International Journal of Differential Equations and Applications
Volume **22**, No. 1 (2023), pages: 67-79
ISSN (Print): 1311-2872; ISSN (Online): 1314-6084;
url: https://www.ijdea.eu

ijdea.eu

ON A HYPOTHETICAL OSCILLATOR: INVESTIGATIONS IN THE LIGHT OF THE MELNIKOV'S APPROACH, SOME SIMULATIONS

Nikolay Kyurkchiev[1,2] and Tsvetelin Zaevski[2]

[1]Faculty of Mathematics and Informatics
University of Plovdiv Paisii Hilendarski
24, Tzar Asen Str., 4000 Plovdiv, BULGARIA

[2] Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8, 1113 Sofia, BULGARIA

ABSTRACT: A variety of microelectromechanical (MEM) oscillators is governed by a version of the Mathieu equation that harbors both linear and cubic nonlinear time-varying stiffness terms [7]–[11]. Obviously, these studies can be successfully continued. In this paper, we propose a generalized hypothetical differential model. Considerations in the light of the Melnikov's approach are also given. We focus on some interesting simulations based on the proposed new model as well as demonstrate specialized modules for investigating the dynamics of these oscillators. This is an integral part of a planned much more general Web-based application for scientific computing.

Key Words: Mathieu equation, microelectromechanical oscillators, Melnikov's approach, generalized hypothetical oscillator

Received: June 28, 2023 **Revised:** October 18, 2023
Published: October 21, 2023 **doi:** 10.12732/ijdea.v22i1.6
Academic Publications, Ltd. https://acadpubl.eu

1. INTRODUCTION

In [8], the authors study the dynamics of the following *microelectromechanical oscillator*

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\beta x - \delta x^3 - \epsilon\left(\gamma(1+\cos(\omega t))x + \eta(1+\cos(\omega t))x^3\right) - \epsilon\alpha y, \end{cases} \tag{1}$$

where $0 \leq \epsilon < 1$. For other results see [7]–[11]. Obviously, these studies can be successfully continued. In this article, we propose a generalized hypothetical differential model taking a special attention on the Melnikov's approach. We present some simulations in the light of the so defined new model. We demonstrate also some specialized modules for investigating the dynamics of these oscillators. The derived results can be used as an integral part of a much more general application for scientific computing – for some details see [13]–[17].

The plan of the paper is as follows. We state our model in section 2. Our results in the light of the Melnikov's approach can be found in 3. Some simulations are presented in section 4. We conclude by section 5.

2. THE MODEL

We consider the following new generalized hypothetical model

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = bx - \displaystyle\sum_{i=0}^{\left[\frac{n}{2}\right]-1} b_i x^{n-2i} - \epsilon\left(\alpha y + (1+\cos(\omega t))\left(ax + \sum_{i=0}^{\left[\frac{m}{2}\right]-1} a_i x^{m-2i}\right)\right), \end{cases} \tag{2}$$

where $0 \leq \epsilon < 1$ and n and m are odd.

Let us consider the case $n = m = 3$. We can fix without community restriction b and b_0 at $b = b_0 = 1$. For $\epsilon = 0$, the resulting Hamiltonian of the system (2) is

$$H(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4.$$

There is a saddle at the origin, centers at $(\pm 1, 0)$, and double homoclinic orbit given by (see Fig.1)

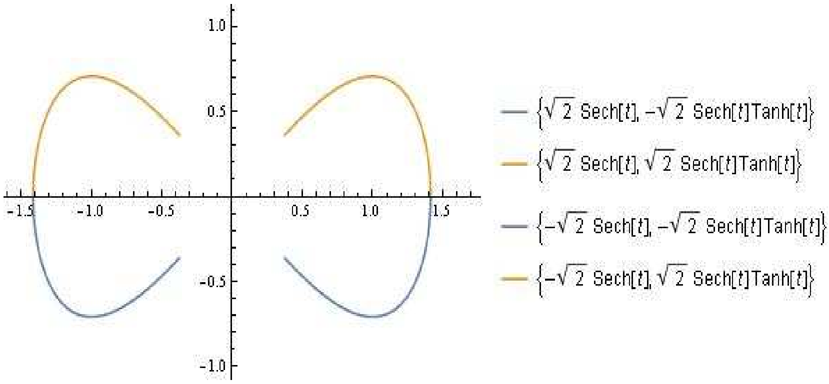

Figure 1: Double homoclinic orbit.

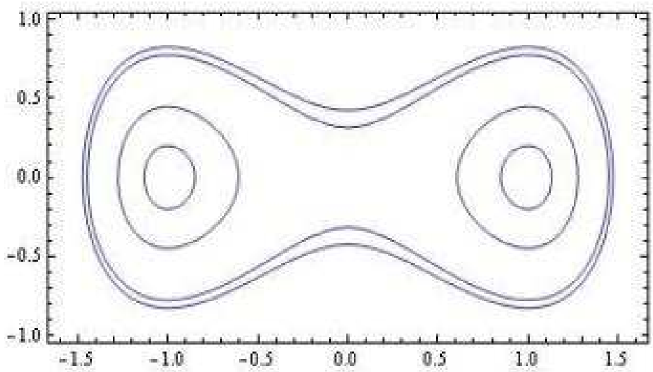

Figure 2: The level curve L_h.

$$x_0(t) = \pm\sqrt{2}sech\, t$$
$$y_0(t) = \mp\sqrt{2}sech\, t\, tanh\, t.$$

We refer to [1]–[4] for more details. The level curve, see [12], can be derived through the formula

$$L_h = \{H(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4 = h\};\ -\frac{1}{4} < h < 0 \cup h > 0$$

and it is depicted on Fig. 2.

3. RESULTS IN THE LIGHT OF THE MELNIKOV'S APPROACH

The Melnikov's method gives us an analytic tool for establishing the existence of transfer homoclinic points of the Poincare map for a periodic orbit of a perturbed dynamical system.

Let us consider again the case $m = n = 3$. The first Melnikov function is given by

$$M(t_0) = \int_{-\infty}^{\infty} y_0(t)\left(\alpha y_0(t) + (1+\cos(\omega(t+t_0)))(a x_0(t) + a_0 x_0^3(t))\right) dt.$$

From a numerical point of view, the task of finding a multiple root of $M(t_0)$ is more interesting given that the parameters appearing in the proposed differential model are subject to a number of restrictions (of a physical and practical nature).

The following proposition stands.

Proposition 1. If $m = n = 3$ and $A = k$ for

$$A = \frac{8\alpha}{\pi} sinh\left(\frac{\pi\omega}{2}\right); \;\; k = \omega^2(6a + a_0(4+\omega^2)),$$

then the Melnikov function $M(t_0)$ has a root with multiplicity two.

Proof. We have

$$\begin{aligned} M(t_0) &= \int_{-\infty}^{\infty} y_0(t)\left(\alpha y_0(t) + (1+\cos(\omega(t+t_0)))(a x_0(t) + a_0 x_0^3(t))\right) dt \\ &= \int_{-\infty}^{\infty} \alpha(\sqrt{2} sech\, t\, tanh\, t)^2\, dt + \\ &+ \int_{-\infty}^{\infty} \sqrt{2} sech\, t\, tanh\, t\, (1+\cos(\omega(t+t_0)))\left(-a\sqrt{2} sech\, t + a_0\left(-\sqrt{2} sech\, t\right)^3\right) dt \end{aligned}$$

The integrals can be evaluated (the second by the method of residuals) to yield

$$\begin{aligned} M(t_0) &= \tfrac{4}{3}\alpha - \tfrac{1}{6}\pi\omega^2(6a + a_0(4+\omega^2)) csch\left(\tfrac{\pi\omega}{2}\right)\sin(t_0\omega) \\ &= -\tfrac{1}{6}\pi\omega^2(6a + a_0(4+\omega^2)) csch\left(\tfrac{\pi\omega}{2}\right)\left(\sin(t_0\omega) - \tfrac{A}{k}\right) \end{aligned}$$

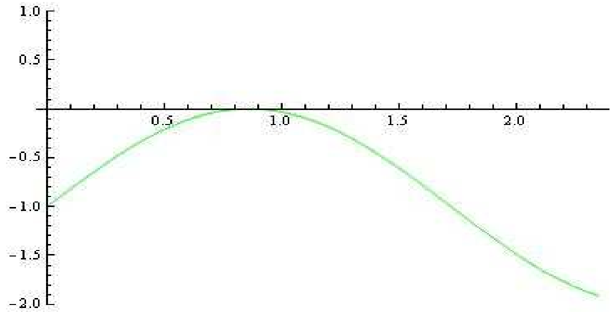

Figure 3: The root t_0 of multiplicity two (Example 1).

This completes the proof of Proposition 1.

Example 1. Let us consider the example when $\alpha = 2.4$, $a_0 = 0.18$, $\omega = 1.8255$, $a = 2.46$. The root of multiplicity two is $t_0 \approx 0.86$ – see Fig.3.

Next we investigate the case $n = 3$, $m = 5$. The following proposition is valid.

Proposition 2. If $n = 3$, $m = 5$, and $B = k_1$ for

$$B = \frac{120\alpha}{\pi} sinh\left(\frac{\pi\omega}{2}\right); \;\; k_1 = \omega^2(90a + (4+\omega^2)(15a_1 + a_0(16+\omega^2))),$$

then the Melnikov function $M(t_0)$ has a root with multiplicity two.

Proof. For the first Melnikov function we have

$$M(t_0) \;\; = \;\; -\tfrac{\pi\omega^2}{90}\left(90a + (4+\omega^2)(15a_1 + a_0(16+\omega^2))\right) csch\left(\tfrac{\pi\omega}{2}\right) \times$$

$$\times \left(sin(t_0\omega) - \tfrac{B}{k_1}\right).$$

This completes the proof of Proposition 2.

Example 2. For $\alpha = 1.15$, $a_0 = 0.09$, $a_1 = 0.15$, $a = 1.01$, $\omega = 1.9946$ the root of multiplicity two is $t_0 \approx 0.75$ – see Fig.4.

The reader can consider the corresponding approximation problem for arbitrarily chosen n and m. For $n > 3$ the study of the critical levels of the Hamiltonian $H(x, y)$ is very complicated. In this regard, we recommend the excellent study by Gavrilov and Iliev [24]. Here we will only note that even with the weakened choice $n = 3$ and m - a sufficiently large number, some difficulties (of a user nature) are encountered when calculating the Melnikov integrals using known Computer Algebraic Systems. For example, with $n = 3$, $m = 7$ we get

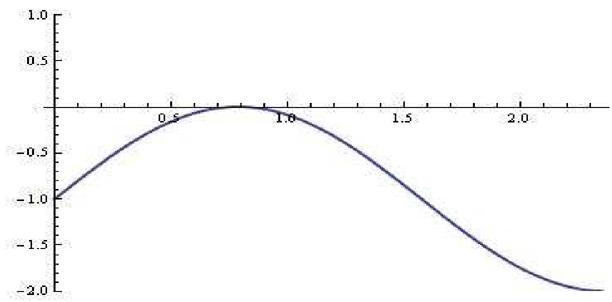

Figure 4: The root t_0 of multiplicity two (Example 2).

$$M(t_0) = \frac{4\alpha}{3} - \frac{\pi\omega^2}{2520}\left(2520a + (4+\omega^2)(420a_2 + 28a_1(16+\omega^2) + a_0(16+\omega^2)(36+\omega^2))\right) \times$$
$$\times\, csch\left(\frac{\pi\omega}{2}\right)\sin(t_0\omega),$$

but with a pre-set limit by us: $|Im(\omega)| \leq 2$.

It is easily understood that

$$M(t_0) \;=\; -\tfrac{\pi\omega^2}{2520}\left(2520a + (4+\omega^2)(420a_2 + 28a_1(16+\omega^2) + a_0(16+\omega^2)(36+\omega^2))\right)$$
$$\times\left(\sin(t_0\omega) - \tfrac{C}{k_2}\right).$$

Thus we proved the following statement

Proposition 3. If $n = 3$, $m = 7$, and $C = k_2$ for

$$C = \frac{3360\alpha}{\pi} sinh\left(\frac{\pi\omega}{2}\right);$$

$$k_2 = \omega^2(2520a + (4+\omega^2)(420a_2 + 28a_1(16+\omega^2) + a_0(16+\omega^2)(36+\omega^2))),$$

then the Melnikov function $M(t_0)$ has a root with multiplicity two.

4. SOME SIMULATIONS

Here we will focus on some interesting simulations

1 For given $n = m = 5$, $a = 1$, $\alpha = 2.2$, $a_0 = -1.35$, $a_1 = -2.9$, $b = b_0 = b_1 = 1$, $\epsilon = 0.01$, $\omega = 1.9$ the simulations on the system (2) for $x_0 = 0.9$; $y_0 = 0.7$ are depicted on Fig. 5.

2 For given $n = m = 5$, $a = 1$, $\alpha = 2.9$, $a_0 = -3.35$, $a_1 = -2.2$, $b = b_0 = b_1 = 1$, $\epsilon = 0.01$, $\omega = 2.1$ the simulations on the system (2) for $x_0 = 0.7$; $y_0 = 0.6$ are depicted on Fig. 6.

3 For given $n = m = 7$, $a = 1$, $\alpha = 2.9$, $a_0 = -3.35$, $a_1 = -2.2$, $b = 0.1$, $b_0 = 0.3$, $b_1 = 0.01$, $b_2 = 0.2$, $\epsilon = 0.01$, $\omega = 2.1$ the simulations on the system (2) for $x_0 = 0.7$; $y_0 = 0.6$ are depicted on Fig. 7.

4 For given $n = m = 9$, $a = 1$, $\alpha = 2.4$, $a_0 = -1.35$, $a_1 = -0.25$, $a_2 = -0.55$, $a_3 = 0.12$, $b = 0.1$, $b_0 = 0.2$, $b_1 = 0.05$, $b_2 = 0.15$, $b_3 = 0.09$, $\epsilon = 0.01$, $\omega = 1.85$ the simulations on the system (2) for $x_0 = 0.8$; $y_0 = 0.6$ are depicted on Fig. 8.

5. CONCLUDING REMARKS

We have investigated in this paper a generalized differential model for microelectromechanical oscillators in the light of the Melnikov's approach. The derived results can be used to estimate the associated total energy potential of the considered differential system. As we mentioned above, the method of residues has been used when calculating the Melnikov integrals. Modern computer algebraic systems for scientific calculations provide this opportunity for the users. The upgrade of the Web Application planned by us foresees the use of an algorithm (hidden to the user) to define the limit, for example $|Im(\omega)| \leq Const.$

Also, some specialized modules for investigating the dynamics of some new hypothetical oscillators have been demonstrated. They will be an integral part of the mentioned above Web-based application for scientific computing. Let us note that the theoretical apparatus for studying the circuit implementation (design, fabricating, etc.) of the considered differential model is extremely complex and requires a serious investigation before being adapted for its possible inclusion in our planned Web platform.

Specialists working in this scientific direction have the floor.

ACKNOWLEDGMENTS

The first author is financed by the project No FP23-FMI-002 "Intelligent software tools and applications in research in Mathematics, Informatics, and Pedagogy of Education" of the Scientific Fund of the Paisii Hilendarski University of Plovdiv, Bulgaria.

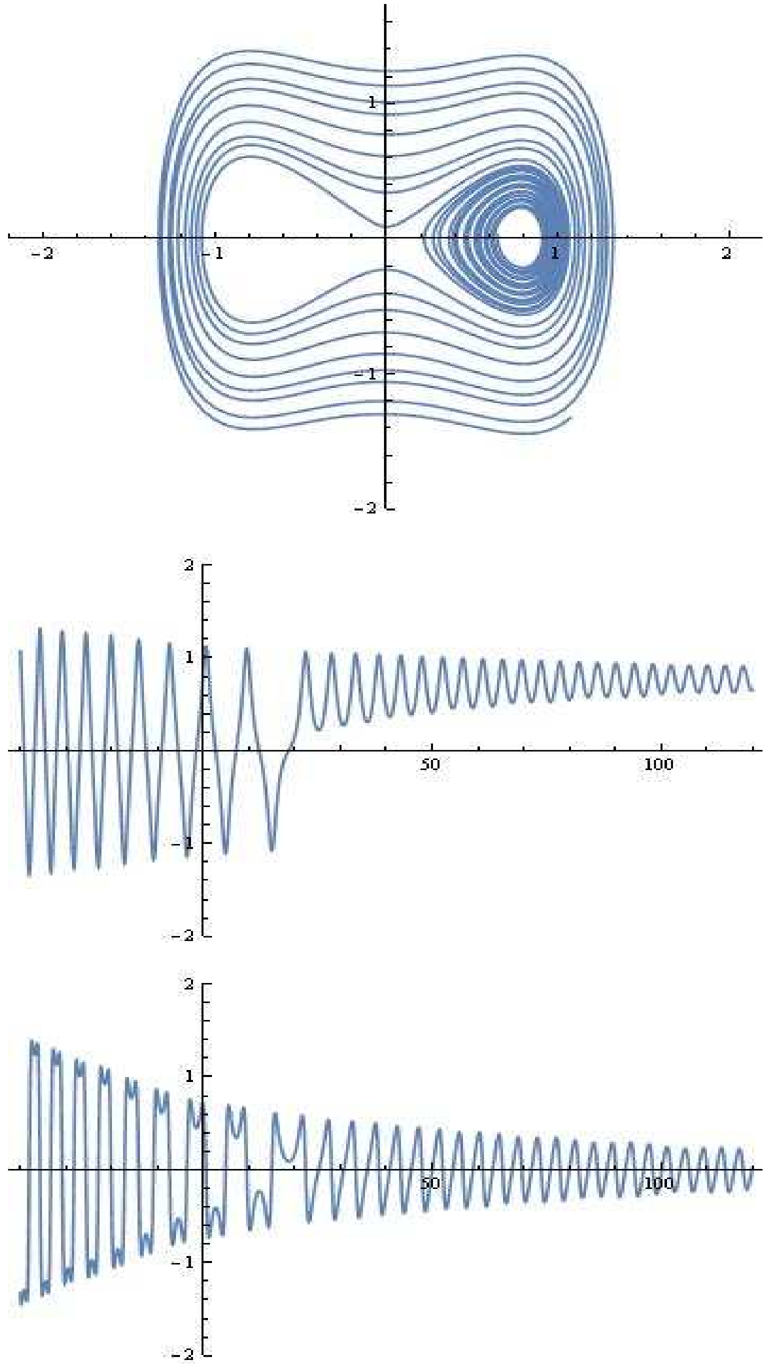

Figure 5: a) phase space; b) x-time series; c) y-time series; (example 1).

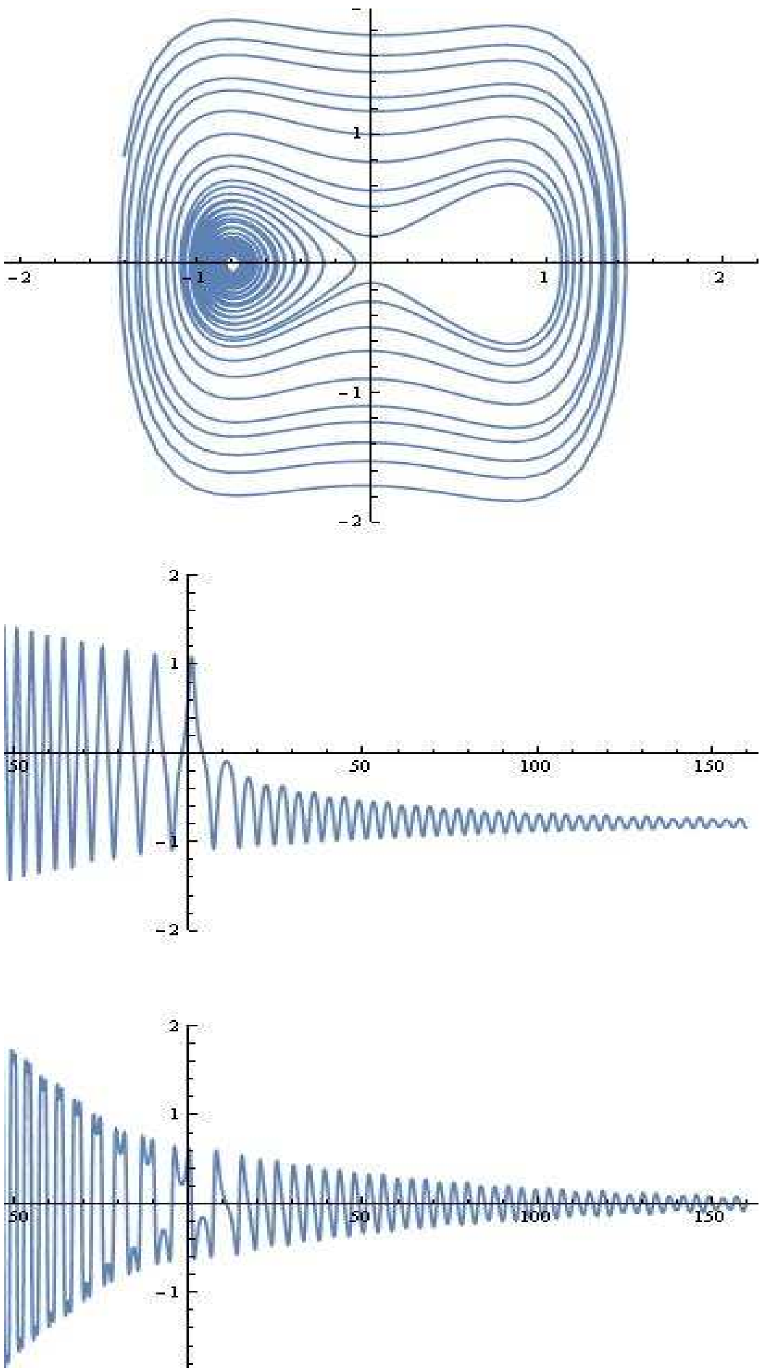

Figure 6: a) phase space; b) x-time series; c) y-time series; (example 2).

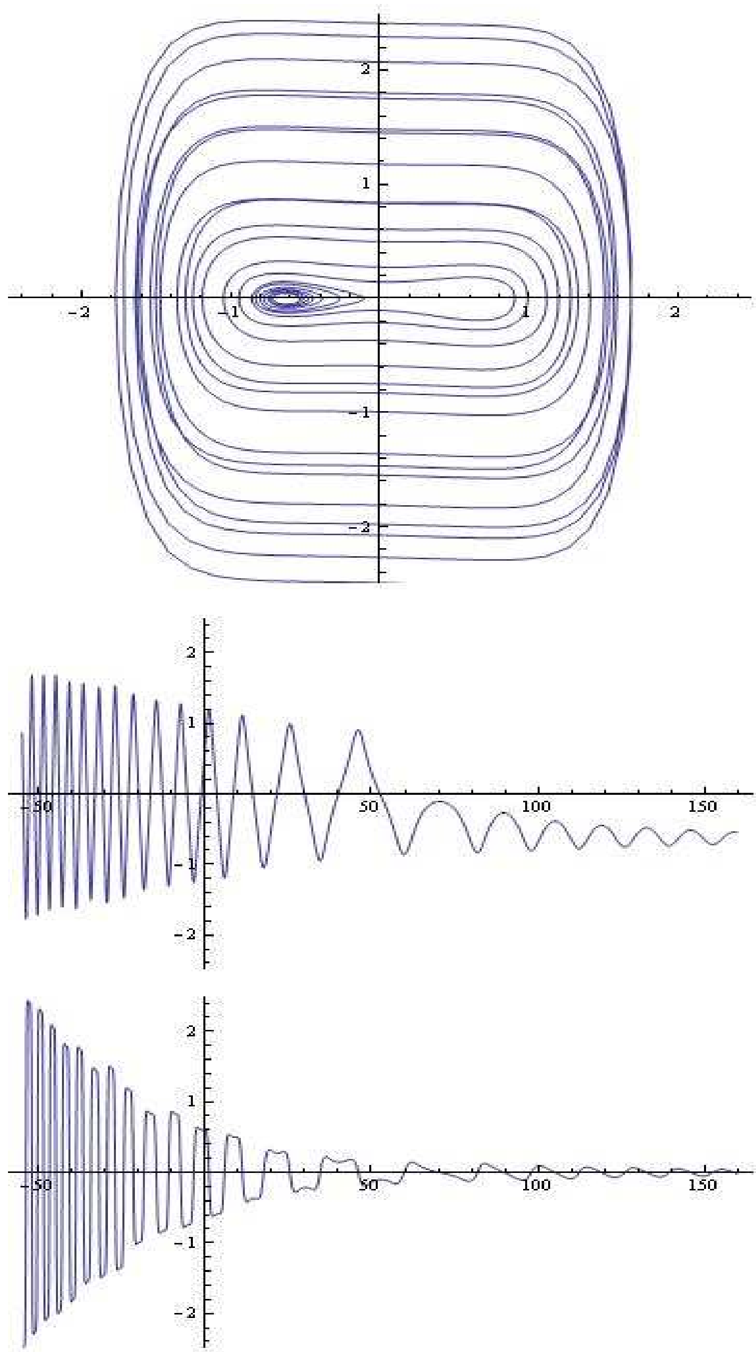

Figure 7: a) phase space; b) x-time series; c) y-time series; (example 3).

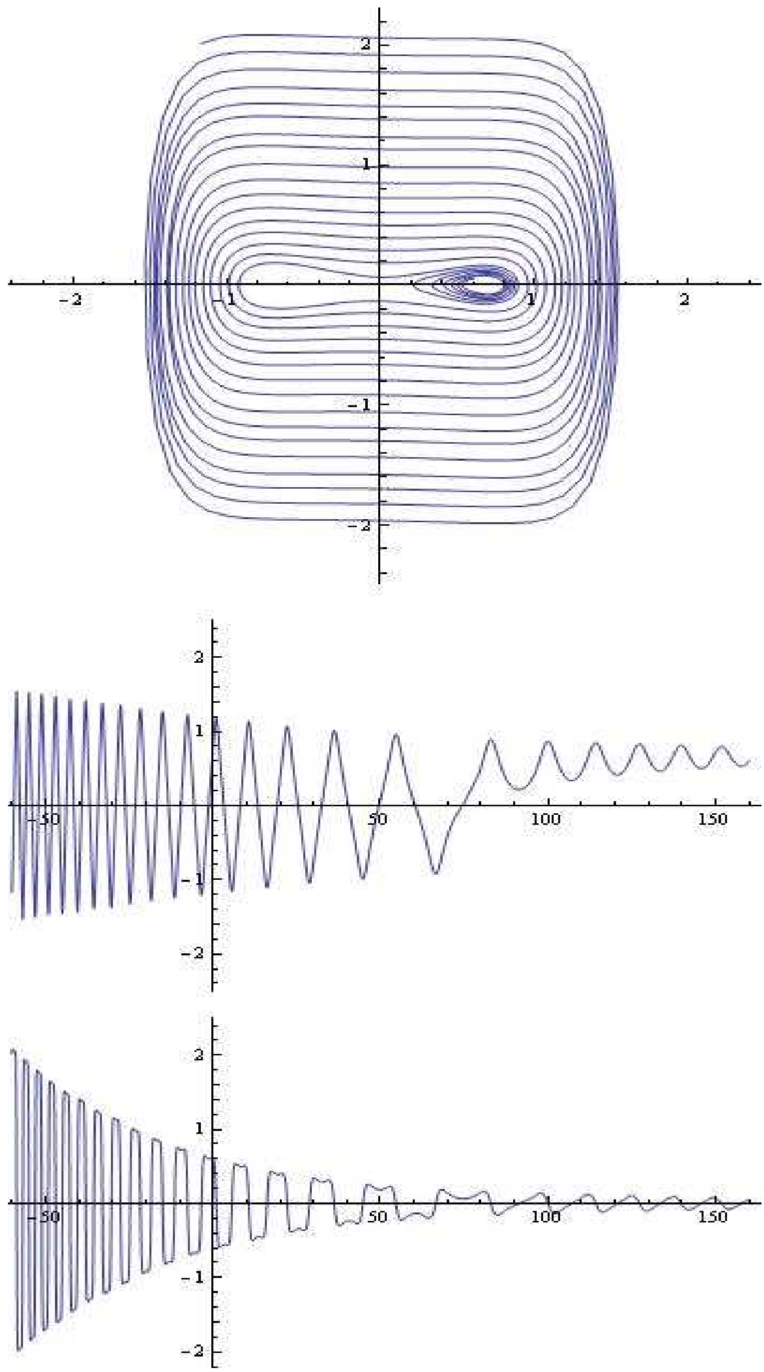

Figure 8: a) phase space; b) x-time series; c) y-time series; (example 4).

The second author was supported by the project KP-06-N32/8 with the Bulgarian National Science Fund.

REFERENCES

[1] J. Guckenheimer, P. Holmes, Nonlinear oscillations, dynamical systems, and bifurcations of vector fields, Sprinder-Verlag, New York, 1983.

[2] L. Perko, *Differential Equations and Dynamical Systems*, Springer–Verlag, New York (1991).

[3] P. Holmes, J. Marsden, Horseshoes in perturbation of Hamiltonian systems with two degrees of freedom, *Comm. Math. Phys.*, 82 (1982), 523–544.

[4] P. Holmes, J. Marsden, A partial differential equation with infinitely many periodic orbits: chaotic ascillations, *Archive for Rational Mechanics and Analysis*, (1981).

[5] V. K. Melnikov, On the stability of a center for time–periodic perturbation, *Tr. Mosk. Mat. Obs.*, **12** (1963).

[6] T. Blows, L. Perko, *SIAM Rev.*, **36**, 341 (1994).

[7] R. Zounes, R. Rand, Subharmonic resonance in the nonlinear Narhieu equation, *Int. J. Non - Linear Mechanics*, 37 (1), 2002, 43–73.

[8] B. DeMartini, H. Butterfield, J. Moehlis, K. Turner, Chaos for a microelectromechanical oscillator governed by the nonlinear Mathieu equation, *J. of Microelectrmechanical Systems*, 16 (6), 2007.

[9] A. Luo and F. Wang, Nonlinear dynamics of a micro-electro-mechanical system with time-varying capacitors, *J. Vib. Acoust.*, vol. 126, no. 1, 77–83, 2004.

[10] S. Wiggins, Introduction to Applied Nonlinear Dynamical Systems and Chaos, 2nd ed. New York: Springer-Verlag, 2003.

[11] B. DeMartini, J. Moehlis, K. Turner, J. Rhoads, S. Shaw, W. Zhang, Modelling of parametrically excited microelectromechanical oscillator dynamics with application to filtering, In Proc, IEEE Sens, Conf., Irvine, CA, 2005, 345–348.

[12] L. Gavrilov, I. Iliev, The limit cycles in a generalized Rayleigh–Lienard oscillator, *Discrete and Continuous Dynamical Systems*, 43 (6) (2023), 2381–2400.

[13] Kyurkchiev, V., Iliev, A., Rahnev, A., Kyurkchiev, N., On a class of orthogonal polynomials as corrections in Lienard differential system. Applications, *Algorithms*, 2023.

[14] Kyurkchiev, N., Iliev, A., On the hypothetical oscillator model with second kind Chebyshev's polynomial–correction: number and type of limit cycles, simulations and possible applications, *Algorithms* 2022, 15, 12.

[15] Kyurkchiev, V., Kyurkchiev, N., Iliev, A., Rahnev, A., On some extended oscillator models: a technique for simulating and studying their dynamics. Plovdiv University Press: Plovdiv, 2022; ISBN 978-619-7663-13-6.

[16] V. Kyurkchiev, N. Kyurkchiev, A. Iliev, A. Rahnev, *On the Lienard System with Some "Corrections of Polynomial-type": Number of Limit Cycles, Simulations and Possible Applications. Part II.*, Plovdiv, Plovdiv University Press (2022), ISBN: 978-619-7663-41-9.

[17] M. Vasileva, V. Kyurkchiev, A. Iliev, A. Rahnev, N. Kyurkchiev, Applications of some orthogonal polynomials and associated polynomials of higher kind, Plovdiv, Plovdiv University Press (2023), ISBN: 978-619-7663-72-3.

[18] X. Li, Y. Shen, J. Sun, S. Yang, New periodic attractors is slow-fast Duffing system with periodic parametric excitation, *Scientific Reports*, 2019, 9:11185.

[19] P. J. Holmes, Chaos in Duffing's equation: Strange Attractors?, (in Encyclopedia of Vibration, (2001).

[20] N. Sanchez, A. Nayfeh, Prediction of bifurcations in a parametrically excited Duffing oscillator, *Int. Journal of Nonlinear Mechanics*, 25 (2–3), 1990, 163–176.

[21] G. Michon, L. Manin, R. Parker, R. Dufour, Duffing oscillator with parametric excitation: analytical and experimental investigation on a Belt–Pulley systems (Preprint).

[22] X. Yue, Ge Lv, Y. Zhang, Rare and hidden attractors in a periodically forced Duffing system with absolute nonlinearity, *Chaos, Solitons and Fractals*, 150 (2021).

[23] K. Tang, K. Man, G. Zhong, G. Chen, Generating chaos via $x|x|$, IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications, 48 (5), 2001, 636–641.

[24] L. Gavrilov, I. Iliev, Complete hyperelliptic integrals of the first kind and their non–oscillation, *Trans. of the Amer. Math. Soc.* (2003), 23 pp.