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INITIAL VALUE PROBLEMS FOR MIXED LINEAR DIFFERENCE-DIFFERENTIAL EQUATIONS IN NONRECTANGULAR DOMAINS

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Abstract: There is developed a special type of operational calculus that is applied to an initial value problem for mixed differential-difference equations in nonrectangular domains. Application of proposed type kind of Laplace transformation permits to reduce this problem to solution of a difference equation that gives a new way to investigate such problems.

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1. Statements of Initial Value Problems for Mixed Linear Difference-differential Equations in Nonrectangular Domains and for Ordinary Differential Equations With Parameter

Consider the linear mixed difference-differential equation

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$$
\begin{aligned}
&a_0(s)y^{(m)}(t,s)+a_1(s)y^{(m-1)}(y,s)\\
&+\cdots+a_m(s)y(t,s)+b_0(s)y^{(n)}(t,s-h)+b_1(s)y^{(n-1)}(t,s-h)\\
&+\cdots+b_n(s)y(t,s-h)=f(t,s),
\end{aligned}
\tag{1}
$$

where $y^{(j)}(t,s) := \frac{\partial^j y}{\partial t^j}, \quad h>0, S>0$. Let $\gamma(s)$ be a given differentiable function on $[-h,S]$ such that $\gamma(0)=0$ and $\gamma'(s)<0$. On the set $E_0 = E_0^1 \bigcup E_0^2, E_0^1 = \{(t,s)| -h \le s \le 0, \gamma(s+h) \le t < \infty], E_0^2 = \{(t,s)|\gamma(s+h) \le t \le \gamma(s)\}$ there is given an initial function $\varphi(t,s)$ and the solution of equation (1) must satisfy the initial value condition

$$
y(t,s)=\varphi(t,s) \quad \text{for } (t,s)\in E_0. \tag{2}
$$

In last decades there appeared some applications of mixed functional differential equations to problems of economics, biology and ecology. Equation (1) is a particular case of such equations. The general theory of mixed functional differential equations also was developed in the last time. A survey of the theory of mixed functional differential equations is published in [1]. Theory of mixed difference-differential equations is developed in [2]. The method of steps is applied to the solution of an initial value problem in a nonrectangular domain in [2]. But by application of this method to solution of equation (1) one must solve a new ordinary differential equation on every step. It is developed here a variant of operational calculus which permits to reduce the solution of a linear mixed difference-differential on a nonrectangular domain to solution of a linear difference equation. This gives a possibility to solve such equations and to investigate such problems in a more simple way.

Consider now the ordinary differential equation

$$
F(t,s,y(t,s),\dots,y^{(m)}(t,s))=0 \tag{3}
$$

with boundary conditions depending on a parameter s. (Without additional suppositions it is possible to consider the case of $s\in\mathbb{R}^n$). Assume that on $[0,S]$ there are given continuous functions $\gamma(s)$ and $\mu_j(s), j=0,\dots,m-1$. Solution of equation (3) must satisfy the initial conditions

$$
y(\gamma(s),s)=\mu_0(s),\dots,y^{(m-1)}(\gamma(s),s)=\mu_{m-1}(s). \tag{4}
$$

Such a problem arises if a process (propagation of a disease or of some information etc.) begins at a place that corresponding to the value s is described by equation (3) with boundary conditions (4). Then this process is considered at other places with different values of s and it begins at other initial moments depending on s. Then the general view is described by the solution of problem (3)–(4) in a nonrectangular region of the D type.

A particular case of equation (3) is the linear equation with parameter

$$a_0(s)y^{(m)}(t,s) + \cdots + a_m(s)y(t,s) = f(t,s) \tag{5}$$

and then the problem (5)–(4) is a particular case of the problem (1)–(2). Remark that the negativeness of the function $\gamma(s)$ is not substantial here and it can be obtained, if necessary, by a shift along the t-axis.

2. $\gamma-$Operational Calculus

Let $\gamma(s)$, be a given function continuously differentiable on $[-h, S]$ and such that $\gamma(0) = 0$ and $\gamma'(s) < 0$ on $[0, S]$.

Definition. $\gamma-$original is called such a function $y(t,s)$, which is defined on $D\{\gamma(s) \leq t \leq \infty\}$, piecewise continuous with respect to s and t, mapping $\mathbb{R} \to \mathbb{C}$, and satisfies the following conditions:

1. There is given on E_0 a piecewise continuous with respect to s and t function $\varphi(t,s)$ and it is supposed that

$$y(t,s) = \varphi(t,s) \quad \text{for} \quad (t,s) \in E_0, \quad y(t,s) = 0 \quad \text{for} \quad t < \gamma(s+h). \tag{6}$$

If there are made some additional suppositions concerning smoothness of y, then the same suppositions will be made concerning smoothness of the function φ.

2. *There exist such constants $M > 0$ and q that*

$$|y(t,s)| < Me^{qt} \quad \textit{for} \quad (t,s) \in D. \tag{7}$$

Definition. $\gamma-$transform corresponding to the $\gamma-$original $y(t,s)$ is called the function

$$Y(p,s) = \int_{\gamma(s)}^{\infty} y(t,s)e^{-pt}\, dt, \quad p \in \mathbb{C}. \tag{8}$$

We will use the following notation: $y(t,s) \Longleftrightarrow Y(p,s)$.

Theorem 1. *The transform $Y(p,s)$ of the original $y(t,s)$ exists in the half-plane $\Re p > q$ and is an analytical function of p. The integral (8) converges uniformly for any $q_1 > q$ in the half-plane $\Re p \geq q_1$.*

Proof. The proof is similar to the proof of corresponding theorem in [3], Ch.1, §5. □

Theorem 2. *Suppose that* $y_1(t,s) \Longleftrightarrow Y_1(p,s)$ *and* $y_2(t,s) \Longleftrightarrow Y_2(p,s)$, $C_1(s)$ *and* $C_2(s)$ *are arbitrary functions on* s. *Then*

$$C_1(s)y_1(t,s) + C_2(s)y_2(t,s) \Longleftrightarrow C_1(s)Y_1(p,s) + C_2(s)Y_2(p,s) \tag{9}$$

Proof. The prood follows from linearity of integral (8). □

Theorem 3. *Suppose that* $a > 1$ *and* $y(t,s)$ *is an original. Then*

$$\begin{aligned} y(at,s) &\Longleftrightarrow \frac{1}{a}Y(\frac{p}{a},s) + \frac{1}{a}\int_{a\gamma(s)}^{\gamma(s)} \varphi(t,s)e^{-\frac{p}{a}t}\,dt, \text{ if } \quad a\gamma(s) \geq \gamma(s+h), \\ y(at,s) &\Longleftrightarrow \frac{1}{a}Y(\frac{p}{a},s) + \frac{1}{a}\int_{\gamma(s+h)}^{\gamma(s)} \varphi(t,s)e^{-\frac{p}{a}t}\,dt, \text{ if } \quad a\gamma(s) < \gamma(s+h). \end{aligned} \tag{10}$$

Proof. Assume at first that $a\gamma(s) > \gamma(s+h)$. Then changing variables $at = \tau$ we get

$$\begin{aligned} y(at,s) &\Longleftrightarrow \int_{\gamma(s)}^{\infty} y(at,s)e^{-pt}\,dt = \int_{a\gamma(s)}^{\infty} y(\tau,s)e^{-\frac{p}{a}\tau}\frac{1}{a}\,d\tau \\ &= \frac{1}{a}\int_{a\gamma(s)}^{\gamma(s)} \varphi(\tau,s)e^{-\frac{p}{a}\tau}\,d\tau + \frac{1}{a}\int_{\gamma(s)}^{\infty} y(\tau,s)e^{-\frac{p}{a}\tau}\,d\tau \\ &= \frac{1}{a}Y(\frac{p}{a},s) + \frac{1}{a}\int_{a\gamma(s)}^{\gamma(s)} \varphi(\tau,s)e^{-\frac{p}{a}\tau}\,d\tau. \end{aligned}$$

Assume now that $a\gamma(s) \leq \gamma(s)$. Then it follows from (6) that

$$\int_{a\gamma(s)}^{\gamma(s)} \varphi(\tau,s)e^{-\frac{p}{a}\tau}\,d\tau = \int_{\gamma(s+h)}^{\gamma(s)} \varphi(\tau.s)e^{-\frac{p}{a}\tau}\,d\tau.$$ □

Theorem 4. *If* $y(t,s)$ *and* $y'(t,s)$ *are originals, then*

$$y'(t,s) \Longleftrightarrow pY(p,s) - y(\gamma(s),s)e^{-p\gamma(s)}\,. \tag{11}$$

Proof.

$$y'(t,s) \Longleftrightarrow \int_{\gamma(s)}^{\infty} y'(t,s)e^{-pt}\,dt = \int_{\gamma(s)}^{\infty} e^{-pt}dy(t,s) =$$

$$y(t,s)e^{-pt}|_{\gamma(s)}^{\infty} + p\int_{\gamma(s)}^{\infty} y(t,s)e^{-pt}\,dt = pY(p,s) - y(\gamma(s),s)e^{-p\gamma(s)}$$

for $|e^{-pt}| = e^{-\Re pt}$ and $\Re p > q$. □.

Theorem 5. *If $y(t,s),\dots,y^{(n)}(t,s)$ are originals, then*

$$y^{(n)}(t,s) \Longleftrightarrow p^nY(p,s) - [p^{n-1}y(\gamma(s),s) + p^{n-2}y'(\gamma(s),s) + \dots + y^{(n-1)}(\gamma(s),s)]e^{-p\gamma(s)}\,. \quad (12)$$

Proof. Assertion of the theorem we get applying the method of mathematical induction to (11). □

The proofs of following theorems 6 – 10 are analogous to the proofs of corresponding theorems of standard operational calculus. (See [4], Ch. 1, §5.)

Theorem 6. *Suppose that $y(t,s) \Longleftrightarrow Y(p,s)$ and $z \in \mathbb{C}$. Then*

$$e^{zt}y(t,s) \Longleftrightarrow Y(p-z,s). \quad (13)$$

Theorem 7. *Suppose that $y(t,s) \Longleftrightarrow Y(p,s)$. Then*

$$-ty(t,s) \Longleftrightarrow \frac{d}{dp}Y(p,s). \quad (14)$$

Theorem 8. *Suppose that $y(t,s) \Longleftrightarrow Y(p,s)$. Then*

$$\int_{\gamma(s)}^{t} y(t,s)\,dt \Longleftrightarrow \frac{Y(p,s)}{p}\,. \quad (15)$$

Theorem 9. *Suppose that $y(t,s) \Longleftrightarrow Y(p,s)$ and $a > q$. Then*

$$y(t,s) = \frac{1}{2\pi}\int_{a-i\infty}^{a+i\infty} Y(p,s)e^{pt}\,dp. \quad (16)$$

Theorem 10. *Suppose that $y(t,s) \Longleftrightarrow Y(p,s)$ and for $Y(p,s)$ are true suppositions of the Jordan lemma. (See [3], Ch. 1, §4). Then*

$$y(t,s) = \sum_k \operatorname*{res}_{p_k} Y(p,s)e^{pt}, \tag{17}$$

where the sum is spread over all singularities p_k of the function $Y(p,s)$.

Theorem 11. *If $f(t,s)$ is an original and $f(t,s) \Longleftrightarrow e^{-p\gamma(s)}\Phi(p,s)$, where $\Phi(p,s)$ is a meromorphic function, bounded at infinity, then*

$$f(t,s) = \sum_k \operatorname*{res}_{p_k} \Phi(p,s)e^{p(t-\gamma(s))}, \tag{18}$$

where the sum is spread over all singularities of $\Phi(p,s)$.

Proof. Suppose that $a > q$. It follows from Theorem 6 that

$$e^{-at}f(t,s) \Longleftrightarrow e^{-(p+a)\gamma(s)}\Phi(p+a,s),$$

and in consequence of

$$|e^{-at}f(t,s)| < Me^{-(a-q)t},$$

for this original it can be applied Theorem 9 in the form

$$e^{-at}f(t,s) = \frac{1}{2\pi i}\int_{-r-i\infty}^{-r+i\infty} e^{-(p+a)\gamma(s)}\Phi(p+a,s)e^{pt}\,dp, \tag{19}$$

where r is a number satisfying the inequality $0 < r < a-q$.

Let C_R be an arc from the Jordan lemma

$$C_R = \{p|p = Re^{i\varphi}, \frac{\pi}{2}+\varepsilon < \varphi < \frac{3\pi}{2}-\varepsilon\},$$

where $\varepsilon = \arcsin\frac{r}{R} > 0$ for the arc C_R is bounded by the interval of the line $\Re p = -r < 0$. Denote $\alpha = -\cos(\frac{\pi}{2}+\varepsilon)$. Everywhere on the arc C_R it is true the inequality $\cos\varphi \le -\alpha < 0$. From conditions of the theorem it follows existence of such $M_1 > 0$ that

$$|e^{-a\gamma(s)}\Phi(p+a,s))| < M_1 \quad \text{at} \quad |p| > M_1.$$

On the arc C_R at $|p| > M_1$ it is true the estimate

$$\beta(R) = |e^{-(Re^{i\varphi}+a)\gamma(s)}\Phi(p+a,s)| < M_1e^{-R\cos\varphi\gamma(s)} \le M_1e^{R\alpha\gamma(s)}$$

and from $\alpha > 0$ and $\gamma(s) < 0$ it follows that $\beta(R) \to 0$ uniformly with respect to φ at $R \to \infty$. Therefore all conditions of the Jordan lemma are fulfilled and it follows from Theorem 10 that

$$e^{-at} f(t,s) = \sum_k \operatorname*{res}_{p_k} e^{(p+a)(t-\gamma(s))-at} \Phi(p+a,s)$$
$$= e^{-at} \sum_k \operatorname*{res}_{p_k} e^{(p+a)(t-\gamma(s))} \Phi(p+a,s). \quad (20)$$

In consequence of that we have

$$f(t,s) = \sum_k \operatorname*{res}_{p_k} e^{(p+a)(t-\gamma(s))} \Phi(p+a,s) = \sum_k \operatorname*{res}_{p_k} e^{p(t-\gamma(s))} \Phi(p,s), \quad (21)$$

where the sum is spread over all singularities of the function $\Phi(p,s)$. □

The Correspondence Formulae between $\gamma-$originals and $\gamma-$transforms

The correspondence formulae between the γ-originals and their γ-transforms can be derived in the usual way with help of previous theorems, but useful for this purpose is the following theorem.

Theorem 12. *If $y(t)$ is a Laplace original, then*

$$y(t-\gamma(s)) \Longleftrightarrow e^{-p\gamma(s)} Y_L(p).$$

Proof.

$$y(t-\gamma(s)) \Longleftrightarrow \int_{\gamma(s)}^{\infty} y(t-\gamma(s)) e^{-pt}\, dt =$$
$$\int_0^{\infty} y(\tau) e^{-p(t+\gamma(s)}\, d\tau = e^{-p\gamma(s)} Y_L(p).$$ □

Denote $\eta(s) = \begin{cases} 1 & , \text{ if } t \geq \gamma(s) \\ 0 & , \text{ if } t < \gamma(s). \end{cases}$

Using Theorem 12 we get the following formulae:

I. $\eta(t,s) \Longleftrightarrow \frac{e^{-p\gamma(s)}}{p}$.

II. $t-\gamma(s) \Longleftrightarrow \frac{e^{-p\gamma(s)}}{p^2}$.

III $(t-\gamma(s))^n \Longleftrightarrow \frac{n!e^{-p\gamma(s)}}{p^{n+1}}$.

IV. $e^{zt}(t-\gamma(s))^n \Longleftrightarrow \frac{n!e^{-(p-z)\gamma(s)}}{(p-z)^{(n+1)}}$.

V. $\cos\omega(t-\gamma(s)) \Longleftrightarrow \frac{pe^{-p\gamma(s)}}{p^2+\omega^2}$.

VI. $\sin\omega(t-\gamma(s)) \Longleftrightarrow \frac{\omega e^{-p\gamma(s)}}{p^2+\omega^2}$.

VII. $(t-\gamma(s))\cos\omega(t-\gamma) \Longleftrightarrow \frac{(p^2-\omega^2)e^{-p\gamma(s)}}{(p^2+\omega^2)^2}$.

VIII. $(t-\gamma(s))\sin\omega(t-\gamma) \Longleftrightarrow \frac{2p\omega e^{-p\gamma(s)}}{(p^2+\omega^2)^2}$.

IX. $\mathrm{ch}\omega(t-\gamma(s)) \Longleftrightarrow \frac{pe^{-p\gamma(s)}}{p^2-\omega^2}$.

X. $\mathrm{sh}\omega(t-\gamma(s)) \Longleftrightarrow \frac{\omega e^{-p\gamma(s)}}{p^2-\omega^2}$.

XI. $(t-\gamma(s))\mathrm{ch}\omega(t-\gamma(s)) \Longleftrightarrow \frac{(p^2+\omega^2)e^{-p\gamma(s)}}{(p^2-\omega^2)^2}$.

XII. $(t-\gamma(s))\mathrm{sh}\omega(t-\gamma(s)) \Longleftrightarrow \frac{2p\omega e^{-p\gamma(s)}}{(p^2-\omega^2)^2}$.

This table can be easily continued with help of standard table of correspondence of Laplace transforms and originals and Theorem 12.

3. Solution of the Initial Value Problem for Mixed Linear Difference-differential Equations in a Nonrectangular Domain by Using $\gamma-$Operational Calculus

Let us prove some lemmas useful for application of γ-transformations to solution of the equation (1).

Lemma 1. *Let $y(t,s)$ be an original and $h>0$. Then*

$$\int_{\gamma(s)}^{\infty} y(t,s-h)e^{-pt}\,dt = Y(t,s-h) + \int_{\gamma(s)}^{\gamma(s-h)} \varphi(t,s-h)e^{-pt}\,dt. \tag{22}$$

Proof.

$$\int_{\gamma(s)}^{\infty} y(t,s-h)e^{-pt}\,dt = \int_{\gamma(s-h)}^{\infty} y(t,s-h)e^{-pt}\,dt +$$

$$\int_{\gamma(s)}^{\gamma(s-h)} \varphi(t,s-h)e^{-pt}\,dt = Y(p,s-h) + \int_{\gamma(s)}^{\gamma(s-h)} \varphi(t,s-h)e^{-pt}\,dt. \qquad \square$$

Lemma 2. *Let $y(t,s)$ and $y'(t,s)$ be originals and $h>0$. Then*

$$\int_{\gamma(s)}^{\infty} y'(t,s-h)e^{-pt}\,dt = p[Y(p,s-h) + \int_{\gamma(s)}^{\gamma(s-h)} \varphi(t,s-h)e^{-pt}\,dt] - y(\gamma(s),s-h)e^{-p\gamma(s)}. \qquad (23)$$

Proof.

$$\int_{\gamma(s)}^{\infty} y'(t,s-h)e^{-pt}\,dt = \int_{\gamma(s)}^{\infty} e^{-pt}dy(t,s-h) =$$

$$y(t,s-h)e^{-pt}\Big|_{\gamma(s)}^{\infty} + p\int_{\gamma(s)}^{\infty} y(t,s-h)e^{-pt}\,dt =$$

$$p[Y(p,s-h) + \int_{\gamma(s)}^{\gamma(s-h)} \varphi(t,s-h)e^{-pt}\,dt] - y(\gamma(s),s-h)e^{-p\gamma(s)}. \qquad \square$$

Lemma 3. *Let $y(t,s),\dots,y^{(n)}(t,s)$ be originals and $h>0$. Then*

$$\begin{aligned}\int_{\gamma(s)}^{\infty} y^{(n)}(t,s-h)e^{-pt}\,dt &= p^nY(p,s-h) + \\ &\int_{\gamma(s)}^{\gamma(s-h)} [p^n\varphi(t,s-h) + p^{n-1}\varphi'(t,s-h) + \dots \\ &+ p\varphi^{(n-1)}(\gamma(s),s-h)]e^{-pt}\,dt - \\ &[p^{n-1}y(\gamma(s),s-h) + p^{n-2}y'(\gamma(s),s-h) + \dots \\ &+ y^{(n-1)}(\gamma(s),s-h)]e^{-p\gamma(s)}. \end{aligned} \qquad (24)$$

Proof. By applying the method of mathematical induction to (23) we get the assertion of the lemma. $\square$

Consider again the problem (1)–(2). Define the sets

$$E_j = \{(t,s)|\gamma_j(s) < t < \gamma_{j+1}(s)\},\ \gamma_j(s) = \gamma(s-jh).$$

Solution of this problem can be received by the method of steps (see [2]). But on every step one is obliged to solve a new ordinary differential equation of the order m and to take in account the consistency of initial value conditions for this equation.

Apply now the $\gamma-$transformation to solve this problem. Multiply equation (1) by e^{-pt} and integrate it from $\gamma(s)$ to ∞. Suppose that $y(t,s) \Longleftrightarrow Y(p,s)$ and $f(t,s) \Longleftrightarrow F(p,s)$. Using now Theorem 5 and Lemma 3 we get for $Y(p,s)$ the difference equation

$$\begin{aligned} Y(p,s)(a_0p^m + a_1p^{m-1} + \cdots + a_m) + Y(p,s-1) \\ \times (b_0p^n + b_1p^{m-1} + \cdots + b_n) = F(p,s) + \Gamma(p,s) - I(p,s), \end{aligned} \tag{25}$$

where

$$\begin{aligned} &\Gamma(p,s) = [y(\gamma(s),s)(a_0p^{m-1} + a_1p^{m-2} + \cdots + a_{m-1}) + \\ &y'(\gamma(s),s)(a_0p^{m-2} + a_1p^{m-3} + \cdots + a_{m-2}) + \cdots + y^{(m-1)}(\gamma(s),s)a_0 + \\ &y(\gamma(s),s-h)(b_0p^{n-1} + b_1p^{n-2} + \cdots + b_{n-1}) + \\ &y'(\gamma(s),s-h)(b_0p^{n-2} + b_1p^{n-3} + \cdots + b_{n-2}) \\ &+ \cdots + y^{(n-1)}(\gamma(s),s-h)b_0]e^{-p\gamma(s)} \end{aligned} \tag{26}$$

and

$$\begin{aligned} I(p,s) = \int_{\gamma(s)}^{\gamma(s-h)} [\varphi(t,s-h)(b_0p^n + b_1p^{n-1} + \cdots + b_n) + \varphi'(t,s-h)(b_0p^{n-1} \\ + b_1p^{n-2} + \cdots + b_{n-1}) + \cdots + \varphi^{(n-1)}(t,s-h)pb_0]e^{-pt}\,dt. \end{aligned} \tag{27}$$

When we solve equation (1) using method of steps, it is necessary on every step to solve an ordinary differential equation with a parameter and with constant coefficients. As it is known, the solution can be represented as a sum of exponents multiplied by polynomials and permits an exponential estimate of growth as $t \to \infty$. It follows from this that every solution of equation (1) in case when $f(t,s) \equiv 0$ is a $\gamma-$original. It can be derived from (25)–(27) that the $\gamma-$transform of solution of equation (1) can be represented in the form $e^{-p\gamma(s)}\Phi(p,s)$, where $\Phi(t,s)$ is a meromorphic function bounded in the vicinity of infinity. That means that all conditions of Theorem 11 are fulfilled. Therefore, besides the table of correspondence between originals and their transforms, for calculation of originals — solutions of equation (1) — it can be used Theorem 11 and the theory of residues.

Example. Solve equation $y'(t,s) + 2y(t,s-1) = 1$ in the domain $D = \{(t,s)|t > -s, 0 \le s \le 2\}$ with initial value condition $\varphi(t,s) = 1$ at $(t,s) \in E_0$, $E_0 = \{(t,s)| -s-1 < t < -s\} \bigcup \{(t,s)|0 \le s \le 2,\ -s-1 \le t < \infty\}$. Going to transforms we get $pY(p,s) - e^{ps} + 2Y(p,s-1) = e^{ps}/p$. At $s \in (0,1)$ we receive from the initial value condition $Y(p,s-1) = \int_{-s}^{\infty} e^{-pt}\,dt = e^{ps}/p$, for $\Re p > 0$. From this we get $Y(p,s) = -e^{ps}/p^2 + e^{ps}/p$. It follows from **I** and **II**, that $y(t,s) = -(t+s)+1$. Let now $s \in (1,2)$. Then $Y(p,s-1) = -\frac{e^{p(s-1)}}{p}$ $Y(p,s) = 2\frac{e^{p(s-1)}}{p^3} + \frac{e^{ps}}{p^2} + \frac{e^{ps}}{p}$. It follows from **III** and **IV,** that $y(t,s) = (t+s-1)^2 + (t+s) + 1$.

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References

[1] G.A. Kamenskii, A review of the theory of mixed functional differential equations, *Problems of Nonlinear Analysis in Engineering Systems*, **2**, No. 8 (1998), 1-16.

[2] E.P. Ivanova, G.A. Kamenskii, *Initial Value Problems for Difference-differential Equations*, Moscow, MAI Press (1995), In Russian.

[3] V.A. Ditkin, A.P. Prudnikov, *Operational Calculus*, Moscow, "Vysshaja Shkola" (1966), In Russian.