,n_0))\otimes\Phi_1^*(n_1,n_0))]=\\ &((\Phi_2(n,n_0)\otimes\Phi_1(n,n_0))[(N_0^*\otimes M_0)(\Phi_2(n_0,n_0))\otimes\Phi_1^*(n_0,n_0))(\Phi_2(n_0,j+1))\\ &\otimes\Phi_1^*(n_0,j+1))(\Phi_2(j+1,n_0))\otimes\Phi_1^*(j+1,n_0))]+((\Phi_2(n,n_0)\otimes\Phi_1(n,n_0))\\ &[(N_1^*\otimes M_1)(\Phi_2(n_1,n_0))\otimes\Phi_1^*(n_1,n_0))(\Phi_2(n_0,k+1))\otimes\Phi_1^*(n_0,k+1))\\ &(\Phi_2(k+1,n_0))\otimes\Phi_1^*(k+1,n_0))]=\\ &G(n,j+1)\Phi_2(j+1,n_0)\otimes\Phi_1(j+1,n_0)-\\ &\qquad G(n,k+1)\Phi_2(k+1,n_0)\otimes\Phi_1(k+1,n_0)\end{aligned}$$

Since every Fundamental Matrix $((\Phi_2(n,n_0)\otimes\Phi_1(n,n_0))$ of $L\widehat{T}=0$ can be represented as $((\Phi_2(n,n_0)\otimes\Phi_1(n,n_0))=((\Phi_2(j,n_0)\otimes\Phi_1(j,n_0))(C_1\otimes C_2)$ for some constant matrices

C_1 and C_2 (i) follows.
Similarly (ii) can be proved. □

3. DICHOTOMY AND STRONG DICHOTOMY OF DISCRETE MATRIX SYLVESTER SYSTEMS

This section is concerned with Dichotomy and Strong Dichotomy of Discrete Matrix Sylvester Systems. This sections presents the definitions of Dichotomic solution space, Projections of Fundamental Matrix Solutions and characterizations of Strong and Exponential Dichotomy of Discrete Matrix Sylvester Systems. This section also present the bounds for the angle between solutions spaces and characterization of separability of boundary conditions, ranks and projections of Fundamental Matrix Solutions and stability constant of Matrix Sylvester Systems. Finally bounds of Green's matrix over the entire solution space are presented.

Definition 5. We say that the solution space Ω of $L\widehat{T} = 0$ is dichotomic, if $\exists$ a splitting $\Omega = \Omega_1 + \Omega_2$, and a constant K such that

$$\Phi \in \Omega_1 \Longrightarrow \frac{\Phi(n)}{\Phi(j)} \leq K, \text{ for } n \geq j$$

$$\Phi \in \Omega_2 \Longrightarrow \frac{\Phi(n)}{\Phi(j)} \leq K, \text{ for } n \leq j$$

Note 6. *If P_1 and P_2 are projections for the corresponding fundamental matrix solutions $\Phi_2(n, n_0)$ and $\Phi_1(n, n_0)$ of* (6) *and* (7) *respectively, then $(P_1 \otimes P_2)$ is the projection matrix corresponding to $\Phi_2(n, n_0) \otimes \Phi_1(n, n_0)$.*

Equivalently, if for every fundamental matrix $\Phi_2(n, n_0) \otimes \Phi_1(n, n_0)$ $\exists$ a projection $P_1 \otimes P_2 \in R^{s^2 \times s^2}$, with

$$\Omega_1 = \Big\{(\Phi_2(n, n_0) \otimes \Phi_1(n, n_0))(P_1 \otimes P_2)C/C \in \mathbb{R}^{s^2} \tag{19}$$

$$\Omega_2 = \Big\{(\Phi_2(n, n_0) \otimes \Phi_1(n, n_0))(I_{s^2} - P_1 \otimes P_2)C/C \in \mathbb{R}^{s^2} \tag{20}$$

then we say that the two-point boundary value problem is dichotomic.

Definition 7. We say that the solution space of $L\widehat{T} = 0$ is strong dichotomic, if $\exists$ a constant K and a projection $P_1 \otimes P_2 \in \mathbb{R}^{s^2 \times s^2}$ such that for a fixed fundamental matrix $(\Phi_2(n, n_0) \otimes \Phi_1(n, n_0))$,

$$|(\Phi_2(n, n_0) \otimes \Phi_1(n, n_0))(P_1 \otimes P_2)(\Phi_2(n_0, j) \otimes \Phi_1(n_0, j))| \leq K, n \geq j$$

$$|(\Phi_2(n,n_0)\otimes\Phi_1(n,n_0))((I_{s^2}-(P_1\otimes P_2))(\Phi_2(n_0,j)\otimes\Phi_1(n_0,j))|\leq K,$$
$$n\leq j$$

Definition 8. The solution space of $L\widehat{T}=0$ is said to be exponentially dichotomic, if $\exists$ a constant $K>0$, positive constant λ,μ and a projection $P_1\otimes P_2\in\mathbb{R}^{s^2\times s^2}$ such that for a fixed fundamental matrix $(\Phi_2(n,n_0)\otimes\Phi_1(n,n_0))$,

$$|(\Phi_2(n,n_0)\otimes\Phi_1(n,n_0))(P_1\otimes P_2)(\Phi_2(n_0,j)\otimes\Phi_1(n_0,j))|\leq Ke^{\lambda(j-n)},$$
$$n\geq j$$
$$|(\Phi_2(n,n_0)\otimes\Phi_1(n,n_0))((I_{s^2}-(P_1\otimes P_2))(\Phi_2(n_0,j)\otimes\Phi_1(n_0,j))|\leq$$
$$Ke^{\lambda(j-n)},\ n\leq j$$

Lemma 9. *Let Ω_1 and Ω_2 be defined as in equations* (19) *and* (20)

$$\Phi\in\Omega_1\Longrightarrow\frac{\Phi(n)}{\Phi(j)}\leq$$
$$|(\Phi_2(n,n_0)\otimes\Phi_1(n,n_0))(P_1\otimes P_2)(\Phi_2(n_0,j)\otimes\Phi_1(n_0,j))|,\ \textit{for}\ n\geq j$$
$$\Phi\in\Omega_2\Longrightarrow\frac{\Phi(n)}{\Phi(j)}\leq|(\Phi_2(n,n_0)\otimes\Phi_1(n,n_0))((I_{s^2}-(P_1\otimes P_2))$$
$$(\Phi_2(n_0,j)\otimes\Phi_1(n_0,j))|,\ \textit{for}\ n\leq j$$

Proof. Let $\Phi\in\Omega_1$ then $\exists$ a constant $c\in\mathbb{R}^{s^2}$ such that $\Phi(n)=(\Phi_2(n,n_0)\otimes\Phi_1(n,n_0))(P_1\otimes P_2)c$. Thus for all $n\geq j$, we have

$$\begin{aligned}\frac{|\Phi(n)|}{|\Phi(j)|}&=\frac{|(\Phi_2(n,n_0)\otimes\Phi_1(n,n_0))(P_1\otimes P_2)c|}{|(\Phi_2(j,n_0)\otimes\Phi_1(j,n_0))(P_1\otimes P_2)c|}\\&=\frac{|(\Phi_2(n,n_0)\otimes\Phi_1(n,n_0))(P_1\otimes P_2)(P_1\otimes P_2)K(j)c|}{|(\Phi_2(j,n_0)\otimes\Phi_1(j,n_0))(P_1\otimes P_2)c|},\\&\text{where }\ K(j)=(\Phi_2(n_0,j)\otimes\Phi_1(n_0,j))(\Phi_2(j,n_0)\otimes\Phi_1(j,n_0)\\&\leq|(\Phi_2(n,n_0)\otimes\Phi_1(n,n_0))(P_1\otimes P_2)(\Phi_2(n_0,j)\otimes\Phi_1(n_0,j))|.\end{aligned}$$

Second inequality can be proved in the similar lines. □

Hence we can observe that dichotomy follows from strong dichotomy.

Definition 10. The angle $0\leq\theta(n)\leq\pi/2$ between Ω_1 and Ω_2 is defined by

$$\underset{|x|=1,|y|=1,x\in\Omega_1,y\in\Omega_2}{Max}|x^*y|$$

Theorem 11. *Let $|(\Phi_2(n,n_0)\otimes\Phi_1(n,n_0))(P_1\otimes P_2)(\Phi_2(n_0,j)\otimes\Phi_1(n_0,j))|\leq K$ for some K. Then $Cot\,\theta(n)\leq K$.*

Proof. Let $x \in \Omega_1$ and $y \in \Omega_2$ with $|x| = |y| = 1$ be such that $Cos\,\theta(n) = |x^* y|$. If x and y are orthogonal then the result is obvious. So assume that x and y are not orthogonal. Now define $\overline{x} = x, \overline{y} = -(x^* y)^{-1} y$. Clearly $\overline{x}$ is orthogonal to $\overline{x} + \overline{y}$ and hence

$$\cot\theta(n) = \frac{|\overline{x}|}{|\overline{x} + \overline{y}|}. \tag{21}$$

Since $\overline{x} \in \Omega_1$ and $\overline{y} \in \Omega_2$, we have $\overline{x} = (\Phi_2(n, n_0) \otimes \Phi_1(n, n_0))(P_1 \otimes P_2)c$ and $\overline{y} = (\Phi_2(n, n_0) \otimes \Phi_1(n, n_0))(I_{s^2} - (P_1 \otimes P_2))c$ for some $c \in \mathbb{R}^{s^2}$. Now

$$\begin{aligned}
\cot\theta(n) &= \frac{|(\Phi_2(n, n_0) \otimes \Phi_1(n, n_0))(P_1 \otimes P_2)c|}{|(\Phi_2(n, n_0) \otimes \Phi_1(n, n_0))c|} \\
&= \frac{|(\Phi_2(n, n_0) \otimes \Phi_1(n, n_0))(P_1 \otimes P_2)K(j)c|}{|(\Phi_2(n, n_0) \otimes \Phi_1(n, n_0))c|}, \\
&\text{where } K(j) = (\Phi_2(n_0, j) \otimes \Phi_1(n_0, j))(\Phi_2(j, n_0) \otimes \Phi_1(j, n_0)) \\
&= \underset{|\widehat{Q}|}{Max} |(\Phi_2(n, n_0) \otimes \Phi_1(n, n_0))(P_1 \otimes P_2)(\Phi_2(n_0, j) \otimes \Phi_1(n_0, j))\widehat{Q}| \le K.
\end{aligned}$$

□

Note 12. *From the above theorem, it can be observed that the angle between two spaces Ω_1 and Ω_2 can not become smaller than some threshold value $\cot^{-1} K$.*

In general the boundary condition equation 4 must represent s^2-linearly independent conditions on $\widehat{T}(n_0)$ and $\widehat{T}(n_1)$. Thus it is necessary that

$$rank[(N_0 \otimes M_0), (N_1 \otimes M_1)] = s^2 \tag{22}$$

Suppose that $rank(N_1 \otimes M_1)] = r_1 < s^2$, then $\exists\ s^2 \times s^2$ non singular matrix R representing an appropriate linear combination of the rows of $(N_1 \otimes M_1)$ such that

$$R(N_1 \otimes M_1) = \begin{pmatrix} 0 \\ X_{n_1} \end{pmatrix} \quad \begin{matrix} \}s^2 - r_1 \\ \}r_1 \end{matrix}, \qquad rank(X_{n_1}) = r_1$$

If we introduce the partitions

$$R(\widehat{Q}) = \begin{pmatrix} \widehat{Q}_{n_0} \\ \widehat{Q}_{n_1} \end{pmatrix} \quad \begin{matrix} \}s^2 - r_1 \\ \}r_1 \end{matrix}, \qquad R(N_0 \otimes M_0) = \begin{pmatrix} X(n_0) \\ X_{n_1} \end{pmatrix} \quad \begin{matrix} \}s^2 - r_1 \\ \}r_1 \end{matrix}$$

Where rank(X_{n_0})$= s^2 - r_1$ then we find that

$$R[(N_0 \otimes M_0)\widehat{T}_{n_0} + (N_1 \otimes M_1)\widehat{T}_{n_1}] = R\widehat{T}$$

is equivalent to

$$\left.\begin{array}{rcl} X_{n_0}\widehat{T}_{n_0} & = & \widehat{Q}_{n_0} \\ X_{n_1 n_0}\widehat{T}_{n_0} + X_{n_1}\widehat{T}_{n_1} & = & \widehat{Q}_{n_1} \end{array}\right\} \tag{23}$$

Obviously, if rank($(N_0 \otimes M_0)$)$= r_2 < s^2$, by analogous procedure we obtain

$$\left.\begin{array}{rcl} X_{n_0}\widehat{T}_{n_0} + X_{n_0 n_1}\widehat{T}_{n_1} & = & \widehat{Q}_{n_0} \\ X_{n_1}\widehat{T}_{n_1} & = & \widehat{Q}_{n_1} \end{array}\right\} \tag{24}$$

Both the form given by (23) and (24) consists of partially separated boundary conditions. In most of the applications we get $X_{n_0 n_1} = X_{n_1 n_0} = 0$ so that boundary conditions are separated.

Theorem 13. *If the boundary conditions are separable in the sense that $rank(N_0^* \otimes M_0) = s^2 - r_1$, $rank(N_1^* \otimes M_1) = r_1$ then $\exists$ a projection P such that*

$$|\Phi_2(n, n_0) \otimes \Phi_1(n, n_0)P(\Phi_2(n_0, j) \otimes \Phi_1(n_0, j)| \leq \gamma,\ n \geq j$$
$$|\Phi_2(n, n_0) \otimes \Phi_1(n, n_0)(I_{s^2} - P)(\Phi_2(n_0, j) \otimes \Phi_1(n_0, j)| \leq \gamma,\ n \leq j$$

where γ is the stability constant given by (16).

Proof. We first show that $P = (N_0^* \otimes M_0)(\Phi_2(n_0, n_0) \otimes \Phi_1(n_0, n_0))$ is a projection. Let E be an orthogonal matrix such that the last $s^2 - r_1$ rows of $(E \otimes I_s)(N_1^* \otimes M_1)$ are zero. Then

$$(E \otimes I_s)[(N_0^* \otimes M_0)(\Phi_2(n_0, n_0) \otimes \Phi_1(n_0, n_0))+$$
$$(N_1^* \otimes M_1)(\Phi_2(n_1, n_0) \otimes \Phi_1(n_1, n_0))] = I_{s^2}$$

On equating the last $s^2 - r_1$ rows of the above equation, we find that

$$\widetilde{P} = (E \otimes I_s)(N_0^* \otimes M_0)(\Phi_2(n_0, n_0) \otimes \Phi_1(n_0, n_0))(E \otimes I_s)^*$$

has the following structure'

$$\widetilde{P} = \begin{pmatrix} \widetilde{P}_{11} & \widetilde{P}_{12} \\ 0 & I_{s^2 - r_1} \end{pmatrix}$$

Since $rank(\widetilde{P}) = s^2 - r_1$, it follows that $\widetilde{P}_{11} = 0$ and hence $(\widetilde{P})^2 = \widetilde{P}$. Thus $P^2 = (E \otimes I_s)^* \widetilde{P}^2 (E \otimes I_s) = P$. Thus $P = (N_0^* \otimes M_0)(\Phi_2(n_0, n_0) \otimes \Phi_1(n_0, n_0))$ is a projection. From (18) it follows that

$$G(n, j+1) =$$

$$\begin{cases} (\Phi_2(n,n_0)\otimes\Phi_1(n,n_0))P(\Phi_2(n_0,j+1)\otimes\Phi_1(n_0,j+1)), j<n \\ -(\Phi_2(n,n_0)\otimes\Phi_1(n,n_0))(I_{s^2}-P)(\Phi_2(n_0,j+1)\otimes\Phi_1(n_0,j+1)), n<j \end{cases}$$

□

From the above theorem it follows that the boundary conditions are separable and a strong dichotomy exists when $K=\gamma$. From Lemma 9 it follows that the same result holds for weaker version of the dichotomy. By observing the growth of solutions over entire interval, separable boundary conditions are constructed. Let the singular value decomposition of $(\Phi_2(n_1,n_0)\otimes\Phi_1(n_1,n_0))(\Phi_2(n_0,n_0)\otimes\Phi_1(n_0,n_0))$ be given by

$$(\Phi_2(n_1,n_0)\otimes\Phi_1(n_1,n_0))(\Phi_2(n_0,n_0)\otimes\Phi_1(n_0,n_0)) = U\mathbb{D}V^*$$

where U and V are orthogonal matrices and $\mathbb{D}$ is a positive diagonal matrix with ordered elements. Let

$$\mathbb{D} = diag(d_1^{-1}, d_2^{-1}, ..., d_r^{-1}, d_{r+1}, ..., d_{s^2})$$

with $0<d_i\leq 1, i=1,2,...,s^2$

$$\mathbb{D}_1 = diag(d_1, d_2, ..., d_{r_1}, 1, 1, ..., 1)$$
$$\mathbb{D}_2 = diag(1, 1, ..., 1, d_{r+1}, ..., d_{s^2})$$

$$\widetilde{P} = \begin{pmatrix} 0 & 0 \\ 0 & I_{s^2-r_1} \end{pmatrix} \tag{25}$$

Now the separated boundary conditions are defined as

$$(\widetilde{N_0}^* \otimes \widetilde{M_0}) = \widetilde{P}V^* \text{ and } (\widetilde{N_1}^* \otimes \widetilde{M_1}) = (I_{s^2}-\widetilde{P})U^* \tag{26}$$

From the structure of $\widetilde{P}$ it can easily be verified that

$$(\widetilde{N_0}^* \otimes \widetilde{M_0})(\widetilde{\Phi}_2(n_0,n_0)\otimes\widetilde{\Phi_1}(n_0,n_0)) + (\widetilde{N_1}^* \otimes \widetilde{M_1})(\widetilde{\Phi}_2(n_1,n_0)\otimes\widetilde{\Phi_1}(n_1,n_0)) = I_{s^2} \tag{27}$$

where

$$\begin{aligned} &\widetilde{\Phi}_2(n,n_0)\otimes\widetilde{\Phi_1}(n,n_0) \\ &= (\Phi_2(n,n_0)\otimes\Phi_1(n,n_0))(\Phi_2(n_0,n_0)\otimes\Phi_1(n_0,n_0))V\mathbb{D}_1 \\ &= (\Phi_2(n,n_0)\otimes\Phi_1(n,n_0))(\Phi_2(n_0,n_1)\otimes\Phi_1(n_0,n_1))U\mathbb{D}_2 \end{aligned} \tag{28}$$

The corresponding Green's matrix is

$$\widetilde{G}(n,j+1) = \begin{cases} (\widetilde{\Phi}_2(n,n_0)\otimes\widetilde{\Phi_1}(n,n_0))(\widetilde{N_0}^*\otimes\widetilde{M}_0)(\widehat{\Phi}_2(n_0,n_0)\otimes\widehat{\Phi}_1(n_0,n_0)) \\ \qquad (\widehat{\Phi}_2(n_0,j+1)\otimes\widehat{\Phi}_1(n_0,j+1)),\ n>j \\ -(\widetilde{\Phi}_2(n,n_0)\otimes\widetilde{\Phi_1}(n,n_0))(\widetilde{N_1}^*\otimes\widetilde{M}_1)(\widehat{\Phi}_2(n_1,n_0)\otimes\widehat{\Phi}_1(n_1,n_0)) \\ \qquad (\widehat{\Phi}_2(n_0,j+1)\otimes\widehat{\Phi}_1(n_0,j+1)),\ n<j \end{cases} \tag{29}$$

Now the properties of the fundamental matrix $(\widetilde{\Phi}_2(n,n_0)\otimes\widetilde{\Phi_1}(n,n_0))$ in terms of Green's matrix (18) are established.

Result 14. *For the fundamental matrix* $(\widetilde{\Phi}_2(n,n_0)\otimes\widetilde{\Phi_1}(n,n_0))$ *defined as in* (28) *the following relations hold good*

$$\begin{aligned} &(i)(\widetilde{\Phi}_2(n,n_0)\otimes\widetilde{\Phi_1}(n,n_0)) \\ &= G(n,j+1)(\widetilde{\Phi}_2(j+1,n_0)\otimes\widetilde{\Phi_1}(j+1,n_0)) \\ &- G(n,k+1)(\widetilde{\Phi}_2(k+1,n_0)\otimes\widetilde{\Phi_1}(k+1,n_0)), n_0\le j<n\le k\le n_1 \\ &(ii)(\widetilde{\Phi}_2(n_0,n)\otimes\widetilde{\Phi_1}(n_0,n)) \\ &= (\widetilde{\Phi}_2(n_0,k+1)\otimes\widetilde{\Phi_1}(n_0,k+1))G(k+1,n) \\ &- (\widetilde{\Phi}_2(k+1,n_0)\otimes\widetilde{\Phi_1}(j+1,n_0))G(n,k+1), n_0\le j<n\le k\le n_1 \\ &(iii)(\widetilde{\Phi}_2(n,n_0)\otimes\widetilde{\Phi}_1(n,n_0))(\widetilde{\Phi}_2(n_0,k)\otimes\widetilde{\Phi}_1(n_0,k))G(k,j+1) = \\ &\quad G(n,j+1) \end{aligned}$$

Proof. (i) From (i) of Result 4, we have

$$\begin{aligned}(\Phi_2(n,n_0)\otimes\Phi_1(n,n_0)) =& G(n,j+1)(\Phi_2(j+1,n_0)\otimes\Phi_1(j+1,n_0)) \\ &- G(n,k+1)(\Phi_2(k+1,n_0)\otimes\Phi_1(k+1,n_0))\end{aligned}$$

Since $(\Phi_2(n,n_0)\otimes\Phi_1(n,n_0))V\mathbb{D}_1$ is non singular, from equation (28) we have

$$\begin{aligned}&(\Phi_2(n,n_0)\otimes\Phi_1(n,n_0)) = \\ &(\widetilde{\Phi}_2(n,n_0)\otimes\widetilde{\Phi}_1(n,n_0))((\Phi_2(n,n_0)\otimes\Phi_1(n,n_0))V\mathbb{D}_1)^{-1}\end{aligned}$$

we have

$$\begin{aligned}&(\widetilde{\Phi}_2(n,n_0)\otimes\widetilde{\Phi}_1(n,n_0))((\Phi_2(n,n_0)\otimes\Phi_1(n,n_0))V\mathbb{D}_1)^{-1} = \\ &G(n,j+1)(\widetilde{\Phi}_2(j+1,n_0)\otimes\widetilde{\Phi}_1(j+1,n_0))((\Phi_2(n_0,n_0)\otimes\Phi_1(n_0,n_0)) \\ &V\mathbb{D}_1)^{-1} - G(n,k+1)(\widetilde{\Phi}_2(k+1,n_0)\otimes\widetilde{\Phi}_1(k+1,n_0))\end{aligned}$$

$$((\Phi_2(n_0,n_0)\otimes\Phi_1(n_0,n_0))V\mathbb{D}_1)^{-1}$$

Thus

$$\begin{aligned}(\widetilde{\Phi}_2(n,n_0)\otimes\widetilde{\Phi}_1(n,n_0)) =& G(n,j+1)(\widetilde{\Phi}_2(j+1,n_0)\otimes\widetilde{\Phi}_1(j+1,n_0))-\\ & G(n,k+1)(\widetilde{\Phi}_2(k+1,n_0)\otimes\widetilde{\Phi}_1(k+1,n_0))\end{aligned}$$

(ii) From (ii) of Result 4 and equation (28),by proceeding in the same manner as that of (i) we can prove (ii).
(iii) Since

$$\begin{aligned}&(\widetilde{\Phi}_2(n,n_0)\otimes\widetilde{\Phi}_1(n,n_0)) =\\ &((\Phi_2(n,n_0)\otimes\Phi_1(n,n_0))((\Phi_2(n_0,n_0)\otimes\Phi_1(n_0,n_0))V\mathbb{D}_1)\end{aligned}$$

From the above relation,equation(18) and equation(28) we have

$$\begin{aligned}&(\widetilde{\Phi}_2(n,n_0)\otimes\widetilde{\Phi}_1(n,n_0))(\widetilde{\Phi}_2(n_0,k)\otimes\widetilde{\Phi}_1(n_0,k))G(k,j+1)\\ =&((\Phi_2(n,n_0)\otimes\Phi_1(n,n_0))((\Phi_2(n_0,n_0)\otimes\Phi_1(n_0,n_0))V\mathbb{D}_1)\\ &\mathbb{D}_1{}^{-1}V^{-1}((\Phi_2(n_0,n_0)\otimes\Phi_1(n_0,n_0))((\Phi_2(n_0,k)\otimes\Phi_1(n_0,k))\\ &((\Phi_2(k,n_0)\otimes\Phi_1(k,n_0))(N_0^*\otimes M_0)((\Phi_2(n_0,n_0)\otimes\Phi_1(n_0,n_0))\\ &((\Phi_2(n_0,j+1)\otimes\Phi_1(n_0,j+1))\\ =&((\Phi_2(n_0,n_0)\otimes\Phi_1(n_0,n_0))(N_0^*\otimes M_0)((\Phi_2(n_0,n_0)\otimes\Phi_1(n_0,n_0))\\ &((\Phi_2(n_0,j+1)\otimes\Phi_1(n_0,j+1))\\ =&G(n,j+1), n_0\leq j<n\leq k\leq n_1-1\end{aligned}$$

Similar proof can be given for $n<j$ case. □

Now we establish a relationship between the Green's matrices $\widetilde{G}$ and G defined in equations (29)and (18)respectively.

Theorem 15.

$$\begin{aligned}&\widetilde{G}(n,j+1) = G(n,j)-\\ &\quad(\widetilde{\Phi}_2(n,n_0)\otimes\widetilde{\Phi}_1(n,n_0))[(\widetilde{N_0}\otimes M_0)G(n_0,j)+(\widetilde{N_1}\otimes M_1)G(n_1,j)]\end{aligned}$$

Proof. By using (ii) and (iii) of Result 14 for $n>j$

$$\begin{aligned}&\widetilde{G}(n,j+1)\\ =&(\widetilde{\Phi}_2(n,n_0)\otimes\widetilde{\Phi}_1(n,n_0))(\widetilde{N_0}\otimes\widetilde{M_0})(\widetilde{\Phi}_2(n_0,n_0)\otimes\widetilde{\Phi}_1(n_0,n_0))\end{aligned}$$

$$
\begin{aligned}
&(\widetilde{\Phi}_2(n_0,j)\otimes\widetilde{\Phi}_1(n_0,j))\\
=&(\widetilde{\Phi}_2(n,n_0)\otimes\widetilde{\Phi}_1(n,n_0))(\widetilde{N_0}\otimes\widetilde{M_0})(\widetilde{\Phi}_2(n_0,n_0)\otimes\widetilde{\Phi}_1(n_0,n_0))\\
&[(\widetilde{\Phi}_2(n_0,n_1)\otimes\widetilde{\Phi}_1(n_0,n_1))G(n_1,j)-(\widetilde{\Phi}_2(n_0,n_0)\otimes\widetilde{\Phi}_1(n_0,n_0))G(n_0,j)]\\
=&(\widetilde{\Phi}_2(n,n_0)\otimes\widetilde{\Phi}_1(n,n_0))\{I_{s^2}-(\widetilde{N_1}\otimes\widetilde{M_1})(\widetilde{\Phi}_2(n_1,n_0)\otimes\widetilde{\Phi}_1(n_1,n_0))\}\\
&(\widetilde{N_0}\otimes M_0)(\widetilde{\Phi}_2(n_0,n_0)\otimes\widetilde{\Phi}_1(n_0,n_0))(\widetilde{\Phi}_2(n_0,n_1)\otimes\widetilde{\Phi}_1(n_0,n_1))G(n_1,j)\\
&-(\widetilde{\Phi}_2(n,n_0)\otimes\widetilde{\Phi}_1(n,n_0))(\widetilde{N_0^*}\otimes\widetilde{M_0})(\widetilde{\Phi}_2(n_0,n_0)\otimes\widetilde{\Phi}_1(n_0,n_0))G(n_0,j)\\
&(\text{from (27)})=\{(\widetilde{\Phi}_2(n,n_0)\otimes\widetilde{\Phi}_1(n,n_0))(\widetilde{\Phi}_2(n_0,n_1)\otimes\widetilde{\Phi}_1(n_0,n_1))\\
&G(n_1,j)\}-(\widetilde{\Phi}_2(n,n_0)\otimes\widetilde{\Phi}_1(n,n_0))(\widetilde{N_0^*}\otimes\widetilde{M_0})G(n_0,j)=G(n,j)-\\
&(\widetilde{\Phi}_2(n,n_0)\otimes\widetilde{\Phi}_1(n,n_0))[(\widetilde{N_0^*}\otimes\widetilde{M_0})G(n_0,j)+(\widetilde{N_1^*}\otimes\widetilde{M_1})G(n_1,j)]\\
&(\text{by using (iii) of Result 14})
\end{aligned}
$$

By proceeding in the same way as that of the above the theorem can be proved for $n < j$. □

From equations (26) and (28) we have

$$
\begin{aligned}
&(\widetilde{N_0^*}\otimes\widetilde{M_0})(\widetilde{\Phi}_2(n_0,n_0)\otimes\widetilde{\Phi}_1(n_0,n_0))=\overline{P}=\\
&\overline{P}V^*((\Phi_2(n_0,n_0)\otimes\Phi_1(n_0,n_0))((\Phi_2(n_0,n_0)\otimes\Phi_1(n_0,n_0))V\mathbb{D}_1
\end{aligned}
\tag{30}
$$

$$
(\widetilde{N_1^*}\otimes\widetilde{M_1})(\widetilde{\Phi}_2(n_1,n_0)\otimes\widetilde{\Phi}_1(n_0,n_0))=(I_{s^2}-\overline{P})
\tag{31}
$$

By substituting equations (30)and (31) in the equation (29)the Green's matrix for the boundary conditions

$$
(\widetilde{N_0^*}\otimes\widetilde{M_0})\widehat{T}(n_0)+(\widetilde{N_1^*}\otimes\widetilde{M_1})\widehat{T}(n_1)=\widehat{W}
$$

is obtained as

$$
\begin{aligned}
&\widetilde{G}(n,j+1)=\\
&\begin{cases}
(\widetilde{\Phi}_2(n,n_0)\otimes\widetilde{\Phi_1}(n,n_0))\overline{P}(\widetilde{\Phi}_2(n_0,j+1)\otimes\widetilde{\Phi}_1(n_0,j+1)),\ n>j\\
-(\widetilde{\Phi}_2(n,n_0)\otimes\widetilde{\Phi_1}(n,n_0))(I_{s^2}-\overline{P})(\widetilde{\Phi}_2(n_0,j+1)\otimes\widetilde{\Phi}_1(n_0,j+1)),\\
\qquad n<j
\end{cases}
\end{aligned}
$$

We now give the following estimates

Theorem 16. *Let γ be the suprimum of $|G(n,j)|$ over n,j then*

$$
(i)|(\widetilde{\Phi}_2(n_1,n_0)\otimes\widetilde{\Phi_1}(n_1,n_0))\overline{P}(\widetilde{\Phi}_2(n_0,j)\otimes\widetilde{\Phi}_1(n_0,j))|=|G(n_1,j)|\leq 2\gamma
$$

$$(ii)|(\widetilde{\Phi}_2(n_0,n_0)\otimes\widetilde{\Phi_1}(n_0,n_0))(I_{s^2}-\overline{P})(\widetilde{\Phi}_2(n_0,j)\otimes\widetilde{\Phi}_1(n_0,j))| = |G(n_0,j)| \leq 2\gamma$$
$$(iii)|(\widetilde{\Phi}_2(n,n_0)\otimes\widetilde{\Phi_1}(n,n_0))| \leq 2\gamma\ell$$
$$where\ \ell = Max\{|\widetilde{\Phi}_2(n_0,n_0)|,|\widetilde{\Phi}_1(n_0,n_0)|,|\widetilde{\Phi}_2(n_1,n_0)|,|\widetilde{\Phi}_1(n_1,n_0)|\}$$

Proof. (i) Consider

$$|G(n_1,j)| = (\widetilde{\Phi}_2(n_1,n_0)\otimes\widetilde{\Phi_1}(n_1,n_0))\overline{P}(\widetilde{\Phi}_2(n_0,j)\otimes\widetilde{\Phi}_1(n_0,j))| =$$
$$(\widetilde{\Phi}_2(n_1,n_0)\otimes\widetilde{\Phi_1}(n_1,n_0))\overline{P}[(\widetilde{\Phi}_2(n_0,n_1)\otimes\widetilde{\Phi_1}(n_0,n_1))G(n_1,j)-$$
$$(\widetilde{\Phi}_2(n_0,n_0)\otimes\widetilde{\Phi_1}(n_0,n_0))G(n_0,j)] \leq |G(n_1,j)|+$$
$$|(\widetilde{\Phi}_2(n_1,n_0)\otimes\widetilde{\Phi_1}(n_1,n_0))\overline{P}(\widetilde{\Phi}_2(n_0,n_0)\otimes\widetilde{\Phi_1}(n_0,n_0))G(n_0,j)| \leq$$
$$\gamma + |U\mathbb{D}_2\overline{P}\mathbb{D}_1^{-1}V^{-1}|\gamma[\text{By putting } t=n_0 \text{ and } t=n_1 \text{ in equation (28)}]$$
$$= \gamma + |U\mathbb{D}\overline{P}V^{-1}|\gamma < \gamma+\gamma = 2\gamma$$

(ii) Proof is same as that of (i)

(iii)From Result14 it follows that

$$|(\widetilde{\Phi}_2(n,n_0)\otimes\widetilde{\Phi_1}(n,n_0))| =$$
$$|G(n,n_0)(\widetilde{\Phi}_2(n_0,n_0)\otimes\widetilde{\Phi_1}(n_0,n_0)) - G(n,n_1)(\widetilde{\Phi}_2(n_1,n_0)\otimes\widetilde{\Phi_1}(n_1,n_0))|$$
$$\leq |G(n,n_0)||\widetilde{\Phi}_2(n_0,n_0)||\widetilde{\Phi_1}(n_0,n_0)| + |G(n,n_1)||\widetilde{\Phi}_2(n_1,n_0)||\widetilde{\Phi_1}(n_1,n_0)|$$
$$\leq \gamma(|\widetilde{\Phi}_2(n_0,n_0)||\widetilde{\Phi_1}(n_0,n_0)| + |\widetilde{\Phi}_2(n_1,n_0)||\widetilde{\Phi_1}(n_1,n_0)|) \leq 2\gamma\ell$$

□

Result 17.

$$(i)|\widetilde{G}(n,j)| \leq \gamma + 4\gamma^2\ell$$
$$(ii)|\widetilde{G}(n,j)| \leq \gamma + 2\gamma^2\xi, where\ \xi = \eta\, Max\{|\widetilde{N_0^*}|,|\widetilde{M}_0|,|\widetilde{N_1^*}|,|\widetilde{M}_1|\}.$$

Proof. (i) From Theorem 15, we have

$$|\widetilde{G}(n,j)| = |G(n,j) - [\widetilde{\Phi}_2(n,n_0)\otimes\widetilde{\Phi_1}(n,n_0)][(\widetilde{N_0^*}\otimes\widetilde{M}_0)G(n_0,j)+$$
$$(\widetilde{N_1^*}\otimes\widetilde{M}_1)G(n_1,j)]| = |G(n,j) - [\widetilde{\Phi}_2(n,n_0)\otimes\widetilde{\Phi_1}(n,n_0)][\overline{P}V^*G(n_0,j)$$
$$+(I_{s^2}-\overline{P})U^*G(n_1,j)]| \leq \gamma + 2\gamma^2\ell + 2\gamma^2\ell = \gamma + 4\gamma^2\ell$$
$$\{From\, Theorem 16\}$$

(ii)

$$|\widetilde{G}(n,j)| = |G(n,j) - [\widetilde{\Phi}_2(n,n_0)\otimes\widetilde{\Phi_1}(n,n_0)][(\widetilde{N_0^*}\otimes\widetilde{M}_0)G(n_0,j)+$$

$$(\widetilde{N_1^*} \otimes \widetilde{M_1})G(n_1, j)]| \leq |G(n, j)| + |\widetilde{\Phi}_2(n, n_0) \otimes \widetilde{\Phi}_1(n, n_0)| +$$
$$[(\widetilde{N_0^*} \otimes \widetilde{M_0})G(n_0, j) + (\widetilde{N_1^*} \otimes \widetilde{M_1})G(n_1, j)]| = \gamma + \eta[|\widetilde{N_0^*}||\widetilde{M_0}|\gamma +$$
$$|\widetilde{N_1^*}||\widetilde{M_1}|\gamma] \leq \gamma + \gamma\eta 2Max\{|\widetilde{N_0^*}|, |\widetilde{M_0}|, |\widetilde{N_1^*}|, |\widetilde{M_1}|\} = \gamma + 2\gamma\xi$$

□

Theorem 18.

$$(i)|(\widetilde{\Phi}_2(n, n_0) \otimes \widetilde{\Phi_1}(n, n_0))\overline{P}(\widetilde{\Phi}_2(n_0, j) \otimes \widetilde{\Phi_1}(n_0, j))| \leq \gamma + 4\gamma^2\ell,\ n > j$$
$$(ii)|(\widetilde{\Phi}_2(n, n_0) \otimes \widetilde{\Phi_1}(n, n_0))(I_{s^2} - \overline{P})(\widetilde{\Phi}_2(n_0, j) \otimes \widetilde{\Phi_1}(n_0, j))| \leq \gamma + 4\gamma^2\ell,\ n < j$$
$$(iii)|(\widetilde{\Phi}_2(n, n_0) \otimes \widetilde{\Phi_1}(n, n_0))\overline{P}(\widetilde{\Phi}_2(n_0, j) \otimes \widetilde{\Phi_1}(n_0, j))| \leq \gamma + 2\gamma\xi,\ n > j$$
$$(iv)|(\widetilde{\Phi}_2(n, n_0) \otimes \widetilde{\Phi_1}(n, n_0))(I_{s^2} - \overline{P})(\widetilde{\Phi}_2(n_0, j) \otimes \widetilde{\Phi_1}(n_0, j))| \leq \gamma + 2\gamma\xi,\ n < j$$

Proof. Proof follows form the definition of $\widetilde{G}(n, j)$ and Result17 □

Now we investigate the stability bounds of exponential dichotomy. We replace the condition in equation(16)by

$$|G(n, j)| \leq \gamma e^{\lambda(j-n)}, n > j, \lambda > 0 \tag{32}$$
$$|G(n, j)| \leq \gamma e^{\mu(j-n)}, n < j, \mu > 0 \tag{33}$$

and using similar discussed techniques, we can show that equations (32) and (33)imply an exponential dichotomic space for the two-point boundary value problem.

Theorem 19. *Let*

$$\alpha(n) = \gamma\ell[e^{\lambda(n_0-n)} + e^{\mu(n-n_1)}] \tag{34}$$
$$\beta(n) = \gamma[e^{\lambda(n-n_1)} + e^{\mu(n_0-n)}] \tag{35}$$

$\overline{P}$ *is defined same as in equation(25),* $\xi = \eta Max\{|\widetilde{N_0^*}|, |\widetilde{M_0}|, |\widetilde{N_1^*}|, |\widetilde{M_1}|\}$. *Then the following relations hold good*

$$(i)|(\widetilde{\Phi}_2(n, n_0) \otimes \widetilde{\Phi_1}(n, n_0))\overline{P}(\widetilde{\Phi}_2(n_0, j) \otimes \widetilde{\Phi_1}(n_0, j))|$$
$$\leq \gamma e^{\lambda(j-n)} + \alpha(n)\beta(j),\ n > j$$
$$(ii)|(\widetilde{\Phi}_2(n, n_0) \otimes \widetilde{\Phi_1}(n, n_0))(I_{s^2} - \overline{P})(\widetilde{\Phi}_2(n_0, j) \otimes \widetilde{\Phi_1}(n_0, j))|$$
$$\leq \gamma e^{\mu(n-j)} + \alpha(n)\beta(j),\ n < j$$
$$(iii)|(\widetilde{\Phi}_2(n, n_0) \otimes \widetilde{\Phi_1}(n, n_0))\overline{P}(\widetilde{\Phi}_2(n_0, j) \otimes \widetilde{\Phi_1}(n_0, j))|$$
$$\leq \gamma e^{\lambda(j-n)} + \xi\beta(j),\ n > j$$
$$(iv)|(\widetilde{\Phi}_2(n, n_0) \otimes \widetilde{\Phi_1}(n, n_0))(I_{s^2} - \overline{P})(\widetilde{\Phi}_2(n_0, j) \otimes \widetilde{\Phi_1}(n_0, j))|$$
$$\leq \gamma e^{\mu(n-j)} + \xi\beta(j),\ n < j$$

4. CONDITIONING OF BOUNDARY VALUE PROBLEMS ASSOCIATED WITH OF DISCRETE MATRIX SYLVESTER SYSTEMS

This section deals with Conditioning of Boundary Value Problems associated with of Discrete Matrix Sylvester Systems. After introducing wellposedness and condition number of Boundary Value problems of the solution of the boundary value problem finally presented estimates for the condition number Boundary Value Problems associated with of Discrete Matrix Sylvester Systems in terms of well-known quantities and value of the fundamental matrix.

$$\Delta\widehat{T}(n) = H(n)\widehat{T}(n) + \widehat{F}(n) \tag{36}$$

satisfying

$$(I_s \otimes M_0)\widehat{T}(n_0) + (I_s \otimes M_1)\widehat{T}(n_1) = \widehat{W} \tag{37}$$

is unique then the characteristic matrix

$$D = (I_s \otimes M_0)(\Phi_2(n_0, n_0) \otimes \Phi_1(n_0, n_0))+ (I_s \otimes M_1)(\Phi_2(n_1, n_0) \otimes \Phi_1(n_1, n_0)) \tag{38}$$

must be non singular and in this case the boundary value problem is said to be well posed.

Definition 20. The condition number η of the boundary value problem (36)-(37) is defined as

$$\eta = \underset{n_0 \le n \le n_1}{Sup} \|(\Phi_2(n, n_0) \otimes \Phi_1(n, n_0))D^{-1}\| \tag{39}$$

The number η is independent of the choice of the fundamental matrix.

We consider the variation $\widehat{T}(n)$ of equation (36) with respect to small perturbation in the boundary conditions, the perturbation of equation (37) is of the form

$$(I_s \otimes (M_0 + \delta M_0))\widehat{T}(n_0) + (I_s \otimes (M_1 + \delta M_1))\widehat{T}(n_1) = \widehat{W} + \delta\widehat{W} \tag{40}$$

then perturbed characteristic matrix

$$D_1 = (I_s \otimes (M_0 + \delta M_0))(\Phi_2(n_0, n_0) \otimes \Phi_1(n_0, n_0))+ (I_s \otimes (M_1 + \delta M_1))(\Phi_2(n_1, n_0) \otimes \Phi_1(n_1, n_0)) = D + \delta D$$

where $\delta D = (I_s \otimes \delta M_0)(\Phi_2(n_0, n_0) \otimes \Phi_1(n_0, n_0))+$
$(I_s \otimes \delta M_1)(\Phi_2(n_1, n_0) \otimes \Phi_1(n_1, n_0))$.

Assume that D_1 is non singular. Let $\widetilde{T}(n)$ be the unique solution of equation (36) satisfying equation (40).

Lemma 21.

$$\|(\delta D)D^{-1}\| \leq (\|\delta M_0\| + \|\delta M_1\|)\eta$$

Proof. Consider

$$\begin{aligned}
\|(\delta D)D^{-1}\| &= \|[(I_s \otimes \delta M_0)(\Phi_2(n_0,n_0) \otimes \Phi_1(n_0,n_0))+ \\
&\qquad (I_s \otimes \delta M_1)(\Phi_2(n_1,n_0) \otimes \Phi_1(n_1,n_0))]D^{-1}\| \\
&\leq \|(I_s \otimes \delta M_0)\|\|(\Phi_2(n_0,n_0) \otimes \Phi_1(n_0,n_0))D^{-1}\|+ \\
&\qquad \|(I_s \otimes \delta M_1)\|\|(\Phi_2(n_1,n_0) \otimes \Phi_1(n_1,n_0))D^{-1}\| \\
&\leq (\|\delta M_0\| + \|\delta M_1\|)\|(\Phi_2(n_1,n_0) \otimes \Phi_1(n_1,n_0))D^{-1}\| \\
&\leq (\|\delta M_0\| + \|\delta M_1\|)\eta
\end{aligned}$$

□

Theorem 22. *Let $\epsilon > 0$ be such that $0 < \epsilon < \frac{1}{(1+k)\delta\eta}$ where*

$$\delta = Max\{\|\delta M_0\|, \|\delta M_1\|, \|\delta\widehat{W}\|, \|\delta D\|\} \text{ and}$$
$$k = \sum_{j=n_0}^{n_1-1} \|(\Phi_2(n_0,j+1) \otimes \Phi_1(n_0,j+1))\widehat{F}(j)\|$$

Then the solution $\widetilde{T}(n)$ of equation (36) satisfying equation (40) such that

$$\begin{aligned}
&\delta\eta(1-k)(\|(\Phi_2(n_0,n_0)\|\|(\Phi_1(n_0,n_0)\| + \|(\Phi_2(n_1,n_0)\|\|(\Phi_1(n_01,n_0)\|) \\
&\leq \underset{n\in[n_0,n_1]}{Max} \|\widetilde{T}(n) - \widehat{T}(n)\| \leq \\
&\delta\eta(1+k)(\|(\Phi_2(n_0,n_0)\|\|(\Phi_1(n_0,n_0)\| + \|(\Phi_2(n_1,n_0)\|\|(\Phi_1(n_01,n_0)\|)
\end{aligned}$$

Proof. Any solution $\widehat{T}(n)$ of equation (36)satisfying equation (37) is of the form

$$\widehat{T}(n) = (\Phi_2(n,n_0) \otimes \Phi_1(n,n_0))D^{-1}\widehat{W} + \sum_{j=n_0}^{n_1} G(n,j+1)\widehat{F}(j)$$

where the Green's matric $G(n,j)$ is given by

$$G(n,j) =$$

$$\begin{cases} (\Phi_2(n,n_0)\otimes\Phi_1^*(n,n_0))D^{-1}(I_s\otimes M_0)(\Phi_2(n_0,n_0)\otimes\Phi_1^*(n_0,n_0)) \\ (\Phi_2(n_0,j))\otimes\Phi_1^*(n_0,j)), n_0\le j\le n\le n_1 \\ -(\Phi_2(n,n_0)\otimes\Phi_1^*(n,n_0))D^{-1}(I_s\otimes M_1)(\Phi_2(n_1,n_0)\otimes\Phi_1^*(n_1,n_0)) \\ (\Phi_2(n_0,j))\otimes\Phi_1^*(n_0,j)), n_0\le n\le j\le n_1 \end{cases}$$

and any solution $\widetilde{T}(n)$ of equation (36)satisfying equation (38) is given by

$$\widetilde{T}(n) = (\Phi_2(n,n_0)\otimes\Phi_1(n,n_0))D_1^{-1}(\widehat{W}+\delta\widehat{W}) + \sum_{j=n_0}^{n_1-1} G(n,j+1)\widehat{F}(j)$$

where

$$G(n,j) =$$

$$\begin{cases} (\Phi_2(n,n_0)\otimes\Phi_1^*(n,n_0))D^{-1}(I_s\otimes(M_0+\delta M_0))(\Phi_2(n_0,n_0)\otimes\Phi_1^*(n_0,n_0)) \\ (\Phi_2(n_0,j))\otimes\Phi_1^*(n_0,j)), n_0\le j\le n\le n_1 \\ -(\Phi_2(n,n_0)\otimes\Phi_1^*(n,n_0))D^{-1}(I_s\otimes(M_1+\delta M_1))(\Phi_2(n_1,n_0)\otimes\Phi_1^*(n_1,n_0)) \\ (\Phi_2(n_0,j))\otimes\Phi_1^*(n_0,j)), n_0\le n\le j\le n_1 \end{cases}$$

Now consider

$$\begin{aligned} \|\widetilde{T}(n)-\widehat{T}(n)\| &\le \|(\Phi_2(n,n_0)\otimes\Phi_1^*(n,n_0))[D_1^{-1}(\widehat{W}+\delta\widehat{W})-D_1^{-1}\widehat{W}]\| \\ &+\sum_{j=n_0}^{n-1}\|(\Phi_2(n,n_0)\otimes\Phi_1^*(n,n_0))[D_1^{-1}(I_s\otimes(M_0+\delta M_0))-D^{-1}(I_s\otimes M_0)] \\ &\quad(\Phi_2(n_0,n_0)\otimes\Phi_1^*(n_0,n_0))(\Phi_2(n_0,j+1)\otimes\Phi_1^*(n_0,j+1))\| \\ &+\sum_{j=n}^{n_1-1}\|(\Phi_2(n,n_0)\otimes\Phi_1^*(n,n_0))[D_1^{-1}(I_s\otimes(M_1+\delta M_1))-D^{-1}(I_s\otimes M_1)] \\ &\quad(\Phi_2(n_1,n_0)\otimes\Phi_1^*(n_1,n_0))(\Phi_2(n_0,j+1)\otimes\Phi_1^*(n_0,j+1))\| \end{aligned} \tag{41}$$

In accordance with the linear terms we have the following rough estimates.

$$\begin{aligned} &D_1^{-1}(\widehat{W}+\delta\widehat{W})-D_1^{-1}\widehat{W} = (D+\delta D)^{-1}(\widehat{W}+\delta\widehat{W})-D^{-1}\widehat{W} = \\ &D^{-1}(I_{s^2}+D^{-1}\delta D)^{-1}(\widehat{W}+\delta\widehat{W})-D^{-1}\widehat{W} \cong \\ &D^{-1}(I_{s^2}-D^{-1}\delta D)^{-1}(\widehat{W}+\delta\widehat{W})-D^{-1}\widehat{W} \cong D^{-1}\delta\widehat{W} \end{aligned}$$

Similarly

$$D_1^{-1}(I_s\otimes(M_0+\delta M_0))-D^{-1}(I_s\otimes M_0)\cong D^{-1}(I_s\otimes\delta M_0)$$

$$D_1^{-1}(I_s \otimes (M_1 + \delta M_1)) - D^{-1}(I_s \otimes M_1) \cong D^{-1}(I_s \otimes \delta M_1)$$

By using these estimates in equation(41) we get

$$\begin{aligned}
&\|\widetilde{T}(n) - \widehat{T}(n)\| \leq \|(\Phi_2(n,n_0) \otimes \Phi_1^*(n,n_0))D^{-1}\delta\widehat{W})\| \\
&+ \sum_{j=n_0}^{n-1} \|(\Phi_2(n,n_0) \otimes \Phi_1^*(n,n_0))D^{-1}(I_s \otimes \delta M_0) \\
&\qquad (\Phi_2(n_0,n_0) \otimes \Phi_1^*(n_0,n_0))(\Phi_2(n_0,j+1) \otimes \Phi_1^*(n_0,j+1))\widehat{F}(j)\| \\
&+ \sum_{j=n}^{n_1-1} \|(\Phi_2(n,n_0) \otimes \Phi_1^*(n,n_0))D^{-1}(I_s \otimes \delta M_1) \\
&\qquad (\Phi_2(n_1,n_0) \otimes \Phi_1^*(n_1,n_0))(\Phi_2(n_0,j+1) \otimes \Phi_1^*(n_0,j+1))\widehat{F}(j)\| \\
&\leq \|(\Phi_2(n,n_0) \otimes \Phi_1^*(n,n_0))D^{-1}\delta\widehat{W})\| + \|(\Phi_2(n,n_0) \otimes \Phi_1^*(n,n_0))D^{-1} \\
&[(I_s \otimes \delta M_0)(\Phi_2(n_0,n_0) \otimes \Phi_1^*(n_0,n_0)) + (I_s \otimes \delta M_1) \\
&(\Phi_2(n_1,n_0) \otimes \Phi_1^*(n_1,n_0))\| \sum_{j=n_0}^{n_1-1} \|(\Phi_2(n_0,j+1) \otimes \Phi_1^*(n_0,j+1))\widehat{F}(j)\|
\end{aligned}$$

$$\begin{aligned}
&\leq \|(\Phi_2(n,n_0) \otimes \Phi_1^*(n,n_0))D^{-1}\delta\widehat{W})\| + \|(\Phi_2(n,n_0) \otimes \Phi_1^*(n,n_0))D^{-1}\| \\
&+ \|(I_s \otimes \delta M_0)(\Phi_2(n_0,n_0) \otimes \Phi_1^*(n_0,n_0)) + (I_s \otimes \delta M_1) \\
&(\Phi_2(n_1,n_0) \otimes \Phi_1^*(n_1,n_0))\|k \leq \delta\eta + \\
&\delta\eta k[\|(\Phi_2(n_0,n_0)\|\|(\Phi_1(n_0,n_0)\| + \|(\Phi_2(n_1,n_0)\|\|(\Phi_1(n_1,n_0)\|] \leq \\
&(1+k)\delta\eta\|(\Phi_2(n_0,n_0)\|\|(\Phi_1(n_0,n_0)\| + \|(\Phi_2(n_1,n_0)\|\|\Phi_1(n_1,n_0)\|
\end{aligned}$$

Since

$$\begin{aligned}
&\|\widetilde{T}(n) - \widehat{T}(n)\| \geq \|(\Phi_2(n,n_0) \otimes \Phi_1^*(n,n_0))D^{-1}\delta\widehat{W})\| - \\
&\sum_{j=n_0}^{n_1} \|G_1(n,j+1) - G(n,j+1)\|\widehat{F}(j).
\end{aligned}$$

the reverse inequality can be established. In order to have a more workable quantity for η, it can be estimated by
$\eta = \underset{n_0 \leq n \leq n_1}{Sup} \|(\Phi_2(n,n_0) \otimes \Phi_1^*(n,n_0))\|\|D^{-1}\|$. The estimate in the above theorem depend on well-known quantities and on the value of the fundamental matrix at $n = n_0$ and $n = n_1$. □

5. CONCLUSIONS

In this article we have studied the dichotomy and conditioning for two-point boundary value problems associated with Sylvester discrete dynamical systems, by vectorizing the matrix system with the help of Kronecker product of matrices. Established characterizations of Strong and Exponential Dichotomy of Discrete Matrix Sylvester Systems and constructed the bounds for the angle between solutions spaces. Characterized separability of boundary conditions, ranks and projections of Fundamental Matrix Solutions and stability constant of Discrete Matrix Sylvester Systems. Estimated the bounds of Green's matrix over the entire solution space and established relationship between the stability bounds of the problem and the growth behavior of the fundamental matrix solution.

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