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SIMULATIONS ON THE LIENARD POLYNOMIAL SYSTEM WITH DICKSON-TYPE POLYNOMIAL CORRECTIONS. THE LEVEL CURVES

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ABSTRACT: In this article a model with Dickson polynomials (of first and second kind) as corrections in the Lienard differential system is presented.

The model is considered in the light of Melnikov's approach. The level curves are also studied.

Numerical examples, illustrating our results using *CAS MATHEMATICA* are given.

Key Words: Lienard differential system, Melnikov's polynomial, Dickson polynomials as "correcting factors" in the Lienard polynomial system, level curves

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1. INTRODUCTION

The Melnikov function [2] for the Lienard system [3]

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon\left(a_1 x + a_2 x^2 + \cdots + a_{2n+1} x^{2n+1}\right) \\ \dfrac{dy}{dt} &= -x \end{cases}$$

is defined as

$$M(\alpha,\mu) = -2\pi\alpha^2 \left(\frac{a_1}{2} + \frac{3}{8} a_3 \alpha^2 + \cdots + \binom{2n+2}{n+1} \frac{a_{2n+1}}{2^{2n+2}} \alpha^{2n} \right)$$

The *Melnikov polynomial* is defined as

$$P(r^2,n) = -\frac{1}{2\pi r^2} M(r,\mu).$$

The following result provides the necessary information about the number of limit cycles and their radii

Theorem [4]–[5]. The Lienard system for sufficiently small $\epsilon \neq 0$ has at most n limit cycles asymptotic to circles of radii r_j, $j = 1,2,\ldots,n$ as $\epsilon \to 0$ if and only if the nth degree polynomial in r^2,

$$P(r^2,n) = \frac{a_1}{2} + \frac{3}{8} a_3 r^2 + \cdots + \binom{2n+2}{n+1} \frac{a_{2n+1}}{2^{2n+2}} r^{2n}$$

has n positive roots $r^2 = r_j^2$, $j = 1,2,\ldots,n$.

Let $D(x,a)$ and $E(x,a)$ denote the Dickson polynomials [1] of the first and second kind, respectively. In this paper we consider a new extended Lienard–type system with the polynomials $D(x,a)$ and $E(x,a)$ as correction. The type of limit cycles and level curves are also studied. Numerical examples, illustrating our results using *CAS MATHEMATICA* are given.

2. MAIN RESULTS

2.1. DICKSON POLYNOMIALS $D_N(X,A)$ AND $E(X,A)$ AS CORRECTIONS IN THE LIENARD DIFFERENTIAL SYSTEM

In this Section we consider families of the following type:

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon D_n(x,a) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{1}$$

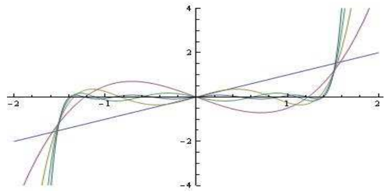

Figure 1: The polynomials $D_n(x,a)$ for $n = 1, 3, 5, 7, 9$ and fixed $a = 0.5$.

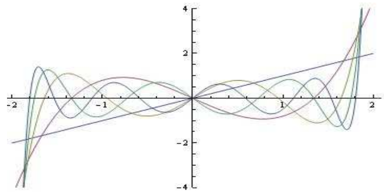

Figure 2: The polynomials $E_n(x,a)$ for $n = 1, 3, 5, 7, 9$ and fixed $a = 0.9$.

and

$$\begin{cases} \dfrac{dx}{dt} & = & y - \epsilon E_n(x,a) \\ \dfrac{dy}{dt} & = & -x \end{cases} \tag{2}$$

where $\epsilon > 0$ and $D_n(x,a)$ and $E_n(x,a)$ are the Dickson polynomials of first and second kind, respectively.

The Dickson polynomials (of the first and second kind) are given by [1]

$$D_n(x,a) := \sum_{i=0}^{[\frac{n}{2}]} \frac{n}{n-i} \left(\begin{array}{c} n-i \\ i \end{array} \right) (-a)^i x^{n-2i}.$$

and

$$E_n(x,a) := \sum_{i=0}^{[\frac{n}{2}]} \left(\begin{array}{c} n-i \\ i \end{array} \right) (-a)^i x^{n-2i}.$$

For example we have (see Fig. 1 and Fig. 2)

$$\begin{cases} D_1(x,a) & = & x \\ D_3(x,a) & = & x^3 - 3ax \\ D_5(x,a) & = & x^5 - 5ax^3 + 5a^2x \\ D_7(x,a) & = & x^7 - 7ax^5 + 14a^2x^3 - 7a^3x \\ D_9(x,a) & = & x^9 - 9ax^7 + 27a^2x^5 - 30a^3x^3 + 9a^4x \end{cases} \tag{3}$$

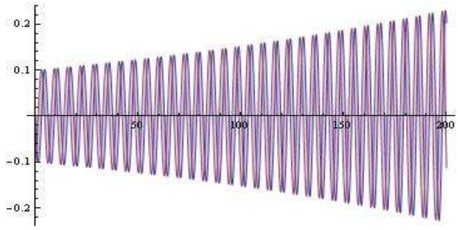

Figure 3: The solutions of the differential system (5).

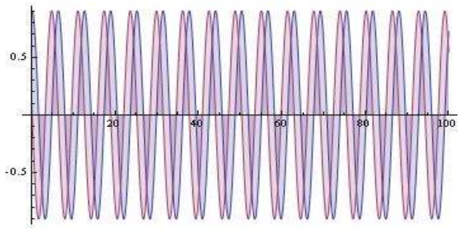

Figure 4: The solutions of the differential system (6).

$$\begin{cases} E_1(x,a) &= x \\ E_3(x,a) &= x^3 - 2ax \\ E_5(x,a) &= x^5 - 4ax^3 + 3a^2x \\ E_7(x,a) &= x^7 - 6ax^5 + 10a^2x^3 - 4a^3x \\ E_9(x,a) &= x^9 - 8ax^7 + 21a^2x^5 - 20a^3x^3 + 5a^4x \end{cases} \tag{4}$$

Polynomials of this type can be used as correction factors in the Lienard differential system. The solutions of the system

$$\begin{cases} \frac{dx}{dt} &= y - \epsilon(D_7(x,a)) \\ \frac{dy}{dt} &= -x \end{cases} \tag{5}$$

for $\epsilon = 0.001; x_0 = 0, y_0 = -0.1,\ a = 2.3$ are depicted on Fig. 3. The solutions of the system

$$\begin{cases} \frac{dx}{dt} &= y - \epsilon(E_7(x,a)) \\ \frac{dy}{dt} &= -x \end{cases} \tag{6}$$

for $\epsilon = 0.001; x_0 = 0.9, y_0 = 0.1,\ a = 0.9$ are depicted on Fig. 4.

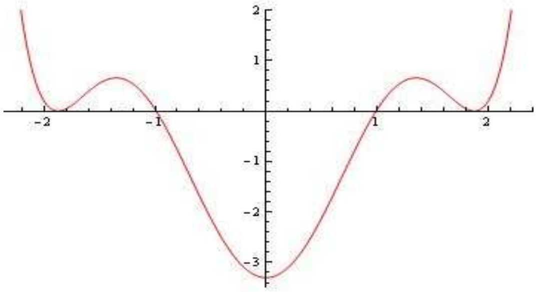

Figure 5: The Melnikov polynomial $P(r^2, 3)$ for $n = 7$ ($D_7(x, a)$; $a = 1$) and $\mu = 6.60337947$ (simple limit cycle: 0.990009 and limit cycle 1.87348 with multiplicity – two.

2.2. THE NEW MODELS IN THE LIGHT OF MELNIKOV'S CONSIDERATIONS

The case $n = 7$ for $D_7(x, a)$; $a = 1$.

Consider the model

$$\begin{cases} \dfrac{dx}{dt} & = & y - \epsilon(-\mu x + 14x^3 - 7x^5 + x^7) \\ \dfrac{dy}{dt} & = & -x \end{cases} \tag{7}$$

where $\mu > 0$, $\epsilon > 0$.

The following is valid

Proposition 1. The Lienard–type system for $n = 7$, and for all sufficiently small $\epsilon \neq 0$ for $\mu = 6.60337947$ has simple limit cycle: 0.990009 and limit cycle 1.87346 with multiplicity – two.

Proof. For the Melnikov polynomial in r^2 (see Fig. 5) we have:

$$P(r^2, 3) = -\frac{\mu}{2} + \frac{21}{4}r^2 - \frac{35}{16}r^4 + \frac{35}{128}r^6. \tag{8}$$

Evidently, for example $\mu = 6.60337947$ we have simple limit cycle and limit cycle with multiplicity – two.

The case $n = 9$ for $D_9(x, a)$; $a = 1$.

Consider the model

$$\begin{cases} \dfrac{dx}{dt} & = & y - \epsilon(\mu x - 30x^3 + 27x^5 - 9x^7 + x^9) \\ \dfrac{dy}{dt} & = & -x \end{cases} \tag{9}$$

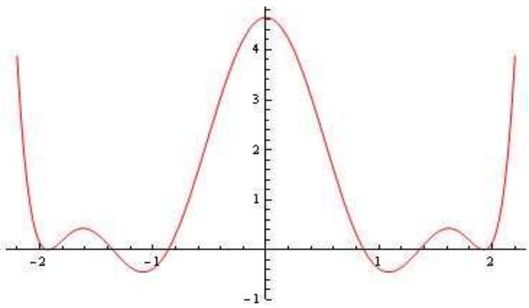

Figure 6: The Melnikov polynomial $P(r^2, 4)$ for $n = 9$ ($D_9(x, a)$; $a = 1$) and $\mu = 9.2958$ (simple limit cycles: 0.865074, 1.35478 and limit cycle 1.92565 with multiplicity – two.

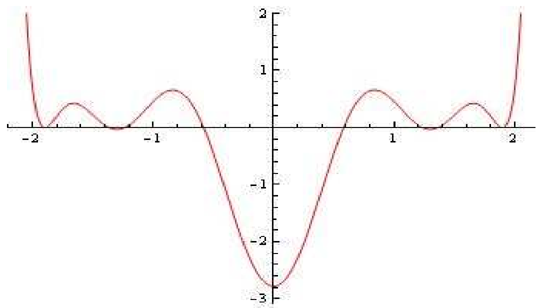

Figure 7: The Melnikov polynomial $P(r^2, 5)$ for $n = 11$ ($E_{11}(x, a)$; $a = 1$) and $\mu = 5.5602243$ (simple limit cycles: 0.58021, 1.23415, 1.36387 and limit cycle 1.895948 with multiplicity – two.

where $\mu > 0$, $\epsilon > 0$.

The following is valid

Proposition 2. The Lienard–type system for $n = 9$, and for all sufficiently small $\epsilon \neq 0$ for $\mu = 9.2958$ has simple limit cycles: 0.865074, 1.35478 and limit cycle 1.92565 with multiplicity – two.

Proof. For the Melnikov polynomial in r^2 (see Fig. 6) we have:

$$P(r^2, 4) = \frac{\mu}{2} - \frac{45}{4}r^2 + \frac{135}{16}r^4 - \frac{315}{128}r^6 + \frac{63}{256}r^8. \tag{10}$$

Evidently, for example $\mu = 9.2958$ we have two simple limit cycles and limit cycle with multiplicity – two.

The case $n = 11$ *for* $E_{11}(x, a)$; $a = 1$.

Consider the model

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon(-\mu x + 35x^3 - 56x^5 + 36x^7 - 10x^9 + x^{11}) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{11}$$

where $\mu > 0$, $\epsilon > 0$.

The following is valid

Proposition 3. The Lienard–type system for $n = 11$, and for all sufficiently small $\epsilon \neq 0$ for $\mu = 5.5602243$ has simple limit cycles: 0.58021, 1.23415, 1.36387 and limit cycle 1.895948 with multiplicity – two.

Proof. For the Melnikov polynomial in r^2 (see Fig. 7) we have:

$$P(r^2, 5) = -\frac{\mu}{2} + \frac{105}{8}r^2 - \frac{35}{2}r^4 + \frac{315}{32}r^6 - \frac{315}{128}r^8 + \frac{231}{1024}r^{10}. \tag{12}$$

Evidently, for example $\mu = 5.5602243$ we have two simple limit cycles and limit cycle with multiplicity – two.

Consider the following model in the light of Zeeman's approach [32]:

$$\begin{cases} \dfrac{dx}{dt} &= c(F(x) - y) \\ \dfrac{dy}{dt} &= \frac{1}{c}x \end{cases} \tag{13}$$

with $c > 0$ and

$$F(x) = x^{11} - 10ax^9 + 36a^2x^7 - 56a^3x^5 + 35a^4x^3 - px.$$

The catastrophe surface

$$(x, y, p) = F(x) - y$$

for the model is depicted on Fig. 8.

2.3. THE LEVEL CURVES

For more details of existing important results on the topic: Limit cycles bifurcations of some generalized polynomial Lienard system see [16]–[31].

Consider the Lienard polynomial systems of the type

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= Poly_i(x) + \epsilon h_i(x)y \end{cases} \tag{14}$$

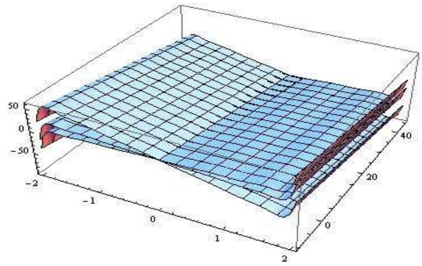

Figure 8: The catastrophe surface $(x, y, p) = F(x) - y$; $a = 0.9$; $p = 0.1, 10, 30$.

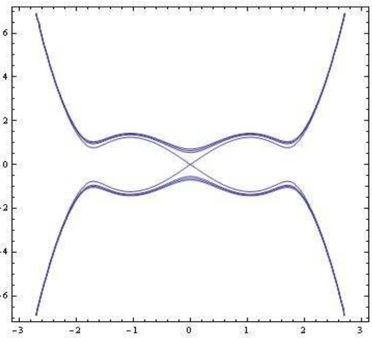

Figure 9: Level curves (the case i1).

where $0 \leq \epsilon < 1$; $h_i(x)$ are specially chosen polynomials, and $Poly_i(x)$ are some of the polynomials discussed in this paper.

The case i) $Poly_i(x)$ coincides with polynomial of the Dickson (of the first kind):

i1)

$$D_5(x, a) = x^5 - 5ax^3 + 5a^2x$$

The Hamiltonian of system (14) ($\epsilon = 0$) is

$$H(x, y) = \frac{y^2}{2} - 5a^2\frac{x^2}{2} + 5a\frac{x^4}{4} - \frac{x^6}{6}.$$

The level curves $L_{h_i} = \{H(x, y) = h_i\}$ for fixed $a = 0.8$ are depicted in Fig. 9

i2)

$$D_7(x, a) = x^7 - 7ax^5 + 14a^2x^3 - 7a^3x$$

The Hamiltonian of system (14) ($\epsilon = 0$) is

$$H(x, y) = \frac{y^2}{2} + 7a^3\frac{x^2}{2} - 14a^2\frac{x^4}{4} + 7a\frac{x^6}{6} - \frac{x^8}{8}.$$

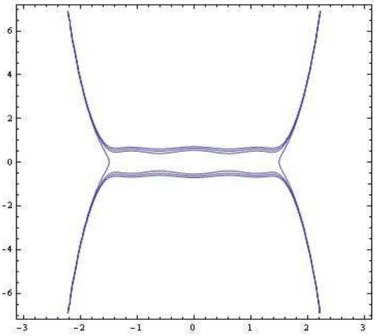

Figure 10: Level curves (the case i2).

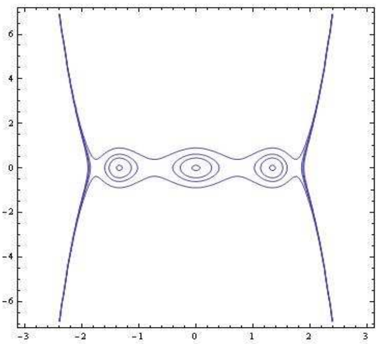

Figure 11: Level curves (the case j1).

The level curves $L_{h_i} = \{H(x,y) = h_i\}$ for fixed $a = 0.5$ are depicted in Fig. 10

The case j) $Poly_i(x)$ coincides with polynomial of the Dickson (of the second kind):

j1)

$$E_7(x,a) = x^7 - 6ax^5 + 10a^2x^3 - 4a^3x$$

The Hamiltonian of system (14) ($\epsilon = 0$) is

$$H(x,y) = \frac{y^2}{2} + 2a^3x^2 - \frac{5}{2}a^2x^4 + ax^6 - \frac{1}{8}x^8.$$

The level curves $L_{h_i} = \{H(x,y) = h_i\}$ for fixed $a = 0.9$ are depicted in Fig. 11

j2)

$$E_{11}(x,a) = x^{11} - 10ax^9 + 36a^2x^7 - 56a^3x^5 + 35a^4x^3 - 6a^5x$$

The Hamiltonian of system (14) ($\epsilon = 0$) is

$$H(x,y) = \frac{y^2}{2} + 3a^5\frac{x^2}{2} - 35a^4\frac{x^4}{4} + 56a^3\frac{x^6}{6} - 36a^2\frac{x^8}{8} + ax^{10}.$$

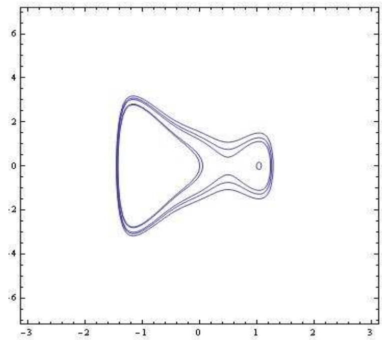

Figure 12: Level curves (the case j2).

The level curves $L_{h_i} = \{H(x,y) = h_i\}$ for fixed $a = 0.9$ are depicted in Fig. 12.

The simulations on the system

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -(x^7 - 7ax^5 + 14a^2x^3 - 7a^3x) + \epsilon(x - x^3 + x^5 - \frac{1}{7}x^7)y \end{cases} \tag{15}$$

for $a = 0.9$ are depicted on Fig. 13.

The simulations on the system

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -(x^7 - 6ax^5 + 10a^2x^3 - 4a^3x) + \epsilon(x - \frac{1}{3}x^3)y \end{cases} \tag{16}$$

for $a = 0.2$ are depicted on Fig. 14.

We note that the component $y(t)$ of the solution can be used for modeling and synthesis of electric circuits.

For other results see [6]–[10], [13]–[15].

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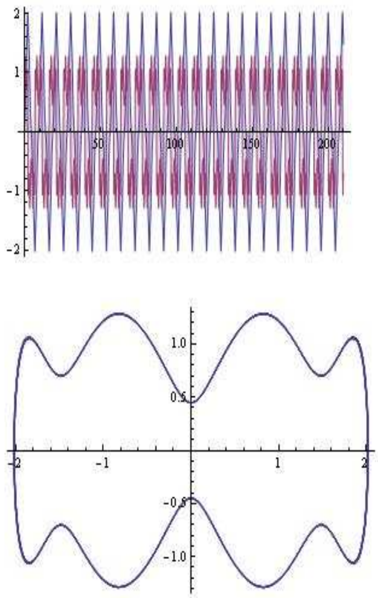

Figure 13: The simulations (system (15)).

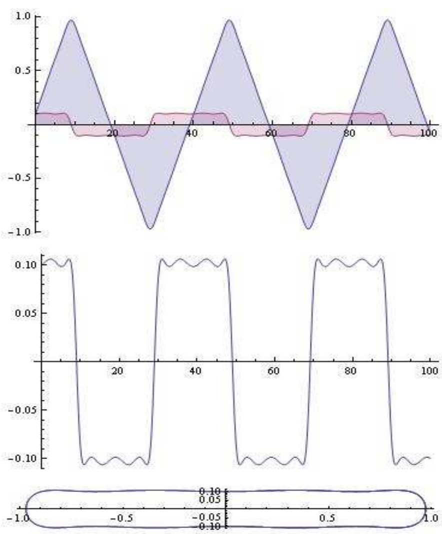

Figure 14: The simulations (system (15)).

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