International Journal of Differential Equations and Applications
Volume **18**, No. 1 (2019), pages: 123-135
ISSN (Print): 1311-2872; ISSN (Online): 1314-6084;
url: https://www.ijdea.eu

ijpam.eu

NONLINEAR DIFFERENCE EQUATIONS WITH SMALL PARAMETERS OF MULTIPLE SCALES

Tahia Zerizer

Mathematics Department
College of Sciences
Jazan University, Jazan, KINGDOM OF SAUDI ARABIA

ABSTRACT: In this article we study a general model of nonlinear difference equations including small parameters of multiple scales. For two kinds of perturbations, we describe algorithmic methods giving asymptotic solutions to boundary value problems. The problem of existence and uniqueness of the solution is also addressed.

AMS Subject Classification: 39A10
Key Words: perturbed difference equations; multiple-scales; boundary value problem; iterative methods; asymptotic expansions

Received: July 17, 2019 **Revised:** December 8, 2019
Published: December 9, 2019 **doi:** 10.12732/ijdea.v18i1.11
Academic Publications, Ltd. https://acadpubl.eu

1. INTRODUCTION

Nowadays, many researchers are interested in solving boundary value problems for difference equations [2, 5, 7, 9, 14, 15], in addition, many practical systems in modern sciences are represented by discrete systems of multiple scales characterized by high-order difference equations with many small parameters of different orders of magnitude [1, 6]. Recently, we have developed a perturbation method for a general model of discrete-time nonlinear systems with a small parameter [18]. It has been successfully used for solving boundary value problems, separation of time scales and model-order

reduction. This paper is aimed at exploiting this approach for the analysis of a general form of nonlinear difference equations with small parameters of multiple scales,

$$f\left(x(t),\cdots,x(t+n),\varepsilon x(t+n+1),\cdots,\varepsilon^m x(t+n+m),\varepsilon,t\right)=0,\quad t\in I_{N-n-m},\quad (1)$$

ε being a small parameter, $|\varepsilon|<\delta<1$, $I_N=\{0,1,\cdots,N\}$, N a positive integer, and x in $F(I_N,U)$, the space of all mappings of I_N into U; $(U,\|.\|)$ a Banach space. The mapping f, defined into U, is supposed to be p-differentiable in its arguments. We aim to solve equation (1) together with the boundary values

$$x(t)=\alpha(t,\varepsilon),\ t=0,\cdots,n-1;\ x(N-t)=\beta(t,\varepsilon),\ t=0,\cdots,m-1,\qquad (2)$$

where $\alpha(t,\varepsilon)$, $\beta(t,\varepsilon)$ have the asymptotic representations

$$\begin{aligned}\alpha(t,\varepsilon)=&\alpha^{(0)}(t)+\varepsilon\alpha^{(1)}(t)+\cdots+\varepsilon^p\alpha^{(p)}(t)+\mathcal{O}(\varepsilon^{p+1}),\\ \beta(t,\varepsilon)=&\beta^{(0)}(t)+\varepsilon\beta^{(1)}(t)+\cdots+\varepsilon^p\beta^{(p)}(t)+\mathcal{O}(\varepsilon^{p+1}).\end{aligned}\qquad (3)$$

The paper is structured as follows. In the next section, sufficient conditions are established for the existence and uniqueness of a sequence solution $(x(t,\varepsilon))_{0\le t\le N}$ for BVP (1)−(2), and an iterative process is described to determine asymptotic approximate solutions:

$$x(t,\varepsilon)=x^{(0)}(t)+\varepsilon x^{(1)}(t)+\varepsilon^2x^{(2)}(t)+\cdots+\varepsilon^p x^{(p)}(t)+\mathcal{O}(\varepsilon^{p+1}).\qquad (4)$$

In Section 3, we introduce the *right-end perturbation* model and similar techniques are developed as in Section 2. We end this paper with a short conclusion.

Throughout this paper, for an abbreviated writing, $D_0^{k_0}D_1^{k_1}\cdots D_l^{k_l}f$ denotes the partial derivative $\frac{\partial^{k_0+k_1+\cdots+k_l}f(x_0,x_1,\cdots,x_l)}{\partial x_0^{k_0}\partial x_1^{k_1}\cdots\partial x_l^{k_l}}$.

2. MAIN RESULTS

In this section we demonstrate the iterative procedure for solving BVP (1)−(2). Note that the general problem (1)–(2) includes the cases studied in [8, 10, 18]; in [16, 17] some classes of nonlinear perturbed difference equations are examined, and in [11, 12, 13, 19], we have considered some systems of perturbed difference equations.

2.1. REDUCED PROBLEM

To define the reduced system, we follow a similar approach to the singular perturbation theory by setting $\varepsilon = 0$ into (1)−(2). This yields to the *reduced problem*

$$\begin{gathered} x(t) = \alpha(t,0) = \alpha^{(0)}(t), \quad t = 0, \cdots, n-1, \\ f\left(x(t), x(t+1), \cdots, x(t+n), 0, \cdots, 0, t\right) = 0, \\ 0 \leq t \leq N - n - m, \end{gathered} \tag{5}$$

$$x(N-t) = \beta(t,0) = \beta^{(0)}(t), \quad t = 0, \cdots, m-1, \tag{6}$$

were the boundary conditions are uncoupled; the values $x(0)$, ..., $x(N-m)$, can be recursively computed from the IVP (5) without needing the final conditions, i.e., there is a *boundary layer behavior* at (6). Obviously, the solution of (5) may be found by straightforward computation, so we need the following hypothesis.

H 1. Suppose $D_n f\left(x(t), x(t+1), \cdots, x(t+n), 0, \cdots, 0, t\right) \neq 0$, for all $x \in U$, $0 \leq t \leq N - n - m$, and that f has range including zero.

Proposition 1. *If hypothesis 1 holds, then problem (5)−(6) has a unique solution, which we denote* $\left(x^{(0)}(t)\right)_{0 \leq t \leq N}$.

Proof. The proof is by induction.

For each fixed t, $0 \leq t \leq N - n - m$, consider the mappings $f_t^{\alpha} : U \to U$,

$$f_t^{\alpha}(x) := f(x^{(0)}(t), \cdots, x^{(0)}(t+n-1), x, 0, \cdots, 0, t).$$

Hypothesis H1 assures that f_0^{α} is one-to-one since for all $x \in U$, we have

$$\frac{df_0^{\alpha}(x)}{dx} = D_n f(\alpha^{(0)}(0), \alpha^{(0)}(1), \cdots, \alpha^{(0)}(n-1), x, 0, \cdots, 0) \neq 0,$$

thus equation

$$f(\alpha^{(0)}(0), \alpha^{(0)}(1), \cdots, \alpha^{(0)}(n-1), x, 0, \cdots, 0) = 0,$$

has a unique root, which we designate as $x^{(0)}(n)$. We consecutively reiterate this inference for $t = 1, 2, \cdots, N - n - m$. Once again, H1 guarantees that for each fixed $t = 1, 2, \cdots, N - n - m$, equation

$$f(x^{(0)}(t), x^{(0)}(t+1), \cdots, x^{(0)}(t+n-1), x, 0, \cdots, 0, t) = 0, \tag{7}$$

has a unique solution in U, we denote $x^{(0)}(t+n)$. □

2.2. PRELIMINARIES

We re-write the BVP (1)−(2) as a system of equations depending on a parameter by denoting $X := (x(0), x(1), \dots, x(N))$, $X_\varepsilon := (\varepsilon, X)$, and introducing the functional

$$\mathcal{H} : (-1,1) \times U^{N+1} \longrightarrow U^{N+1},$$

$$X_\varepsilon \mapsto \mathcal{H}(X_\varepsilon) = (\mathcal{H}_0(X_\varepsilon), \dots, \mathcal{H}_N(X_\varepsilon)),$$

$$\mathcal{H}_t(X_\varepsilon) = x(t) - \alpha(t,\varepsilon), \quad t = 0, \cdots, n-1;$$

$$\mathcal{H}_{t+n}(X_\varepsilon) = f(x(t), \cdots, \varepsilon x(t+n+1), \cdots, \varepsilon^m x(t+n+m), \varepsilon, t)$$
$$t = 0, \cdots, N-n-m;$$

$$\mathcal{H}_{N-t}(X_\varepsilon) = x(N-t) - \beta(t,\varepsilon), \quad t = 0, \cdots, m-1.$$

Therefore, (1)−(2) is analogous to the system $\mathcal{H}(\varepsilon, X) = 0$. Consequently, under certain conditions, we will then be able to apply the Implicit Function Theorem [3] which ensures the existence of a neighborhood $(-\epsilon, \epsilon)$, and mappings $g_0(\varepsilon), g_1(\varepsilon), \dots, g_N(\varepsilon)$, of class C^p on $(-\epsilon, \epsilon)$, such that

$$\begin{gathered} f(g_t(\varepsilon), \cdots, g_{t+n}(\varepsilon), \varepsilon g_{t+n+1}(\varepsilon), \cdots, \varepsilon^m g_{t+n+m}(\varepsilon), \varepsilon, t) = 0, \\ t = 0, \dots, N-n-m, \\ g_t(\varepsilon) = \alpha(t,\varepsilon),\ t = 0, \cdots, n-1, \\ g_{N-t}(\varepsilon) = \beta(t,\varepsilon),\ t = 0, \cdots, m-1. \end{gathered} \tag{8}$$

In order to determine the coefficients of the Taylor expansion

$$g_t(\varepsilon) = g_t(0) + \varepsilon \frac{\dot{g}_t(0)}{1!} + \varepsilon^2 \frac{\ddot{g}_t(0)}{2!} + \cdots + \varepsilon^p \frac{g_t^{(p)}(0)}{p!} + \mathcal{O}(\varepsilon^{p+1}), \tag{9}$$

where the dot denotes the derivative with respect to ε, we repeatedly differentiate the equations (8) with respect to ε. Using the formula of Faà Di Bruno [4], we find the following Lemma. For the sake of simplicity in the notations, we remove some arguments and we denote

$$f(\mathsf{x}_t) := f\left(x^{(0)}(t), \cdots, x^{(0)}(t+n), 0, \cdots, 0, t\right). \tag{10}$$

Lemma 2. *Assume that the functions g_t, and f satisfy (8), and that all the necessary derivatives are defined. Then we have for $p \geq m$,*

$$\sum_{l=0}^{n} D_l f(\mathsf{x}_t) g_{t+l}^{(p)}(0) + \sum_{l=1}^{m} D_{n+l} f(\mathsf{x}_t) g_{t+n+l}^{(p-l)}(0)$$
$$= -\underbrace{\sum_{0} \cdots \sum_{p}}_{} \frac{p! D_0^{p_0} D_1^{p_1} \cdots D_{n+m+1}^{p_{n+m+1}} f(\mathsf{x}_t)}{\prod_{i=1}^{p} (i!)^{k_i} \prod_{i=1}^{p} \prod_{j=0}^{n+m+1} q_{ij}!}$$

$$\times \prod_{i=1}^{p} \left(g_t^{(i)}\right)^{q_{i0}} \cdots \left(g_{t+n}^{(i)}\right)^{q_{in}} \left(\frac{i! g_{t+n+1}^{(i-1)}}{(i-1)!}\right)^{q_{in+1}} \cdots \left(\frac{i! g_{t+n+m}^{(i-m)}}{(i-m)!}\right)^{q_{in+m}} \delta_i. \quad (11)$$

Agreeing that $g_t^{(p)} = 0$ *for* $p < 0$, *the coefficients* k_i, q_{ij} *and* p_j, *are all nonnegative integer solutions of the Diophantine equations*

$$\begin{gathered} \textstyle\sum_0 \to k_1 + 2k_2 + \cdots + pk_p = p, \\ \textstyle\sum_i \to q_{i0} + q_{i1} + \cdots + q_{i\,n+m} + q_{i\,n+m+1} = k_i,\ i = 1, \cdots, p, \\ p_j = q_{1j} + q_{2j} + \cdots + q_{pj},\ j = 0, \cdots, n+m+1, \\ k = p_0 + p_1 + \cdots + p_{n+l} = k_1 + k_2 + \cdots + k_p,\ l = 0, \cdots, m, \end{gathered} \quad (12)$$

in $\underline{\sum_0 \cdots \sum_p}$ *we fix* $k_p = 0$; *the left side of (11) corresponds to* $k_p = 1$ *and*

$$\delta_i = \begin{cases} 1, & i = 1 \quad \vee \quad q_{in+m+1} = 0, \\ 0, & i \geq 2 \quad \wedge \quad q_{in+m+1} \neq 0. \end{cases}$$

Proof. Leibniz's rule for differentiation gives

$$\frac{d^i \left(\varepsilon^l g_t(\varepsilon)\right)}{d\varepsilon^i} = \Sigma_{k=0}^{i} \binom{i}{k} \frac{d^k \varepsilon^l}{d\varepsilon^k} \frac{d^{i-k} g_t(\varepsilon)}{d\varepsilon^{i-k}}, \quad (13)$$

and remark that

$$\frac{d^k \varepsilon^l}{d\varepsilon^k} = \begin{cases} 0, & k > l, \\ \frac{l! \varepsilon^{l-k}}{(l-k)!}, & k \leq l. \end{cases} \quad (14)$$

Combining (13) together with (14) and substituting $\varepsilon = 0$ into (13), yields

$$\frac{d^i \left(\varepsilon^l g_t(\varepsilon)\right)}{d\varepsilon^i}|_{\varepsilon=0} = \frac{i!}{(i-l)!} g_t^{(i-l)}(0), \quad i \geq l. \quad (15)$$

Expanding Faa Di Bruno Formula [4] into (1) with $\varepsilon = 0$ using (15), and noticing that

$$\left(\varepsilon^{(i)}\right)^{q_{in+m+1}}|_{\varepsilon=0} = \begin{cases} 1, & i = 1 \quad \vee \quad q_{in+m+1} = 0, \\ 0, & i \geq 2 \quad \wedge \quad q_{in+m+1} \neq 0. \end{cases} := \delta_i,$$

we find (11) by arranging the equation so that on the left hand side we put the terms corresponding to $k_p^{(0)} = 1$ in the Diophantine equations (12). □

2.3. DESCRIPTION OF THE METHOD

It is agreed that the zero order coefficients in (4) satisfy the *reduced* problem (5)−(6). In order to determine the coefficients of higher order, we substitute

$$x^{(p)}(t) = \frac{g_t^{(p)}(0)}{p!}, \quad t \in I_N, \quad (16)$$

into (11).According to Lemma 2, we deduce the following. The first order coefficients in (4) satisfy the iteration

$$\begin{gathered} x^{(1)}(t)=\alpha^{(1)}(t), \quad t=0,\ldots,n-1, \\ D_n f(\mathsf{x}_t)x^{(1)}(t+n)=-\sum_{l=0}^{n-1} D_l f(\mathsf{x}_t)x^{(1)}(t+l) \\ -D_{n+1}f(\mathsf{x}_t)x^{(0)}(t+n+1)-D_{n+m+1}f(\mathsf{x}_t), \\ t=0,\ldots,N-n-m, \\ x^{(1)}(N-t)=\beta^{(1)}(t), \quad t=0,\ldots,m-1, \end{gathered} \tag{17}$$

where only the initial values are needed; the final values are fixed and do not serve in the iteration. The second order coefficients can be calculated from the recurrence

$$\begin{gathered} x^{(2)}(t)=\alpha^{(2)}(t), \quad t=0,\ldots,n-1, \\ D_n f(\mathsf{x}_t)x^{(2)}(t+n)=-\sum_{l=0}^{n-1} D_l f(\mathsf{x}_t)x^{(2)}(t+l) \\ -D_{n+1}f(\mathsf{x}_t)x^{(1)}(t+n+1)-D_{n+2}f(\mathsf{x}_t)x^{(0)}(t+n+2) \\ -\sum_{0\le l<r\le n} D_l D_r f(\mathsf{x}_t)x^{(1)}(t+l)x^{(1)}(t+r) \\ -\tfrac{1}{2!}\sum_{l=0}^{n} D_l^2 f(\mathsf{x}_t)\left(x^{(1)}(t+l)\right)^2-\tfrac{1}{2!}D_{n+1}^2 f(\mathsf{x}_t)\left(x^{(0)}(t+n+1)\right)^2 \\ -\sum_{l=0}^{n} D_l D_{n+m+1}f(\mathsf{x}_t)x^{(1)}(t+l) \\ -D_{n+1}D_{n+m+1}f(\mathsf{x}_t)x^{(0)}(t+n+1) \\ -\sum_{l=0}^{n} D_l D_{n+1}f(\mathsf{x}_t)x^{(1)}(t+l)x^{(0)}(t+n+1)-\tfrac{1}{2!}D_{n+m+1}^2 f(\mathsf{x}_t), \\ t=0,\ldots,N-n-m, \\ x^{(2)}(N-t)=\beta^{(2)}(t), \quad t=0,\ldots,m-1, \end{gathered} \tag{18}$$

which starts from the initial values while the final values are fixed; the zero and first order coefficients determined from the preceding stage are also needed. In general, to compute the coefficients of order p, we use the forward recurrence

$$\begin{gathered} x^{(p)}(t)=\alpha^{(p)}(t), \quad t=0,\ldots,n-1, \\ \sum_{l=0}^{n+m} D_l f(\mathsf{x}_t)x(t+l)^{(p)}=-\underline{\sum_0\cdots\sum_p}\frac{D_0^{p_0}D_1^{p_1}\cdots D_{n+m+1}^{p_{n+m+1}}f(\mathsf{x}_t)}{\prod_{i=1}^{p}\prod_{j=0}^{n+m+1}q_{ij}!} \\ \prod_{i=1}^{p}\left(x^{(i)}(t)\right)^{q_{i0}}\left(x^{(i)}(t+1)\right)^{q_{i1}}\cdots\left(x^{(i-1)}(t+n+1)\right)^{q_{in+1}} \\ \left(x^{(i-2)}(t+n+2)\right)^{q_{in+2}}\cdots\left(x^{(i-m)}(t+n+m)\right)^{q_{in+m}}\delta_i, \\ t=0,\ldots,N-n-m, \end{gathered} \tag{19}$$

noticing that $x^{(p)}(t):=0$ for $p<0$, and the final values

$$x^{(p)}(N-t)=\beta^{(p)}(t), \quad t=0,\ldots,m-1, \tag{20}$$

are fixed. Once the coefficients have been calculated, we replace in (4) and then we find the desired approximate solution of order p for the BVP (1)−(2). This process is approved in the following theorem.

Theorem 3. *If hypothesis 1 holds, then there exists $\epsilon > 0$, such that for all $|\varepsilon| < \epsilon$, the BVP (1)−(2) has a unique solution $(x(t,\varepsilon))_{t=0,\cdots,N}$ satisfying (4); the coefficients $x^{(0)}(t)$, $x^{(1)}(t)$, $x^{(2)}(t)$, $x^{(n)}(t)$, are the solutions of the problems (5)−(6), (17), (18), (19)−(20), respectively.*

Proof. Consider the functional

$$\widehat{\mathcal{H}} : (-1,1) \times U^{N+1} \longrightarrow (-1,1) \times U^{N+1}, \quad \widehat{\mathcal{H}}\left(X_\varepsilon\right) := \left(\varepsilon, \mathcal{H}\left(X_\varepsilon\right)\right),$$

and let $J_{\widehat{\mathcal{H}}}$ be its Jacobian matrix. We can easily verify that

$$\det J_{\widehat{\mathcal{H}}}\left(X_0\right) = \prod_{t=0}^{N-n-m} D_n f\left(x^{(0)}(t), \cdots, x^{(0)}(t+n), 0, \cdots, 0, t\right), \tag{21}$$

where $X_0 = \left(0, x^{(0)}(0), \ldots, x^{(0)}(N)\right)$; $\left(x^{(0)}(t)\right)_{0\le t\le N}$ being the sequence solution of the reduced problem (5)−(6). Assumption H1 guarantees that

$$\det J_{\widehat{\mathcal{H}}}\left(X_0\right) \neq 0, \tag{22}$$

then $\left[J_{\widehat{\mathcal{H}}}\left(X_0\right)\right]^{-1}$ is well-defined. Since $J_{\widehat{\mathcal{H}}}$ is continuous, there exists $\rho > 0$ such that, for all $X_\varepsilon \in B\left(X_0, \rho\right)$, we have

$$\| J_{\widehat{\mathcal{H}}}\left(X_\varepsilon\right) - J_{\widehat{\mathcal{H}}}\left(X_0\right) \| < \frac{1}{2\| \left[J_{\widehat{\mathcal{H}}}\left(X_0\right)\right]^{-1} \|}.$$

Let $\epsilon = \frac{\rho}{2\| \left[J_{\widehat{\mathcal{H}}}(X_0)\right]^{-1} \|}$, and consider for Y in $B\left(\widehat{\mathcal{H}}\left(X_0\right), \epsilon\right)$, i.e., $\|Y\| < \epsilon$,

$$\mathcal{G}_Y(X_\varepsilon) = X_\varepsilon - \| \left[J_{\widehat{\mathcal{H}}}\left(X_0\right)\right]^{-1} \| \left(\widehat{\mathcal{H}}(X_\varepsilon) - Y\right).$$

It is easy to check that $\mathcal{G}_Y$ is a contraction from $B\left(X_0, \rho\right)$ to itself if $|\varepsilon| < \epsilon$, then $\mathcal{G}_Y$ has a unique fixed point, i.e., there is a unique X_ε in $B\left(X_0, \rho\right)$ such that $Y = \widehat{\mathcal{H}}(X_\varepsilon)$. If $|\varepsilon| < \epsilon$, for $(\varepsilon, 0, \cdots, 0)$ in $B\left(0, \epsilon\right)$, there is a unique $(\varepsilon, g_0(\varepsilon), \cdots, g_N(\varepsilon))$ in $B\left(X_0, \rho\right)$, such that

$$(\varepsilon, 0, \cdots, 0) = \widehat{\mathcal{H}}(\varepsilon, g_0(\varepsilon), \cdots, g_N(\varepsilon)).$$

We just proved that for $|\varepsilon| < \epsilon$, BVP (1)−(2) has a unique solution. Once more, the Implicit Function Theorem guarantees that $g_0(\varepsilon), g_1(\varepsilon), \ldots, g_N(\varepsilon)$, are of class $C^p\left(-\epsilon, \epsilon\right)$ as are $\widehat{\mathcal{H}}$ and $\widehat{\mathcal{H}}^{-1}$, while their derivatives are established in Lemma 2. □

If we suppose in the procedure above that f is a smooth functional (has derivatives of all orders everywhere in U), then the solution can be expressed as an infinite order asymptotic expansion, for that we need the following assumption.

Hypothesis 2. We assume that $\|\alpha^{(t)}(t,\varepsilon)\| \le \frac{A}{\delta^t}$, $\|\beta^{(t)}(t,\varepsilon)\| \le \frac{B}{\delta^t}$, $|\delta| < \epsilon$, A and B are constants, and that f is a smooth functional.

Theorem 4. *If Hypothesis 2 holds, there exists $\epsilon > 0$, for all $|\varepsilon| < \epsilon$, BVP (1)−(2) has a unique solution $(x(t,\varepsilon))_{t=0,\cdots,N}$ satisfying $x(t,\varepsilon) = \sum_{p=0}^{\infty} \varepsilon^p x^{(p)}(t)$, where $x^{(0)}(t)$, $x^{(1)}(t)$, $x^{(2)}(t)$, $x^{(p)}(t)$ are the solutions of problems (5)−(6), (17), (18), (19)−(20), respectively.*

3. RIGHT END PERTURBATION

We can easily develop results, similar to those given in Section 2, to equations presenting a *right-end perturbation*

$$\begin{gathered} f\left(\varepsilon^n x(t), \cdots, \varepsilon x(t+n-1), x(t+n), \cdots, x(t+n+m), \varepsilon, t\right) = 0 \\ 0 \le t \le N-n-m, \end{gathered} \tag{23}$$

subject to the boundary conditions

$$x(t) = \alpha(t,\varepsilon),\ t = 0, \cdots, n-1,\ x(N-t) = \beta(t,\varepsilon),\ t = 0, \cdots, m-1. \tag{24}$$

Deleting the parameter ε in (23)–(24), follows the reduced system

$$x(t) = \alpha(t,0) = \alpha^{(0)}(t), \quad t = 0, \cdots, n-1, \tag{25}$$

$$\begin{gathered} f\left(0, \cdots, 0, x(t+n), \cdots, x(t+n+m), 0, t\right) = 0, \\ 0 \le t \le N-n-m, \\ x(N-t) = \beta(t,0) = \beta^{(0)}(t), \quad t = 0, \cdots, m-1, \end{gathered} \tag{26}$$

where the *boundary layer behavior* is located at the initial values (25). System (25) −(26) actually poses a final value problem because it can be solved backwards using only the final values.

Hypothesis 3. For all $x \in U$, $0 \le t \le N-n-m$, suppose

$$D_n f\left(0, \cdots, 0, x(t+n), \cdots, x(t+n+m), 0, t\right) \neq 0,$$

and that f has range including zero.

Proposition 5. *If hypothesis 3 holds, then BVP (25)−(26) has a unique solution, we denote $\left(x^{(0)}(t)\right)_{0 \le t \le N}$.*

Proof. For each fixed t, $0 \le t \le N-n-m$, consider the functions

$$f_t^{\beta} : U \to U,$$

$$f_t^{\beta}(x) := f(0, \cdots, 0, x^{(0)}(t+n), \cdots, x^{(0)}(t+n+m), 0, t).$$

With hypothesis H3 we conclude that f^{α}_{N-n-m} is one-to-one due to the fact that for all $x \in U$,

$$\frac{df^{\beta}_{N-n-m}(x)}{dx} = D_n f(0, \cdots, x, \beta^{(0)}(m-1), \cdots, \beta^{(0)}(0), 0, N-n-m)$$

is non-zero. Hence, equation

$$f(0, \cdots, 0, x, \beta^{(0)}(m-1), \cdots, \beta^{(0)}(1), \beta^{(0)}(0), 0, N-n-m) = 0,$$

has a unique root, we designate as $x^{(0)}(n)$. Repeating this deduction consecutively for $t = N-n-m-1, N-n-m-2, \cdots, 0$, H3 ensures that equation

$$f\left(0, \cdots, 0, x^{(0)}(t+n), \cdots, x^{(0)}(t+n+m), 0, t\right) = 0, \tag{27}$$

has a unique solution in U, we denote $x^{(0)}(t+n)$. □

As in Section 2.2, under some conditions, we can apply the Implicit Function Theorem, and then we can locally find mappings $g_0(\varepsilon), g_1(\varepsilon), \cdots, \cdots, g_N(\varepsilon)$, such that

$$\begin{gathered} f\left(\varepsilon^n g_t(\varepsilon), \cdots, \varepsilon g_{t+n-1}(\varepsilon), g_{t+n}(\varepsilon), \cdots, g_{t+n+m}(\varepsilon), \varepsilon, t\right) = 0, \\ t = 0, \ldots, N-n-m, \\ g_t(\varepsilon) = \alpha(t, \varepsilon),\ t = 0, \cdots, n-1, \\ g_{N-t}(\varepsilon) = \beta(t, \varepsilon),\ t = 0, \cdots, m-1. \end{gathered} \tag{28}$$

The sequential differentiation of equations (28), and the we use Faa di Bruno's formula give the following Lemma. To write concise formula, we drop arguments, and denote

$$f(\mathrm{x}_t) := f\left(0, \cdots, 0, x^{(0)}(t+n), \cdots, x^{(0)}(t+n+m), 0, t\right). \tag{29}$$

Lemma 6. *Assume that f and g_t satisfy (28), and all the necessary derivatives are defined. Then we have for $p \geq n$,*

$$\begin{aligned} \sum_{l=0}^{n-1} D_l f(\mathrm{x}_t) g_{t+l}^{(p-n+l)} + \sum_{l=n}^{n+m} D_l f(\mathrm{x}_t) g_{t+l}^{(p)} = -\underbrace{\sum_{0} \cdots \sum_{p}}_{} \\ \frac{p! D_0^{p_0} D_1^{p_1} \cdots D_{n+m+1}^{p_{n+m+1}} f(\mathrm{x}_t)}{\prod_{i=1}^{p} (i!)^{k_i} \prod_{i=1}^{p} \prod_{j=0}^{n+m+1} q_{ij}!} \prod_{i=1}^{p} \left(\frac{i! g_t^{(i-n)}}{(i-n)!}\right)^{q_{i0}} \left(\frac{i! g_{t+1}^{(i-n+1)}}{(i-n+1)!}\right)^{q_{i1}} \cdots \times \\ \left(g_{t+n}^{(i)}\right)^{q_{in}} \cdots \left(g_{t+n+m}^{(i)}\right)^{q_{in+m}} \delta_i. \end{aligned} \tag{30}$$

Agreeing that $g_t^{(p)} = 0$ for $p < 0$, the coefficients k_i, q_{ij} and p_j, are all nonnegative integer solutions of (12); in $\underbrace{\sum_0 \cdots \sum_p}$ we fix $k_p^{(0)} = 0$; the case $k_p^{(0)} = 1$ is omitted and corresponds to the left side of equation (30).

We can already proceed the main result of this Section. From (9), (16) and (30), we can deduce for the solution of BVP (23)−(24), that first order coefficients of expansion (4) verify the backwards iteration

$$\begin{gathered} x^{(1)}(t)=\alpha^{(1)}(t), \quad t=0\cdots,n-1, \\ D_nf(\mathrm{x}_t)x^{(1)}(t+n)=-\sum_{l=n+1}^{n+m}D_lf(\mathrm{x}_t)x^{(1)}(t+l) \\ -D_{n-1}f(\mathrm{x}_t)x^{(0)}(t+n-1)-D_{n+m+1}f(\mathrm{x}_t), \\ t=0,\ldots,N-n-m, \\ x^{(1)}(N-t)=\beta^{(1)}(t), \quad t=0,\cdots,m-1. \end{gathered} \tag{31}$$

where only from the $m-1$ final values are used, the initial values are fixed and are not needed is the calculation. In what follows, for a short writing, we remove the arguments. The second order coefficients can be calculated using the backwards iteration

$$\begin{aligned} x^{(2)}(t)=&\alpha^{(2)}(t),\ t=0\cdots,n-1, \\ D_nf(\mathrm{x}_t)x^{(2)}(t+n)=&-\sum_{l=n+1}^{n+m}D_lf(\mathrm{x}_t)x^{(2)}(t+l) \\ &-D_{n-1}f(\mathrm{x}_t)x^{(1)}(t+n-1)-D_{n-2}f(\mathrm{x}_t)x^{(0)}(t+n-2) \\ &-\sum_{n\le l<r\le n+m}D_lD_rf(\mathrm{x}_t)x^{(1)}(t+l)x^{(1)}(t+r) \\ &-\sum_{n\le l\le n+m}D_{n-1}D_lf(\mathrm{x}_t)x^{(0)}(t+n-1)x^{(1)}(t+l) \\ &-\frac{1}{2!}D^2_{n+m+1}f(\mathrm{x}_t)-\frac{1}{2!}\sum_{l=n}^{n+m}D^2_lf(\mathrm{x}_t)\left(x^{(1)}(t+l)\right)^2 \\ &-\sum_{l=n}^{m+m}D_lD_{n+m+1}f(\mathrm{x}_t)x^{(1)}(t+l) \\ &-D_{n-1}D_{n+m+1}f(\mathrm{x}_t)x^{(0)}(t+n-1) \\ &-\frac{1}{2!}D^2_{n-1}f(\mathrm{x}_t)\left(x^{(0)}(t+n-1)\right)^2, \\ &t=0,\ldots,N-n-m, x^{(2)}(N-t)=\beta^{(2)}(t), \\ &t=0,\cdots,m-1, \end{aligned} \tag{32}$$

independently of the initial values. In general, to find the coefficients of order p, $p\ge 2$, we iterate backwards using only the final values in the process

$$\begin{gathered} x^{(p)}(t)=\alpha^{(p)}(t), \quad t=0\cdots,n-1, \\ \sum_{l=0}^{n-1}D_lf(\mathrm{x}_t)x^{(p-n+l)}(t+l)+\sum_{l=n}^{n+m}D_lf(\mathrm{x}_t)x^{(p)}(t+l)= \end{gathered}$$

$$-\sum_{0}\cdots\sum_{p}\frac{D_0^{p_0}D_1^{p_1}\cdots D_{n+m+1}^{p_{n+m+1}}f(\mathrm{x}_t)}{\prod_{i=1}^{p}\prod_{j=0}^{n+m+1}q_{ij}!}\times$$

$$\prod_{i=1}^{p}\left(x^{(i-n)}(t)\right)^{q_{i0}}\left(x^{(i-n+1)}(t+1)\right)^{q_{i1}}\cdots\left(x^{(i-1)}(t+n-1)\right)^{q_{in-1}}$$

$$\times\left(x^{(i)}(t+n)\right)^{q_{in}}\cdots\left(x^{(i)}(t+n+m)\right)^{q_{in+m}}\delta_i,$$

$$x^{(p)}(N-t)=\beta^{(p)}(t),\quad t=0,\cdots,m-1, \quad (33)$$

while the initial values remain fixed, and assuming that $x^{(p)}(t)=0$ for $p<0$. The main result of this Section is stated as follows.

Theorem 7. *If Hypothesis 3 holds, then there exists $\epsilon>0$, such that for all $|\varepsilon|<\epsilon$, the BVP (23)–(24) has a unique solution $(x(t,\varepsilon))_{t=0,\cdots,N}$, which satisfy (4), where $x^{(0)}(t)$, $x^{(1)}(t)$, $x^{(2)}(t)$, $x^{(p)}(t)$, are the solutions of (25)–(26), (31), (32), (33), respectively.*

Proof. We proceed as in Section 2.2, the BVP (23)–(24) is switched into a system of equations depending on ε and the determinant of its Jacobian matrix, at $\left(0,x^{(0)}(0),\cdots,x^{(0)}(N)\right)$, where $\left(x^{(0)}(t)\right)_{0\leq t\leq N}$ is the solution of the reduced problem (25) $-$(26), is equal to

$$\prod_{t=0}^{N-n-m}D_nf\left(0,\cdots,0,x^{(0)}(t+n),\cdots,x^{(0)}(t+n+m),0,t\right). \quad (34)$$

Hypothesis H3 assures that (34) is not zero, we can then apply Implicit Function Theorem for a sufficiently small value of ε. The remainder of the proof is almost identical to the proof of Theorem 3, and Lemma 6 establishes the problems for the coefficients of the asymptotic expansion. □

If one is interested in an infinite order asymptotic expansion of the solution, f should be a smooth functional. In Lemma 6, the formula is valid for any order of the derivative, provided it exists. Using the same approach as for Theorem 7, we prove the following result.

Theorem 8. *If Hypotheses 2 and 3 hold, then there exists $\epsilon>0$, such that for all $|\varepsilon|<\epsilon$, BVP (23)$-$(24) has a unique solution $(x(t,\varepsilon))_{t=0,\cdots,N}$ satisfying $x(t,\varepsilon)=\sum_{p=0}^{\infty}\varepsilon^p x^{(p)}(t)$, where $x^{(0)}(t)$, $x^{(1)}(t)$, $x^{(2)}(t)$, $x^{(p)}(t)$, are solutions of (25)$-$(26), (31), (32), (33), respectively.*

4. CONCLUSION

In this paper, we have formulated a general form of "*singularly perturbed difference equations*" with small parameters of multiple scales. For boundary value problems, we have provided sufficient conditions for the existence and uniqueness of a solution and we have presented an algorithmic method producing asymptotic expansions at any order. The perturbation theory and the asymptotic analysis of difference equations have the same characteristics as the theory of singular perturbation of ODEs: a separation of time scales, reduction of order and a boundary layer phenomenon. By the same procedure we can evaluate IVPs for *left-end perturbation* or FVPs for *right-end perturbation* when the discrete time is given in a finite interval.

REFERENCES

[1] G. A. Cassatella Contra, D. Levi, Discrete Multiscale Analysis: A Biatomic Lattice System, *J. Nonlinear Math. Phys.*, **17**, No. 3 (2010), 357-377.

[2] J. R. Graef, L. Kong, X. Liu, Existence of solutions to a discrete fourth order periodic boundary value problem, *J. Differ. Equ. Appl.*, **22**, No. 8 (2016), 1167-1183.

[3] S. G. Krantz, H. R. Parks, *The Implicit Function Theorem: History, Theory, and Applications*, Birkhauser, Boston (2003).

[4] R. L. Mishkov, Generalization of the Formula of Faa Di Bruno for a Composite Function with a Vector Argument, *Int. J. Math. Math. Sci.*, **24** (2000), 481-491.

[5] X. Liu, T. Zhou, H. Shi, Existence of Solutions to Boundary Value Problems for a Fourth-Order Difference Equation, *Discrete Dynamics in Nature and Society*, **2018** (2018), doi: 10.1155/2018/5278095.

[6] V. Radisavljević-Gajić, M. Milanović, P. Rose, *Three-Stage Discrete-Time Feedback Controller Design. In: Multi-Stage and Multi-Time Scale Feedback Control of Linear Systems with Applications to Fuel Cells*, Mechanical Engineering Series, Springer, Cham (20019).

[7] S. Stević, Solvability of Boundary-Value Problems for a Linear Partial Difference Equations, *Electron. J. Differential Equations*, **2017**, No. 17 (2017), 1-10.

[8] T. Sari, T. Zerizer, Perturbations for linear difference equations, *J. Math. Anal. Appl.*, **1** (2005), 43-52.

[9] J. Tan, Z. Zhou, Boundary value problems for a coupled system of second-order nonlinear difference equations, *Adv. Differ. Equ.*, **2017** (2017), doi: 10.1186/s13662-017-1257-4.

[10] T. Zerizer, Perturbation method for linear difference equations with small parameters, *Adv. Differ. Equ.*, **11** (2006), 1-12.

[11] T. Zerizer, Perturbation method for a class of singularly perturbed systems, *Adv. Dyn. Syst. Appl.*, **9**, No. 2 (2014), 239-248.

[12] T. Zerizer, Problems for a linear two-time-scale discrete model, *Adv. Dyn. Syst. Appl.*, **10**, No. 1 (2015), 85-96.

[13] T. Zerizer, Boundary value problems for linear singularly perturbed discrete systems, *Adv. Dyn. Syst. Appl.*, **10**, No. 2 (2015), 215-224.

[14] T. Zerizer, Boundary value problem for a three-time-scale singularly perturbed discrete system, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, **23** (2016), 263-272.

[15] T. Zerizer, On a class of perturbed discrete nonlinear systems, *GJPAM*, **14**, No. 10 (2018), 1407-1418.

[16] T. Zerizer, A class of nonlinear perturbed difference equations, *Int. J. Math. Anal.*, **12**, No. 5 (2018), 235-243.

[17] T. Zerizer, A class of multi-scales nonlinear difference equations, *Appl. math. sci.*, **12**, No. 19 (2018), 911-919.

[18] T. Zerizer, Nonlinear perturbed difference equations, *J. Nonlinear Sci. Appl.*, **11** (2018), 1355-1362.

[19] T. Zerizer, Boundary value problem for a two-time-scale nonlinear discrete system, *Int. J. Appl. Anal.*, **32**, No. 2 (2019), 239-247.