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SOME INVESTIGATIONS AND SIMULATIONS ON THE PLANAR RAYLEIGH–LIENARD DIFFERENTIAL SYSTEM (WEB PLATFORM UPGRADE)

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ABSTRACT: In this article we demonstrate some specialized modules for investigating the dynamics of the planar Rayleigh–Lienard differential system $\frac{dx}{dt} = y$; $\frac{dy}{dt} = -Poly(x) + \epsilon F(x,y)y$, an integral part of a planned much more general Web–based application for scientific computing. The polynomials $Poly(x)$ as "corrections", can be arbitrary orthogonal polynomials and their associated polynomials. The Web–based application, with the ability to upgrade and include additional modules that existing platforms (with paid and free access) do not provide to users (for example, modules for automatic generation of theorems for the number and type of limit cycles (in the light of Melnikov's considerations), generation of radiation diagrams, etc.) Numerical examples, illustrating our results using *CAS MATHEMATICA* are given.

Key Words: Rayleigh-Lienard differential system, level curves, emitting chart

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1. INTRODUCTION

A class of mixed Lienard–type equations has the following form

$$\ddot{x} + a_1(x)\dot{x}^2 + a_2(x)\dot{x} + a_3(x) = 0 \tag{1}$$

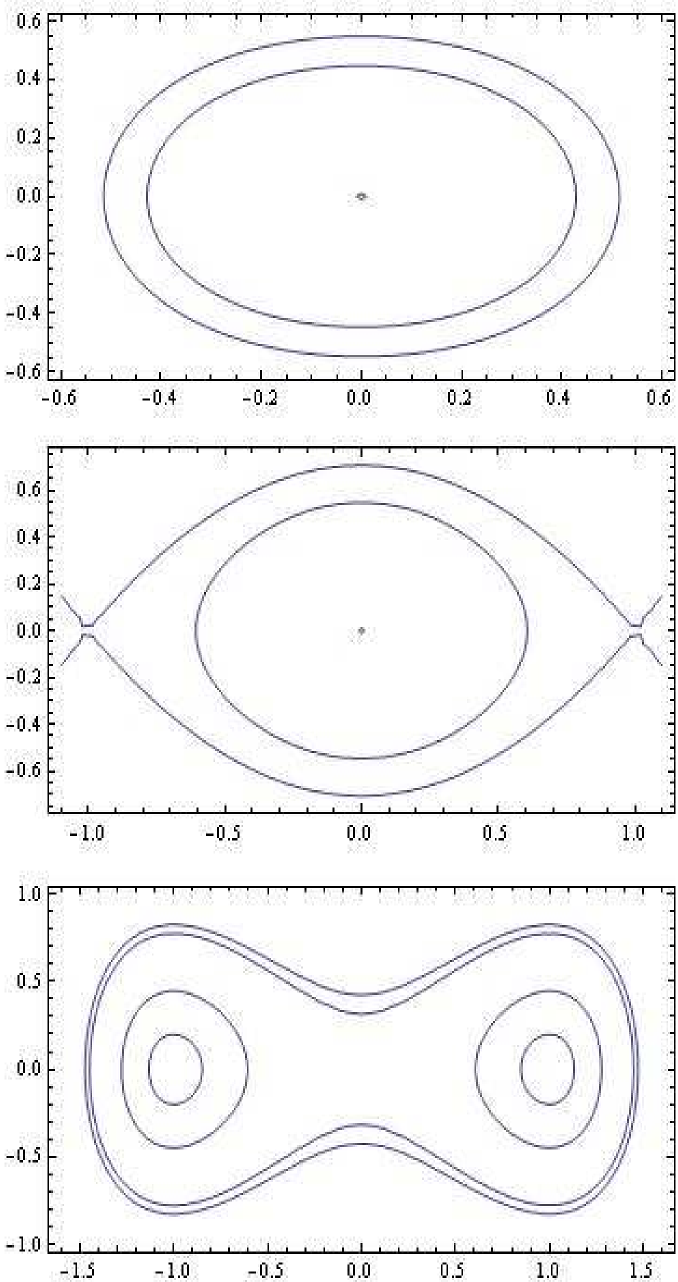

Figure 1: Phase portraits; a) global center; b) truncated pendulum; c) eight loop.

where $a_i(x)$; $i = 1, 2, 3$ are arbitrary functions of x.

When $a_1(x) = 0$, from (1), the classical Lienard equation is obtained.

Gavrilov and Iliev [1] study the limit cycles of the generalized Rayleigh–Lienard equation

$$\ddot{x} + ax + bx^3 - (a_1 + a_2x^2 + a_3\dot{x}^2 + a_4x^4 + a_5\dot{x}^4 + a_6x^6)\dot{x} = 0 \tag{2}$$

assuming that a and b are fixed no-zero constants, and a_j, $j = 1, \ldots, 6$ are small real parameters.

The equivalent planar Rayleigh–Lienard differential system is

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -ax - bx^3 + (a_1 + a_2x^2 + a_3y^2 + a_4x^4 + a_5y^4 + a_6x^6)y \end{cases} \tag{3}$$

Without loss of generality let $|a| = |b| = 1$.

Consider the Hamiltonian of the system (3)

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -(ax + bx^3) \end{cases} \tag{4}$$

The level curves [1]:

$L_h^1 = \{H^1(x, y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{4}x^4 = h\}$ (global center); $h > 0$;

$L_h^2 = \{H^2(x, y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{4}x^4 = h\}$ (truncated pendulum); $0 < h < \frac{1}{4}$;

$L_h^3 = \{H^3(x, y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4 = h\}$ (eight loop); $-\frac{1}{4} < h < 0 \cup h > 0$

are depicted at Fig. 1. *Gavrilov and Iliev* proved the following

Theorem [1]. The cyclicity of each given open period annulus of the Hamiltonian system (4) with respect to the six–parameter deformation (3) is five, except for the exterior period annulus of the eight loop case. In this latter case, the cyclicity is bounded by six.

Consider the classical Lienard differential system

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -Poly(x) + \epsilon F(x)y \end{cases} \tag{5}$$

where $0 \leq \epsilon < 1$; $F(x)$ and $Poly(x)$ are specially chosen polynomials.

Some of previous research [2]–[12], [14]–[17] on this issue encouraged us to begin developing specialized modules as part of a much more general Web–based application for scientific computing, with the ability to upgrade and include additional modules that existing platforms (with paid and free access) do not provide to users (for example, modules for automatic generation of theorems for the number and type of limit cycles (in the light of Melnikov's considerations), generation of radiation diagrams, etc.)

A natural extension of this Web–based platform involves further consideration and simulations on the planar differential system of Rayleigh–Lienard.

2. MAIN RESULTS. SIMULATIONS

Consider a class of planar Rayleigh–Lienard system:

$$\begin{cases} \frac{dx}{dt} & = & y \\ \frac{dy}{dt} & = & -Poly(x) + \epsilon F(x,y)y \end{cases} \tag{6}$$

where $0 \leq \epsilon < 1$; $F(x,y)$ is of the type (see (3))

$$F(x,y) = a_1 + a_2x^2 + a_3y^2 + a_4x^4 + a_5y^4 + a_6x^6$$

(a_j are real parameters) and $Poly(x)$ are specially chosen polynomials as "corrections" in the differential system.

The following capabilities has been implemented:

- the user sets: the polynomials $Poly(x)$, which can be arbitrary orthogonal polynomials and their associated polynomials (such as associated Hermite polynomials, associated Gegenbauer polynomials, associated Legendre polynomials, associated Lommel polynomials, q–Lommel polynomials associated with the Jackson q–Bessel function, continuous and bivariate q–Hermite polynomials, extended Gegenbauer polynomials and their q–analogues, associated Jacoby polynomials, Chebyshev, Dickson and Gegenbauer polynomials of higher kind etc.).

Define the normalized antenna factor

$$\frac{y(b\cos\theta + c)}{m}$$

where θ is the azimuthal angle and c is the phase difference.

We note that the use of $y(\theta)$ - the solution component of the corresponding Rayleigh–Lienard differential system as an antenna factor is very complicated.

In this paper we demonstrate some specialized modules for investigating the dynamics of differential model, an integral part of a planned much more general Web–based application for scientific computing.

I. The associated Gegenbauer polynomials as "corrections" in the planar system (6)

For given $\epsilon = 0.0001$, $b = -0.05$; $c = 0$ and

$$a_1 = 7.9; a_2 = -1.5; a_3 = 23.8; a_4 = 8.6; a_5 = -21.4; a_6 = -1.9$$

the simulations on the system (6) for

1.
$$Poly(x) = 2x^3 - x$$

2.
$$Poly(x) = \frac{16}{3}x^5 - \frac{42}{9}x^3 + x$$

with $x_0 = 0.99$; $y_0 = 0.22$ are depicted on Fig. 2–Fig. 3.

Remark. The detailed study of the dynamics of the differential model in case I.1. can be completed in the sense of the considerations of the above-cited Theorem. Our considerations on the Melnikov homoclinic integral are summarized in Appendix 1.

The Hamiltonian of the system (6) ($\epsilon = 0$) in the case I.2 is

$$\frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{7}{6}x^4 + \frac{8}{9}x^6.$$

For the level curves see Fig. 4.

II. The Dickson polynomials (of the 6-th kind) as "corrections" in the planar system (6)

Let

$$Poly(x) = x^9 - 4x^7 - 3x^5 + 20x^3 - 11x.$$

The Hamiltonian of system (6) ($\epsilon = 0$) is

$$H(x, y) = \frac{y^2}{2} + \frac{1}{10}x^{10} - \frac{1}{2}x^8 - \frac{1}{2}x^6 + 5x^4 - \frac{11}{2}x^2.$$

The level curves $L_{h_i} = \{H(x, y) = h_i\}$ are depicted at Fig. 5.

For given $\epsilon = 0.0001$, $b = 0.85$; $c = 1.35$ and

$$a_1 = 7.9; a_2 = -1.5; a_3 = 23.8; a_4 = 8.6; a_5 = -21.4; a_6 = -1.9$$

the simulation on the system (6) for

$$Poly(x) = x^9 - 4x^7 - 3x^5 + 20x^3 - 11x$$

with $x_0 = 0.45$; $y_0 = 0.45$ is depicted on Fig. 6.

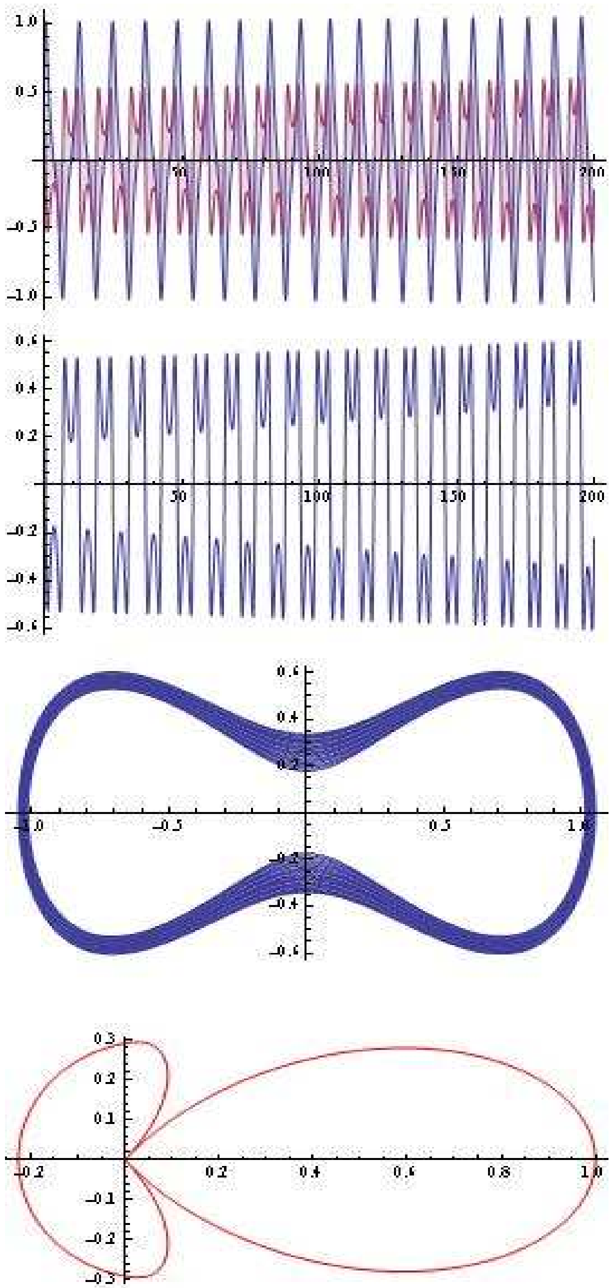

Figure 2: a) The solutions of differential system; b) y-component of the solution; c) Phase portrait; d) radial diagram (the case 1).

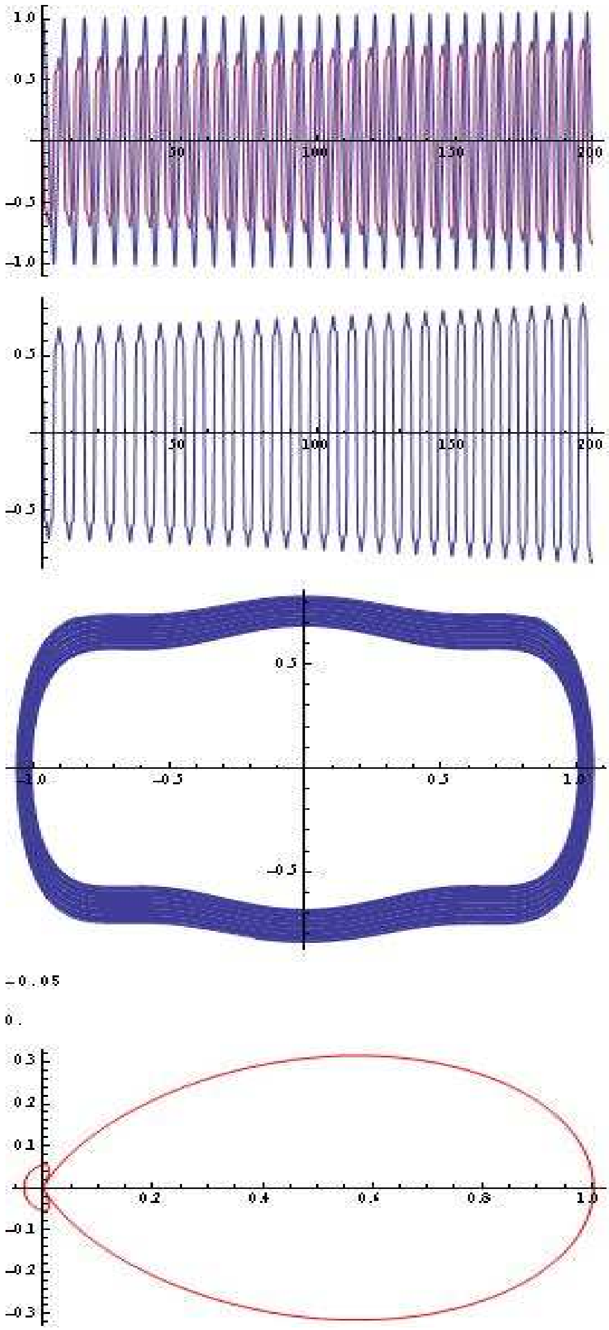

Figure 3: a) The solutions of differential system; b) y-component of the solution; c) Phase portrait; d) radial diagram (the case 2).

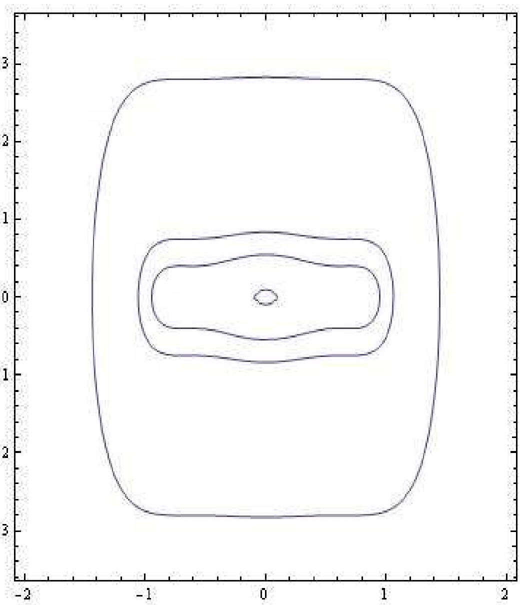

Figure 4: Phase portrait (the case I.2).

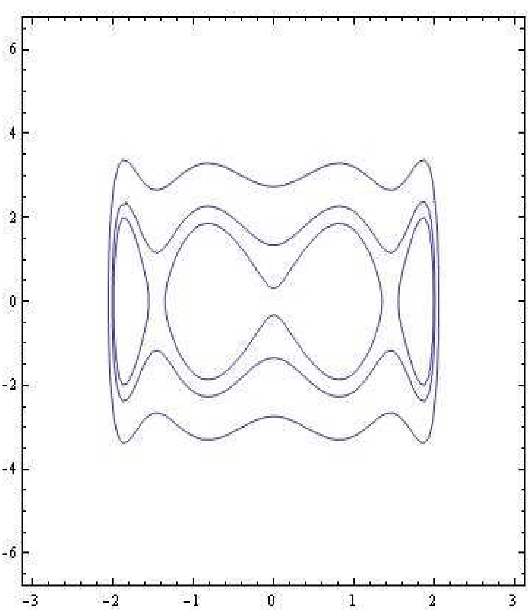

Figure 5: Level curves.

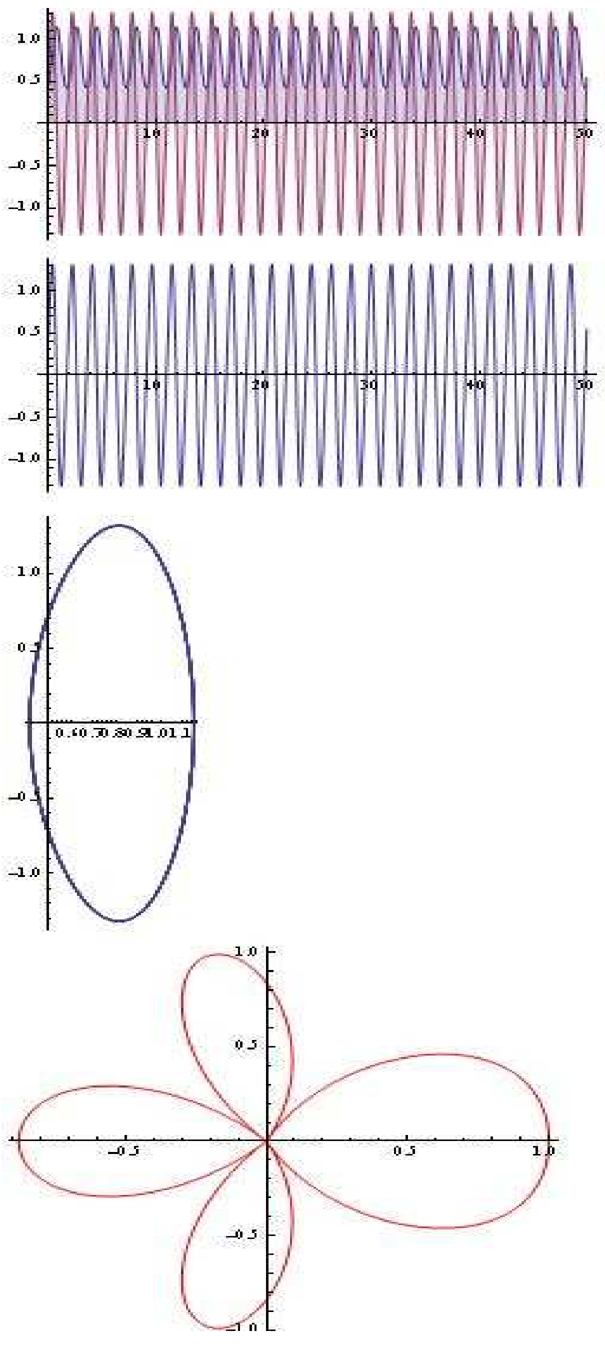

Figure 6: a) The solutions of differential system; b) y-component of the solution; c) Phase portrait; d) radial diagram (example II.).

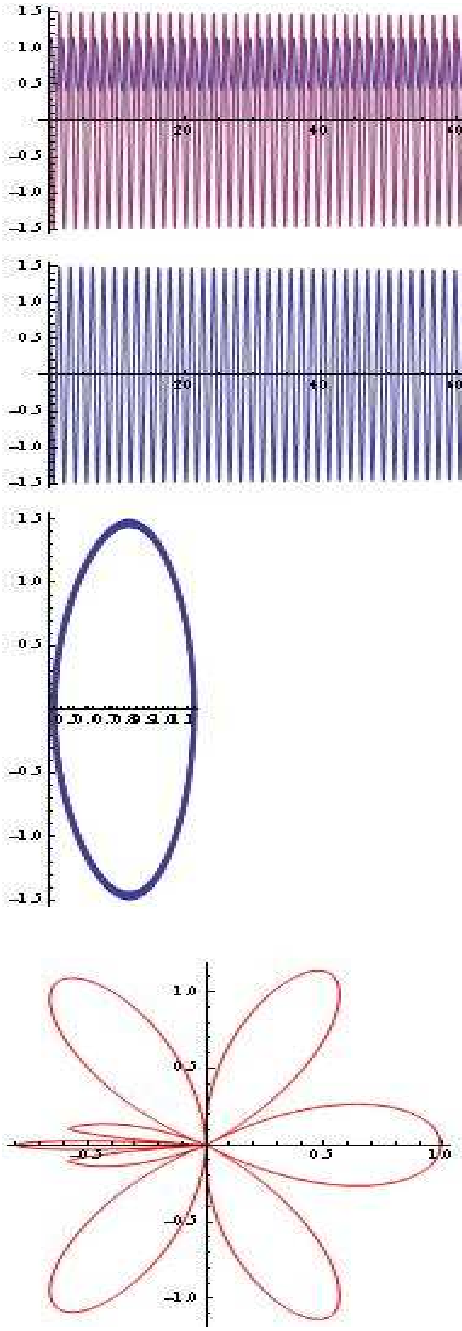

Figure 7: a) The solutions of differential system; b) y-component of the solution; c) Phase portrait; d) radial diagram (example III.).

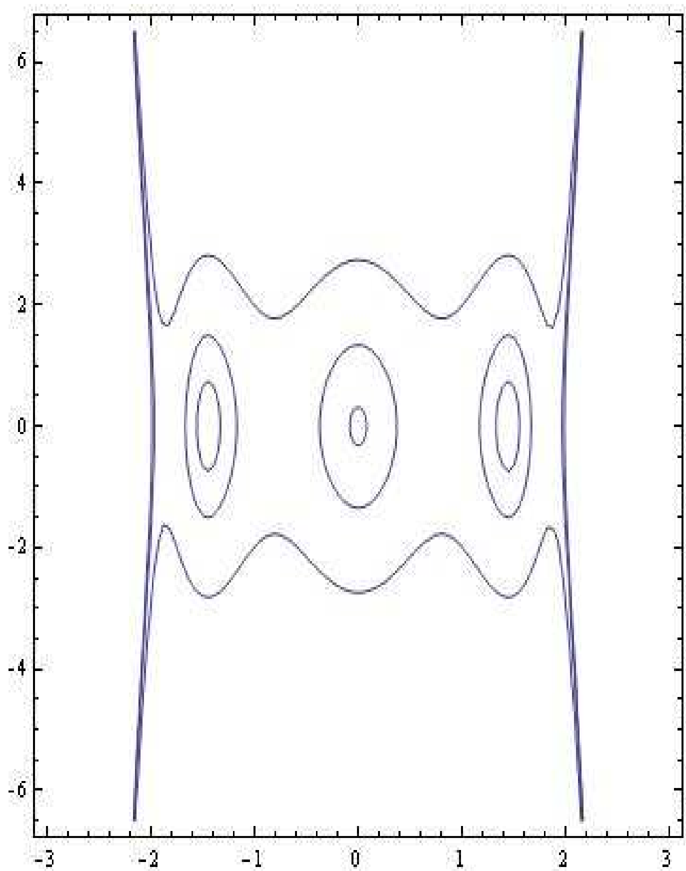

Figure 8: Phase portrait (the case III).

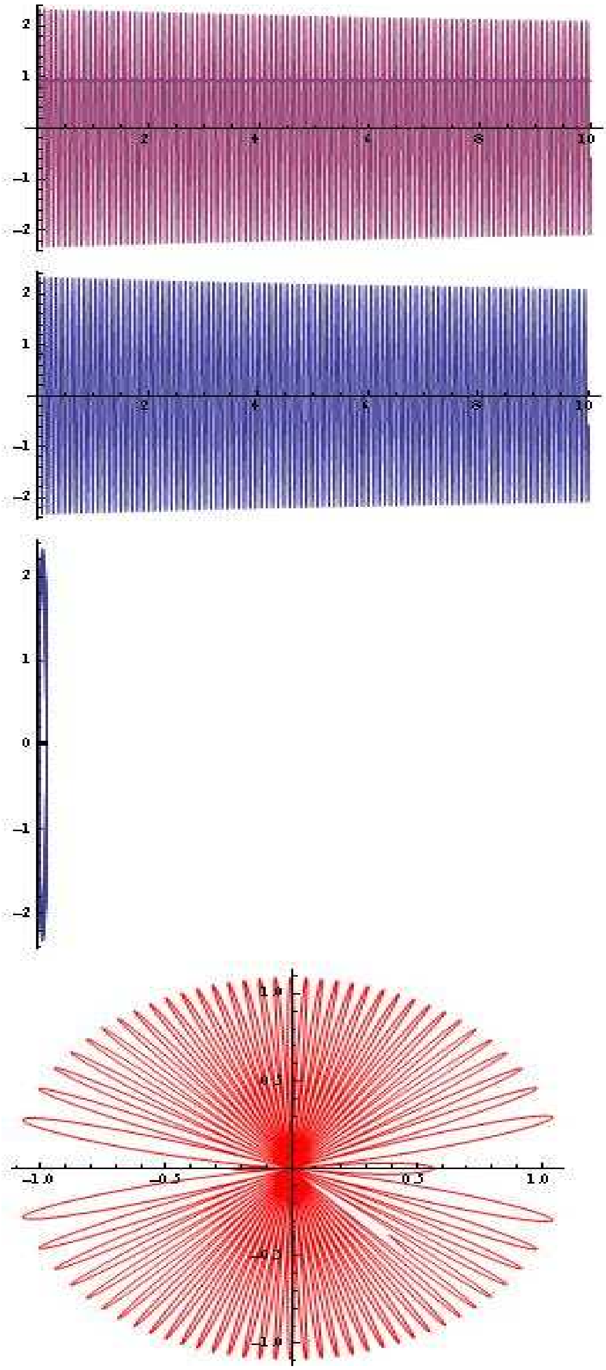

Figure 9: a) The solutions of differential system; b) y-component of the solution; c) Phase portrait; d) radial diagram (example IV.).

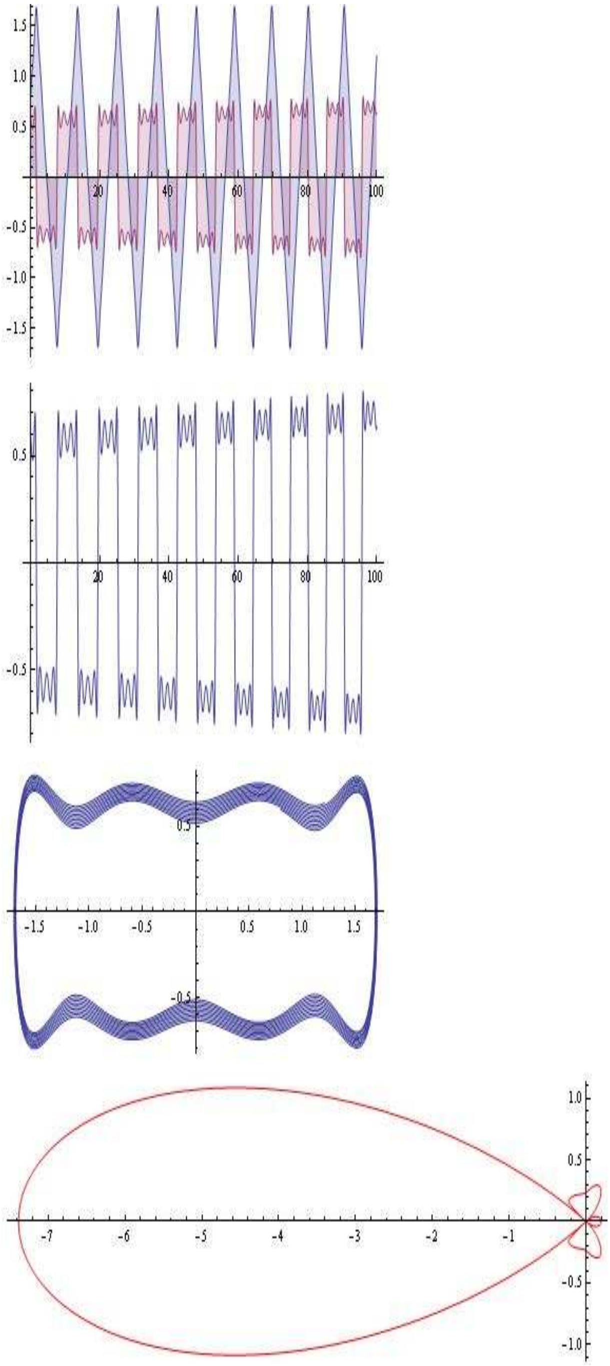

Figure 10: a) The solutions of differential system; b) y-component of the solution; c) Phase portrait; d) radial diagram (example V.).

III. The Dickson polynomials (of the 7-th kind) as "corrections" in the planar system (6)

For given $\epsilon = 0.0001,\ b = 1;\ c = 0.98$ and

$$a_1 = 7.9; a_2 = -1.5; a_3 = 23.8; a_4 = 8.6; a_5 = -21.4; a_6 = -1.9$$

the simulation on the system (6) for

$$Poly(x) = x^9 - 3x^7 - 9x^5 + 30x^3 - 15x$$

with $x_0 = 1.1;\ y_0 = 0.7$ is depicted on Fig. 7.

The Hamiltonian of the system (1.6) ($\epsilon = 0$) in the case III is

$$\frac{1}{2}y^2 + \frac{15}{2}x^2 - \frac{30}{4}x^4 + \frac{9}{6}x^6 + \frac{3}{8}x^8 - \frac{1}{10}x^{10}.$$

For the level curves see Fig. 8.

IV. The associated Hermite polynomials as "corrections" in the planar system (1.6)

For given $\epsilon = 0.0001,\ b = 0.9;\ c = 1.3$ and

$$a_1 = 7.9; a_2 = -1.5; a_3 = 23.8; a_4 = 8.6; a_5 = -21.4; a_6 = -1.9$$

the simulation on the system (6) for

$$\begin{aligned} Poly(x) \quad = \quad & 128x^7 - 192(7 + 2a)x^5 + 160(21 + 2a^2 + 14a)x^3 - \\ & -16(105 + 128a + 42a^2 + 4a^3)x \end{aligned}$$

for fixed $a = 0.5$ with $x_0 = 0.9;\ y_0 = 0.7$ is depicted on Fig. 9.

V. Arbitrary polynomial given by user as "corrections" in the planar system (6)

For given $\epsilon = 0.0001,\ b = 0.99;\ c = 0.79$ and

$$a_1 = 7.9; a_2 = -1.5; a_3 = 23.8; a_4 = 8.6; a_5 = -21.4; a_6 = -1.9$$

the simulation on the system (6) for

$$Poly(x) = 3.90022x^5 + 4.1292x^3 - 1.01442x$$

with $x_0 = 0.8;\ y_0 = 0.6$ is depicted on Fig. 10.

VI. Associated q–Lommel polynomials as "corrections" in the planar system (6)

Let

$$
\begin{aligned}
Poly(x) &= 232x^5(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)- \\
&-8x^3q^3(1+q-q^3-3q^4-2q^5+2q^7+3q^8+q^9-q^{11}-q^{12})+ \\
&+2xq^8(1+q+q^2-q^3-q^4-q^5)
\end{aligned}
$$

For given $\epsilon = 0.001$, $b = 0.7$; $c = 0.15$ and

$$a_1 = 0.05; a_2 = -0.9; a_3 = 1; a_4 = 0.85; a_5 = -1.2; a_6 = -1.5$$

the simulation on the system (6) for $q = 0.7$, i.e.

$$Poly(x) = 14.7431x^5 - 1.76975x^3 + 0.16589x$$

with $x_0 = 0.2$; $y_0 = 0.1$ is depicted on Fig. 11.

VII. Associated Lommel polynomials as "corrections" in the planar system (1.6)
Let

$$Poly(x) = 185794560x^9 - 41287680x^7 + 822528x^5 - 6448x^3 + 50x$$

(associated Lommel polynomial of degree 9).
The Hamiltonian of system (6) ($\epsilon = 0$) is

$$H(x,y) = \frac{y^2}{2} - 18579456x^{10} + 5160960x^8 - 137088x^6 + 1612x^4 - 25x^2.$$

The level curves $L_{h_i} = \{H(x,y) = h_i\}$ are depicted at Fig. 12.

1. For given $\epsilon = 0.0001$, $b = 0.5$; $c = 0.35$ and

$$a_1 = 0;\ a_2 = a_3 = a_4 = a_5 = a_6 = 1$$

the normalized antenna factor is depicted on Fig. 13.

2. For given $\epsilon = 0.0001$, $b = 0.05$; $c = 0.35$ and

$$a_1 = 0;\ a_2 = a_3 = a_4 = a_6 = 1;\ a_5 = -1$$

the normalized antenna factor is depicted on Fig. 14.

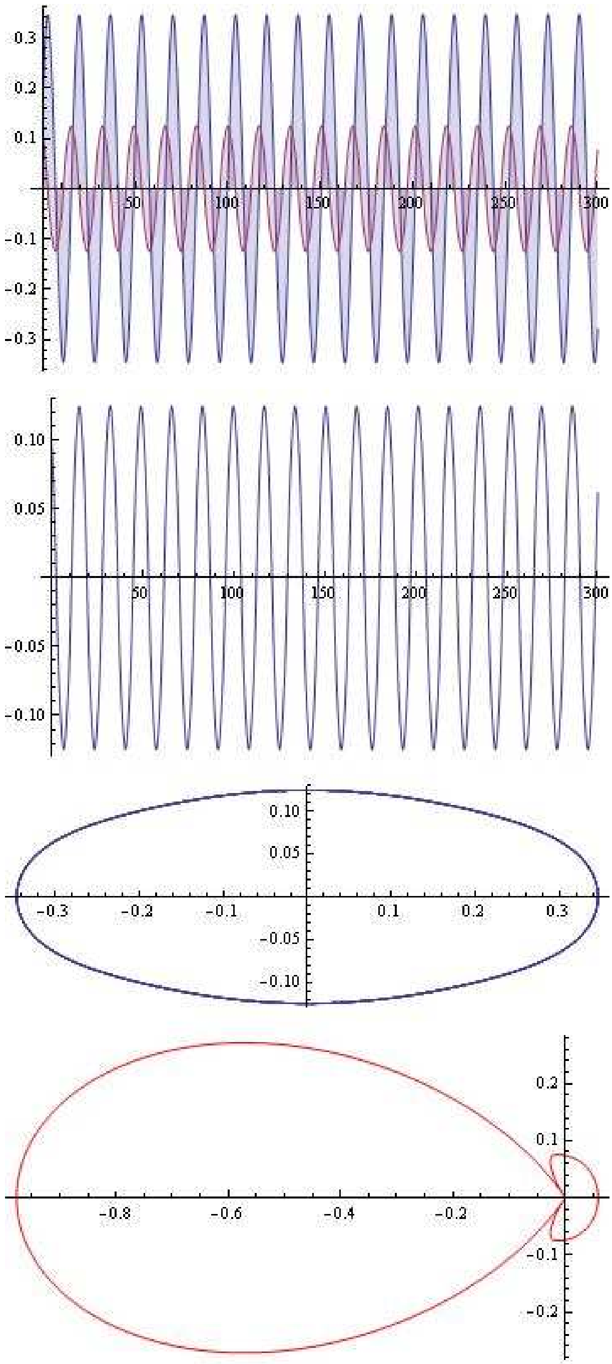

Figure 11: a) The solutions of differential system; b) y-component of the solution; c) Phase portrait; d) radial diagram (example VI.).

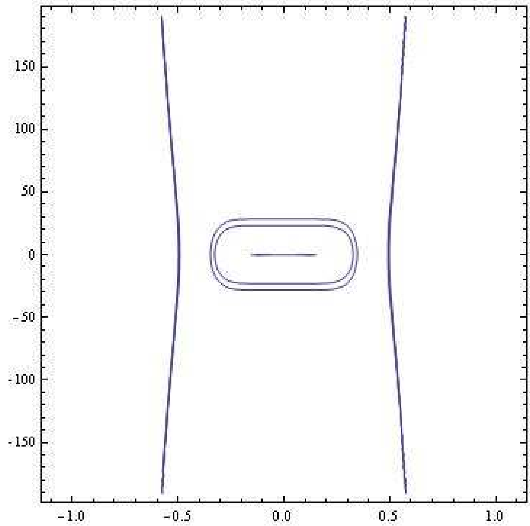

Figure 12: Level curves.

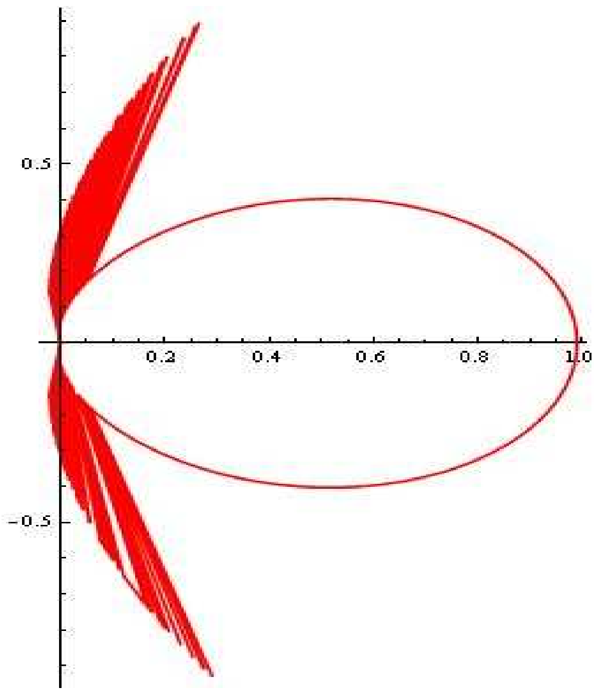

Figure 13: The normalized antenna factor (example VII.1).

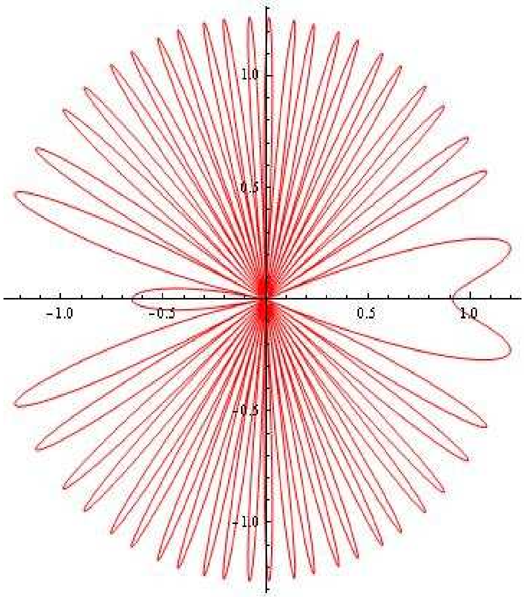

Figure 14: The normalized antenna factor (example VII.2).

3. CONCLUDING REMARKS

We will note that some specifics of the amplitudes of these polynomials as ”corrections” in the Rayleigh–Lienard differential system open up the possibility of modeling signals from the field of antenna–feeder techniques.

We use a algorithm for control and visualization of the ”antenna factor” (with a possibly user–set value of the lateral radiation).

In this sense, the research in this paper and the demonstrated modules contribute to the upgrade of the mentioned planned Web–based platform (for more details see [11]) for scientific computing.

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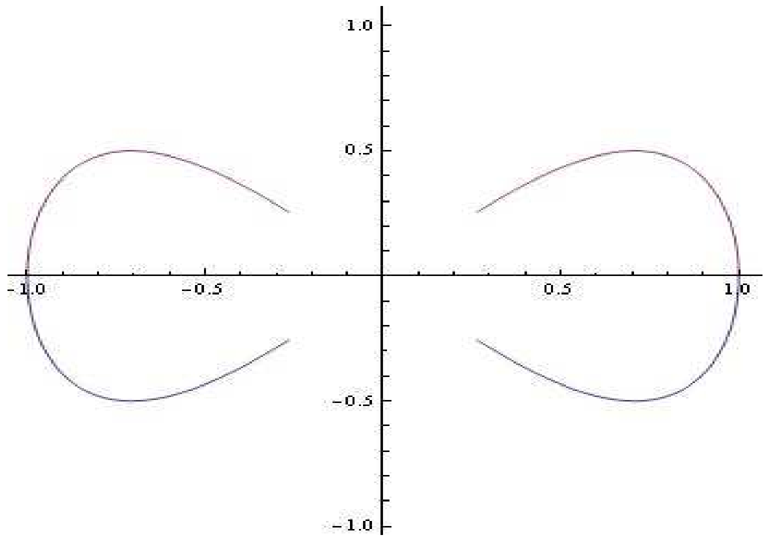

Figure 15: The homoclinic orbit.

4. APPENDIX 1.

The case **I.1**. The system of the type

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= x - 2x^3 + \epsilon(a_1 + a_2x^2 + a_3y^2 + a_4x^4 + a_5y^4 + a_6x^6)y \end{cases}$$

has the following Hamiltonian ($\epsilon = 0$)

$$H(x, y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{2}x^4.$$

The homoclinic orbit is given by (see Fig.15)

$$x_0(t) = \pm sech\, t$$
$$y_0(t) = \mp sech\, t\, tanh\, t.$$

The Melnikov homoclinic integral can be computed explicitly

$$M(a_1; a_2; a_3; a_4; a_5; a_6) =$$

$$= \int_{-\infty}^{\infty} y_0(t) \left(a_1 + a_2 x_0^2(t) + a_3 y_0^2(t) + a_4 x_0^4(t) + a_5 y_0^4(t) + a_6 x_0^6(t)\right) y_0(t)\, dt =$$

$$= \int_{-\infty}^{\infty} (sech\, t\, tanh\, t)^2 \left(a_1 + a_2 (sech\, t)^2 + a_3(-sech\, t\, tanh\, t)^2 + a_4 (sech\, t)^4 + \right.$$

$$\left. + a_5(-sech\, t\, tanh\, t)^4 + a_6 (sech\, t)^6\right)\, dt$$

From a numerical point of view, the task of finding a root of $M(a_1; a_2; a_3; a_4; a_5; a_6)$ is more interesting given that the parameters appearing in the proposed differential model are subject to a number of restrictions (of a physical and practical nature).

We have

$$M(a_1; a_2; a_3; a_4; a_5; a_6) = \frac{2}{3465} \left(1155 a_1 + 462 a_2 + 198 a_3 + 264 a_4 + 40 a_5 + 176 a_6\right).$$

Thus we prove the following

Proposition. If

$$1155 a_1 + 462 a_2 + 198 a_3 + 264 a_4 + 40 a_5 + 176 a_6 = 0$$

then the Melnikov function $M(a_1; a_2; a_3; a_4; a_5; a_6)$ has root.

Remark. The study of corresponding critical levels of $H(x, y) = \frac{1}{2} y^2 - P(x)$ is very complicated.

In this regard, we recommend the excellent study by Gavrilov and Iliev [13].

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