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APPROXIMATION SOLVABILITY OF NONLINEAR VARIATIONAL INEQUALITIES BASED ON GENERAL AUXILIARY PROBLEM PRINCIPLE

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Abstract: First a general class of auxiliary problem principle is introduced and then it is applied to approximation-solvability of the following class of nonlinear variational inequality problems (NVIP):

Find an element $x^* \in K$ such that

$$\langle T(x^*), x - x^* \rangle + f(x) - f(x^*) \geq 0 , \quad \text{for all } x \in K,$$

where $T : K \longrightarrow E^*$ is a mapping from a nonempty closed convex subset K of a reflexive Banach space E into its dual E^*, and $f : K \longrightarrow R$ is a continuous convex functional on K. This general class of auxiliary problem principle is described as follows: for a given iterate $x^k \in K$ and for a constant $\rho > 0$, determine x^{k+1} such that

$$\langle \rho T(x^k) + h'(x^{k+1}) - h'(x^k), x - x^{k+1} \rangle + \rho[f(x) - f(x^{k+1})] \geq 0 , \quad \text{for all } x \in K , \text{ and for } k \geq 0,$$

where $h : K \longrightarrow R$ is m-times continuously Frechet-differentiable ($m \geq 2$) on K.

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1. Introduction

Argyros and Verma [1] used inexact Newton-like iterative procedures to approximate a solution of a class of nonlinear equations in a Banach space setting, because solving a nonlinear equation using Newton-like iterates at each stage is quite expensive one in general. On the top of that, some auxiliary results of this work turn out to be quite useful to further generalizations to the existing versions of the auxiliary problem principle initiated by Cohen [3].

In this paper, based on [1, Remarks 1 & 9], we intend first to introduce a general version of the auxiliary problem principle [3] and then apply it to the approximation-solvability of a class of nonlinear variational inequalities. The obtained results complement the earlier works of Cohen [3] and Verma [18] on the approximation-solvability of nonlinear variational inequalities in different space settings.

Let E be a reflexive Banach space with the duality pairing $\langle\cdot,\cdot\rangle$ and norm $\|.\|$. Let $T : K \longrightarrow E^*$ be any mapping from K, a closed convex subset of E, to E^* (dual of E). Let $f : K \longrightarrow R$ be a continuous convex function on K. We consider a class of nonlinear variational inequality problems (abbreviated as NVIP): find an element $x^* \in K$ such that

$$\langle T(x^*), x - x^*\rangle + f(x) - f(x^*) \geq 0 \quad \text{for all } \ x \in K. \tag{1.1}$$

Now we need to recall the following auxiliary result, most commonly used in the context of the approximation-solvability of the nonlinear variational inequality problems based on the iterative procedures.

Lemma 1.1. *An element $u \in K$ is a solution of the NVIP (1.1) if*

$$\langle T(u), x - u\rangle + f(x) - f(u) \geq 0 \quad \textit{for all } \ x \in K.$$

T is called r-strongly monotone if for each $x, y \in E$, we have

$$\langle T(x) - T(y), x - y\rangle \geq r\|x - y\|^2 \quad \text{for a constant } r > 0.$$

This implies that

$$\|T(x) - T(y)\| \geq r\|x - y\|,$$

that is, T is r-expanding, and when $r = 1$, it is expanding. The mapping T is called β-Lipschitz continuous (or β-Lipschitzian) if there exists a constant $\beta \geq 0$ such that

$$\|T(x) - T(y)\| \leq \beta\|x - y\| \quad \text{for all } \ x, y \in H.$$

2. General Auxiliary Problem Principle

Before we discuss the approximation-solvability of the NVIP (1.1), we introduce a general version of the existing auxiliary problem principle (APP) initiated by Cohen [3] and applied and studied by others. This general class of auxiliary problem principle (GAPP) is described as follows:

Gapp 2.1. *For a given iterate x^k, determine an x^{k+1} such that (for $k \geq 0$):*

$$\langle \rho T(x^k) + h'(x^{k+1}) - h'(x^k), x - x^{k+1} \rangle + \rho[f(x) - f(x^{k+1})] \geq 0 \quad \text{for all } x \in K, \quad (2.1)$$

where $h : E \longrightarrow R$ is m-times continuously Frechet-differentiable ($m \geq 2$) from a Banach space E into its dual E^* and ρ is a positive constant.

When $m = 2$, the GAPP(2.1) reduces to:

Gapp 2.2. *For a given iterate x^k, determine an x^{k+1} such that (for $k \geq 0$):*

$$\langle \rho T(x^k) + h'(x^{k+1}) - h'(x^k), x - x^{k+1} \rangle + \rho[f(x) - f(x^{k+1})] \geq 0 \quad \text{for all } x \in K, \quad (2.2)$$

where $h : E \longrightarrow R$ is 2-times continuously Frechet-differentiable ($m = 2$) from a Banach space E into its dual E^* and ρ a positive constant.

Next, we recall some auxiliary results crucial to the approximation-solvability of the NVIP (1.1) at hand.

Lemma 2.1. *Let E_1 and E_2 be two Banach spaces and $h : E_1 \longrightarrow E_2$ be a nonlinear mapping such that h is m-times continuously Frechet-differentiable ($m \geq 2$ an integer) on E_1. Suppose that there exist an $x^* \in E_1$ and nonnegative numbers α_i ($i = 2, 3, \ldots, m$) such that*

$$\langle h^{(m)}(x) - h^{(m)}(x^*), (x - x^*)^m \rangle \geq 0$$

and

$$\|h^{(i)}(x^*)\| \geq \alpha_i \,.$$

Then we have:

$$h(x) - h(x^*) - \langle h'(x^*), x - x^* \rangle \geq \left[\alpha_2/2! + (\alpha_3/3!)\|x - x^*\| + \cdots + (\alpha_m/m!)\|x - x^*\|^{m-2} \right] \|x - x^*\|^2.$$

Proof. The proof follows using the following identity [1]:

$$
\begin{aligned}
& h(x) - h(x^*) - \langle h'(x^*), x - x^* \rangle \\
& = \int_0^1 \langle [h'(x^* + \theta_1(x - x^*)) - h'(x^*)], x - x^* \rangle \, d\theta_1 \\
& = \int_0^1 \int_0^1 \langle [h''(x^* + \theta_2\theta_1(x - x^*)) - h''(x^*)], \theta_1(x - x^*)^2 \rangle d\theta_1 d\theta_2 \\
& \qquad + \int_0^1 \int_0^1 \langle [h''(x^*), \theta_1(x - x^*)^2 \rangle d\theta_1 d\theta_2 \\
& = \cdots = \int_0^1 \ldots \int_0^1 \langle \{h(m)[x^* + \theta_m\theta_{m-1} \ldots \theta_1(x - x^*)] - h^{(m)}(x^*)\}, \\
& \qquad \theta_{m-1}\theta_{m-2}^2 \ldots \theta_2^{m-2}\theta_1^{m-1}(x - x^*)^m \rangle d\theta_m d\theta_{m-1} \ldots d\theta_2 d\theta_1 + \ldots \\
& + \int_0^1 \int_0^1 \langle [h''(x^*), \theta_1(x - x^*)2 \rangle d\theta_1 d\theta_2 \,.
\end{aligned}
$$

For m = 2 in Lemma 2.1, we arrive at

Lemma 2.2. *Let E_1 and E_2 be two Banach spaces and $h : E_1 \longrightarrow E_2$ be a nonlinear mapping such that h is 2-times continuously Frechet-differentiable on E_1. Suppose that there exist an $x^* \in E_1$ and a nonnegative number α_2 such that*

$$\langle h''(x) - h''(x^*), (x - x^*)^2 \rangle \geq 0$$

and

$$\|h''(x^*)\| \geq \alpha_2 \,.$$

Then we have:

$$h(x) - h(x^*) - \langle h'(x^*), x - x^* \rangle \geq [\alpha 2/2!] \|x - x^*\|^2 \,.$$

We are just about ready to present, based on the GAPP (2.1), the approximation-solvability of the NVIP (1.1).

Theorem 2.1. *Let E be a reflexive Banach space and $T : K \longrightarrow E^*$ an r-strongly monotone mapping from a nonempty closed convex subset K of E into E^*, the dual of E. Let $f : K \longrightarrow R$ be proper, convex and lower semicontinuous (lsc) on K and $h : K \longrightarrow R$ be m-times continuously Frechet-differentiable ($m \geq 2$ an integer) on K. Suppose that there exist an $x \in K$ and nonnegative numbers α_i ($i = 2, 3, \dots, m$) such that*

$$\langle h^{(m)}(x) - h^{(m)}(x'), (x - x')^m \rangle \geq 0 \tag{2.3}$$

and

$$\|h^{(i)}(x')\| \geq \alpha_i \,. \tag{2.4}$$

Then there exists a unique solution x^{k+1} to the GAPP (2.1). If, in addition, $x^* \in K$ is any fixed solution of the NVIP (1.1), T is β-Lipschitz continuous and

$$0 < \rho < (4r/\beta^2)[\alpha_2/2! + (\alpha_3/3!)\|x^{k+1} - x^k\| + \dots + (\alpha_m/m!)\|x^{k+1} - x^k\|^{m-2}] \,,$$

then $\{x^k\}$ converges to x^*.

Proof. To show the sequence $\{x^k\}$ converges to x^*, a solution of the NVIP (1.1), we need to compute the estimates. Let us define a function Λ^* by

$$\Lambda^*(x) := h(x^*) - h(x) - \langle h'(x), x^* - x \rangle \geq [\alpha_2/2! + (\alpha_3/3!)\|x^* - x\| + \dots + (\alpha_m/m!)\|x^* - x\|^{m-2}]\|x^* - x\|^2 \quad \text{for } x \in K \,,$$

where x^* is any fixed solution of the NVIP (1.1). It follows that

$$\Lambda^*(x^{k+1}) = h(x^*) - h(x^{k+1}) - \langle h'(x^{k+1}), x^* - x^{k+1} \rangle \,.$$

Now we can write

$$\begin{aligned} \Lambda^*(x^k) - \Lambda^*(x^{k+1}) &= h(x^{k+1}) - h(x^k) - \langle h'(x^k), x^{k+1} - x^k \rangle \\ &\quad + \langle h'(x^{k+1}) - h'(x^k), x^* - x^{k+1} \rangle \\ &\geq [\alpha_2/2! + (\alpha_3/3!)\|x^{k+1} - x^k\| + \dots \\ &\quad + (\alpha_m/m!)\|x^{k+1} - x^k\|^{m-2}]\|x^{k+1} - x^k\|^2 \\ &\quad + \langle h'(x^{k+1}) - h'(x^k), x^* - x^{k+1} \rangle \\ &\geq [\alpha_2/2! + (\alpha_3/3!)\|x^{k+1} - x^k\| + \dots \\ &\quad + (\alpha_m/m!)\|x^{k+1} - x^k\|^{m-2}]\|x^{k+1} - x^k\|^2 \\ &\quad + \rho\langle T(y^k), x^{k+1} - x^* \rangle + \rho(f(x^{k+1}) - f(x^*)) \,, \end{aligned} \tag{2.5}$$

for $x = x^*$ in (2.1).

If we replace x by x^{k+1} in (1.1) and combine with (2.5), we obtain

$$\begin{aligned}\Lambda^*(x^k) - \Lambda^*(x^{k+1}) &\geq [\alpha_2/2! + (\alpha_3/3!)\|x^{k+1} - x^k\| + \dots \\ &\quad + (\alpha_m/m!)\|x^{k+1} - x^k\|^{m-2}]\|x^{k+1} - x^k\|^2 \\ &\quad + \rho\langle T(x^k), x^{k+1} - x^*\rangle - \rho\langle T(x^*), x^{k+1} - x^*\rangle \\ &= [\alpha_2/2! + (\alpha_3/3!)\|x^{k+1} - x^k\| + \dots \\ &\quad + (\alpha_m/m!)\|x^{k+1} - x^k\|^{m-2}]\|x^{k+1} - x^k\|^2 \\ &\quad + \rho\langle T(x^k) - T(x^*), x^{k+1} - x^k\rangle + \rho\langle T(x^k) - T(x^*), x^k - x^*\rangle\,.\end{aligned}$$

Since T is r-strongly monotone and β-Lipschitz continuous, it implies that

$$\begin{aligned}&\Lambda^*(x^k) - \Lambda^*(x^{k+1}) \\ &\qquad \geq [\alpha_2/2! + (\alpha_3/3!)\|x^{k+1} - x^k\| + \dots \\ &\qquad\quad + (\alpha_m/m!)\|x^{k+1} - x^k\|^{m-2}]\|x^{k+1} - x^k\|^2 \\ &\qquad\quad + \rho r\|x^k - x^*\|^2 - \rho\langle T(x^k) - T(x^*), x^{k+1} - x^k\rangle \\ &\qquad \geq [\alpha_2/2! + (\alpha_3/3!)\|x^{k+1} - x^k\| + \dots \\ &\qquad\quad + (\alpha_m/m!)\|x^{k+1} - x^k\|^{m-2}]\|x^{k+1} - x^k\|^2 \\ &\qquad\quad + \rho r\|x^k - x^*\|^2 - \rho\|T(x^k) - T(x^*)\|\|x^{k+1} - x^k\| \\ &\qquad \geq [\alpha_2/2! + (\alpha_3/3!)\|x^{k+1} - x^k\| + \dots \\ &\qquad\quad + (\alpha_m/m!)\|x^{k+1} - x^k\|^{m-2}\|x^{k+1} - x^k\|^2\end{aligned}$$

$$\begin{aligned}&\quad + \rho r\|x^k - x^*\|^2 - \rho^2/4[\alpha_2/2! + (\alpha_3/3!)\|x^{k+1} - x^k\| + \dots \\ &\quad + (\alpha_m/m!)\|x^{k+1} - x^k\|^{m-2}]\|T(x^k) - T(x^*)|^2 \\ &\quad - [\alpha_2/2! + (\alpha_3/3!)\|x^{k+1} - x^k\| \\ &\quad + (\alpha_m/m!)\|x^{k+1} - x^k\|^{m-2}]\|x^{k+1} - x^k\|^2 + \dots \\ &= \rho r\|x^k - x^*\|^2 - \rho^2/4[\alpha 2/2! + (\alpha 3/3!)\|x^{k+1} - x^k\| + \dots \\ &\quad + (\alpha_m/m!)\|x^{k+1} - x^k\|^{m-2}]\|T(x^k) - T(x^*)|^2 \\ &\geq \rho r\|x^k - x^*\|^2 - \rho^2/4[\alpha_2/2! + (\alpha_3/3!)\|x^{k+1} - x^k\| + \dots \\ &\quad + (\alpha_m/m!)\|x^{k+1} - x^k\|^{m-2}]\beta^2\|x^k - x^*\|^2 \\ &= \rho\{r - \rho/4[\alpha_2/2! + (\alpha_3/3!)\|x^{k+1} - x^k\| + \dots \\ &\quad + (\alpha_m/m!)\|x^{k+1} - x^k\|^{m-2}]\beta^2\}\|x^k - x^*\|^2,\end{aligned}$$

that is,

$$\Lambda^*(x^k) - \Lambda^*(x^{k+1}) \\ \geq \rho\{r - \rho/4[\alpha_2/2! + (\alpha_3/3!)\|x^{k+1} - x^k\| + \dots \\ + (\alpha_m/m!)\|x^{k+1} - x^k\|^{m-2}]\beta^2\}\|x^k - x^*\|^2. \quad (2.6)$$

It follows from (2.6) that (except for $x^k = x^*$) the sequence $\{\Lambda^*(x^k)\}$ is strictly decreasing for

$$0 < \rho < (4r/\beta^2)[\alpha_2/2! + (\alpha_3/3!)\|x^{k+1} - x^k\| + \dots \\ + (\alpha_m\varepsilon)/m!)\|x^{k+1} - x^k\|^{m-2}],$$

and nonnegative by its definition. As a result, the difference of two successive terms tends to zero, which implies in turn that $x^k \longrightarrow x^*$ (strongly) as $k \to \infty$. This completes the proof.

When $m = 2$ in Theorem 2.1, we arrive at:

Theorem 2.2. *Let E be a reflexive Banach space and $T : K \longrightarrow E^*$ an r-strongly monotone mapping from a nonempty closed convex subset K of E into E^*, the dual of E. Let $f : K \longrightarrow R$ be proper, convex and lower semicontinuous (lsc) on K and $h : K \longrightarrow R$ be 2-times continuously Frechet-differentiable on K. Suppose that there exist an $x \in K$ and a nonnegative number α_2 such that*

$$\langle h''(x) - h''(x'), (x - x')^2\rangle \geq 0 \quad (2.7)$$

and

$$\|h''(x')\| \geq \alpha_2 \, . \quad (2.8)$$

Then there exists a unique solution x^{k+1} to the GAPP (2.1). If, in addition, $x^* \in K$ is any fixed solution of the NVIP (1.1), T is β-Lipschitz continuous and

$$0 < \rho < (4r/\beta^2)[\alpha_2/2!],$$

then $\{x^k\}$ converges to x^*.

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