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THE OPTIMAL STABILIZATION PROBLEM

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Abstract: I will review the optimal stabilization problem for smooth control systems.

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1. Setting of the Problem

For a finite-dimensional *control system*

$$\dot{x} = f(x,u),\ x \in X \subset R^n,\ u \in U \subset R^m,\ f(0,0) = 0\,, \tag{1}$$

where X is an open neighbourhood of $0 \in R^n$, $U = \bar{U}$ is a closed neighbourhood of $0 \in R^m$, the boundary $\bar{U} \setminus U$ is piecewise smooth (usually U is a cube), and an *integral functional* of expenditure

$$J = \int_0^\infty \omega(x,u)\,dt,\ \ \omega(x,u) \geq 0,\ \omega(0,0) = 0 \tag{2}$$

$$f,\ \omega \in C^2$$

we want to find a control function $u \in C^0(X,U)$ (feedback), providing a central

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extremal field in the state space $X \subset R^n$ (with the center $0 \in R^n$) of the functional J under nonholonomic restrictions, given by the control system.

Lemma 1. (about smoothness of the solution) *If a solution* $u_{\rm opt} \in C^0(X,U)$ *exists,* $u_{\rm opt} \in C^{0(2)})(X,U)$*, where* $C^{i(j)}(X,U)$ *means the space of functions,* i *times smooth everywhere and* j *times smooth on a dense subset of* X*,* $(j \geq i)$*.* □

Remarks. 1. In general, solutions need not exist and need not be unique. But usually a projected system is required to hold two natural properties (which already imply existence and uniqueness of the solution): first, to be *controllable*, and second, the function ω under the integral should be *nondegenerated* (in the sense, that $\omega(x,u) = 0 \Rightarrow (x,u) = 0$), the last requirement is because the expenditure functional should express real expenditure.

2. The property of controllability simply means that we can move the system from any one point of the state space X to any other point with an admissible control. The controllability problem is not yet solved for arbitrary nonlinear systems and functional spaces of control. Calman's criterion is known for linear systems $\dot{x} = Ax + Bu$, namely, such a system is controllable under piecewise constant control functions, iff $rank[B|AB|...|A^{n-1}B] = n = dim\, X$. For infinitely smooth nonlinear systems there can be formulated a necessary condition for controllability under piecewise constant controls, namely, if $\mathcal{F} = \{D_u := f^i(x,u)\frac{\partial}{\partial x^i},\ u \in U\}$, the family of vector fields, corresponding to the constant controls u, then the module rank over $C^\infty(X,R)$ of the Lie closure $[\mathcal{F},\mathcal{F}]$ must be equal $n = dim\, X$ $(rank_{C^\infty(X,R)}([\mathcal{F},\mathcal{F}]) = n = dim\, X)$.

3. Usually, the restrictions of controllability and nondegeneracy are too strict. Sometimes, degenerated functions ω can be very useful for exact solution of the problem, and controllability sometimes can be replaced by the condition of *absence of first integrals*, which already can be more or less effectively checked (a module of 1-forms M^0 is not integrable, iff the derived module $M^\infty := \bigcap_{i=0}^{\infty} M^i = 0$, where $M^{i+1} = ker(pr_i \circ d) : M^i \to \Omega^1/M^i$). A more delicate (than nonintegrability) property of control system is *differential flatness* over a subspace of the state space (which means that the differential ideal defined by the system is equivalent to the trivial one on this subspace).

We want to find a local minimum of the map $J : C^{0(2)}_{\rm stab}(X,U) \to \tilde{C}^{0(2)}_{\substack{\rm partial\\ \rm functions}}(X,R) :$ $u \mapsto \int\limits_{\substack{0\\ \rm along\\ \dot{x}=f(x,u(x))}}^{\infty} \omega(x,u(x))\, dt$. Since the image of this map $Im\, J$ is just partially ordered there can be several local minima and no global.

Lemma 2. *J is continuous with respect to (C^0, C^0)-topologies on its domain and codomain.* □

Definition 1. A *nondegenerated optimal control system* is provided if:

- the system of linear approximation at the origin $\dot{x} = \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial u}(0,0)u$ is controllable;
- the function ω is not degenerated: $\omega(x,u) = 0 \Longrightarrow (x,u) = 0$.

Corrollary of the Lemma 2. (C^1-robustness of the nondegenerated optimal control system) *The space of stabilizing controls $C^{0(2)}_{\text{stab}}(X,U)$ is open in C^1-topology in the space $C^{0(2)}(X,U)$.* □

Main proposition. A stabilizing feedback $u_{\text{opt}} \in C^{0(2)}_{\text{stab}}(X,U)$ is a local minimum of J iff there exists a function of expenditures V s.t. u_{opt} and V satisfy the *Bellman's equation*

$$min_{\substack{u \in C^0-\text{neighbour-}\\ \text{hood } of\ u_{\text{opt}}}} \{\mathcal{L}_{D_{u(x)}} V + \omega\} = \mathcal{L}_{D_{u_{\text{opt}}}} V + \omega(u_{\text{opt}}) = 0 \tag{3}$$

(i.e. u_{opt} is a local minimum of the function in the curly braces) □

Definition 2. The function V is called a *Bellman-Lyapunov function* (since it satisfies the Bellman equation, and simultaneously has the properties of a Lyapunov function for the optimal control system).

Remark. Sufficiency of the **main proposition** is proved easily, necessity requires to regard a special kind of variations in the stabilizing controls space $C^{0(2)}_{\text{stab}}(X,U)$.

Corrollary. (Hamilton-Jacobi equation) *From this proposition we can come to the Hamilton-Jacobi equation. We are looking for a function $u^* \in C^{0(2)}(T^*X,U)$, which gives a minimum of the expression $\mathcal{L}_{D_u} V + \omega(u)$ at each point of the phase space T^*X, and substitute this function for the expression. Thereby, we obtain the Hamilton-Jacobi equation*

$$\begin{aligned}\varphi(x, \text{grad}V) &:= min_{u \in U}\{\mathcal{L}_{D_u} V + \omega(u)\} \\ &= \mathcal{L}_{D_{u^*(x,\text{grad}V)}} V + \omega(x, u^*(x, \text{grad}V)) = 0\end{aligned} \tag{4}$$

and the canonical Hamilton system

$$(Id\varphi): \begin{cases} \dot{x}^i = \varphi_{V_i} \\ \dot{V}_i = -\varphi_{x^i} \end{cases} \tag{5}$$

Remarks. 1. The equations (3) and (4) are not equivalent. We don't lose any solution when we come to the Hamilton-Jacobi equation. Certain solutions of the Hamilton-Jacobi equation will be candidates for a Bellman-Lyapunov function.

2. We don't have correct initial conditions for the Hamilton-Jacobi equation in the usual sense. We just know $V(0) = 0, \ \mathrm{grad}V(0) = 0$.

Proposition 1.

- *The Lagrangian manifold L^- of a Bellman-Lyapunov function V is:*
  - *a (maximal) Lagrangian manifold on a separatrix of stable (with respect to the origin $(0,0) \in T^*X$) points of the Hamiltonian system $(L^- \longmapsto S^-)$;*
  - *a section surface of the canonical projection $T^*X \xrightarrow{p} X$.*
- *The optimal trajectories in the state space X are projections of the trajectories of the Hamiltonian system on $L^- = L(V)$, or, the same,* $\boldsymbol{D_{u_{\mathrm{opt}}} = p_*(Id\varphi|_{L^-})}$.
- *The optimal feedback $u_{\mathrm{opt}}(x) = u^*|_{L^-}$.* □

In general, there can be many *Hamiltonians* (many Hamilton-Jacobi equations), associated with the optimal stabilization problem. And if the origin in the phase space is degenerated there can also be several stable and unstable manifolds through it (for each Hamiltonian).

Proposition 2. *For a nondegenerated optimal control system:*

- $\exists!$ *Hamiltonian;*
- *the origin $(0,0) \in T^*X$ of the associated Hamiltonian system is a hyperbolic point;*

- *$\exists$ exactly two separatrices S^-, S^+ through the origin (stable and unstable manifolds), $dimS^- = dimS^+$;*
- *$S^- = L^- = L(V)$, $S^+ = L^+ = L(V_1)$, where V is a Bellman-Lyapunov function, $V_1 \in C^{0(2)}(X, R^-)$ (a nonpositively determined function).* □

Corrollary. *For a nondegenerated case the problem has one and only one solution.* □

Now, the main question is, *how to calculate L^-*?

The main algorithm, of course, is to use the Couchy characteristics (but, we should be more careful, since there are no standart initial conditions). However, we know that for a nondegenerated case the origine in the phase space is a stable hyperbolic point. So, there exists an $Id\varphi$-invariant splitting of the tangent bundle $T(T^*X) = T_1(T^*X) \oplus T_2(T^*X)$ in a neighbourhood of the origin $(0,0) \in T^*X$. Subbundles $T_1(T^*X)$, $T_2(T^*X)$ are stable points in the space of all subbundles (where topology is defined pointwise by value of the corner between corresponding tangent subspaces of equal dimension) respectively under Hamiltonian and inverse Hamiltonian flows. Stability means, that there exists a neighbourhood of a subbundle such that any neighbourhood inside it is mapped to a smaller neighbourhood. These subbundles are integrable but can be nonsmooth, so Frobenius theorem does not hold for them in general. Leaves through the origin are exactly stable and unstable manifolds, which can be reached as a limit position of any initial manifold, containing the origin, for which the angle between this manifold and corresponding separatrix is not too big, by integrating the Hamiltonian system (5) in direct (for the unstable manifold) or in opposite direction (for the stable manifold). So, $L^- = L(V) = \lim_{t \to -\infty} L_t$ for the Hamiltonian system (5) with the initial condition $L|_{t=0} = T_{(0,0)}L^-$.

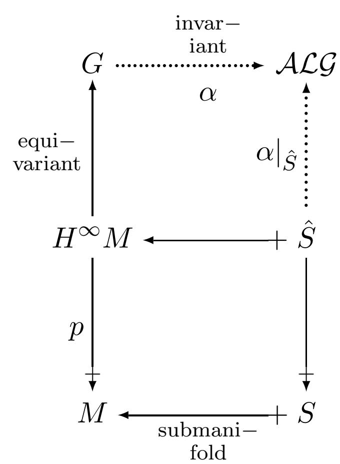
If the problem is *purely smooth* (when U is big with respect to X), or if we make it smooth (by changing control variables with the function of 'arctan'-type), we can use some *properties of the Bellman-Lyapunov function* to describe it exactly (more or less):

1. *Isolatedness of a Bellman-Lyapunov function* (up to an additive constant) in the space of all solutions of the Hamilton-Jacobi equation in the usual jet-topology, or even, in the pointwise convergence topology in the section space of the jet bundle $\mathcal{J}^k(X, R)$, for each $k = 1, ..., \infty$.

Isolatedness implies stability under all automorphisms of the Hamilton-Jacobi equation (or, at least, under all infinitesimal symmetries of the equation).

So, if, for example, D is an evolutionary derivative of $C^\infty(\mathcal{J}^\infty(X,R))$, s.t. $D\varphi = 0$, then there exists a constant c s.t. $DV = c$. It allows us to get additional equations on the jet $j^k(V)$ of Bellman-Lyapunov function of V and to restore it in a good case.

2. We know the *canonical form* of a nondegenerated Bellman-Lyapunov function V in a special coordinate system ($V = \|x\|^2$). So, in principle, we can calculate differential invariants of this function, or joint differential invariants of this function and Hamiltonian with respect to the group of real analytical or smooth transformations. For real analytical transformations the main idea is to prolong the group action onto the infinite frame bundle, where the action is simply transitive, and from where it is possible to construct an equivariant mapping into the group space. A rough scheme is the following:



So, on the frame bundle space we will get, from one side, the lifted original ideal, and from other side, an invariant real subalgebra (preimage of the algebra of invariant forms on the analytical transformation group), which generates the algebra of all differential forms over the functional ring. So, we can calculate all differential invariants of the original ideal.

3. We know that our solution *admits an orthogonal group*. So, the problem can be reduced to integrating of very big overdetermined system of PDEs.

The last question is: *when can the problem be solved explicitely*?

The following cases can be distinguished:

- systems with *quadratic Hamiltonian*, which include not only the classical case with a linear control system and quadratic function ω under the integral J, but many nonlinear systems can get quadratic Hamiltonian by choosing a special functional. For such systems it is possible to write a simple explicit series of first integrals:

$$\varphi,\ \nabla d\varphi(Id\varphi, Id\varphi),\ \nabla d(\nabla d\varphi(Id\varphi, Id\varphi))(Id\varphi, Id\varphi), \dots,$$

  where ∇ is the usual Eucledian covariant derivative (from the module of 1-forms with coefficients in the exterior differential algebra into itself over the identity morphism of this module). In this case the problem can be reduced also to solving the algebraic Riccatti equation for n^2 variables, or even to solving a certain system of quadratic equations for just n variables. The last system is obtained by differentiating a linear generator g with undetermined coefficients along the Hamiltonian vector field $Id\varphi$, that is, we get the sequence of generators of the ideal of the Lagrangian manifold L^-: $g, (Id\varphi)g, ..., (Id\varphi)^{n-1}g, ... \in I(L^-)$, and get explicite relations between them after that.

- control systems which hold *the Huygens principle* (that is, if we start from the level surface of Bellman-Lyapunov function at a moment, then at the next moment we will finish also on the level surface of this function). For this case also a series of equations describing the Lagrangian manifold of the Bellman-Lyapunov function can be written explicitly (they follow from the defining property: $\varphi_{V_i} V_i = F(V)$).

- systems *admitting sufficiently many symmetries*.

- systems with *functionals of special kind*, for which the integrals of Euler-Lagrange equations are known, and consistable with the restrictions given by the control system (very effective method). (E.g., take homogenous linearly independent linear forms $\psi_1, \psi_2, ..., \psi_r$ of equal numbers as the number of controls. Take a subintegral function $\omega = \sum \psi_i^2 + \sum \dot{\psi}_i^2$ (*Kolesnikov's function*). Then, in a common case the first integrals of Euler-Lagrange equations for this functional are consistent and completely define the optimal feedback).

- *degenerated systems* (some additional equations can be written here).

- also it should be noticed that for nondegenerated systems with a smooth Hamiltonian any initial segment of the Taylor's series for the solution can be effectively calculated (for the second derivatives quadratical Riccatti equation should be solved and for derivatives of higher order it will be just linear equations of the algebraic variables).

There are open problems in this area:

- describe all exactly integrable classes of optimal control systems;
- extend this approach to control systems on Banach spaces (where the general situation is similar)
  - how can this approach be extended to the systems with delay argument, which determines just a semiflowon an appropriate Banach space?
- write a complete system of differential invariants of the Bellman-Lyapunov function and joint differential invariants of the Bellman-Lyapunov function and Hamiltonian under the real analytical group of transformations (it is definitely possible to do), and under the group of smooth transformations;
- write conditions of real integrability for finite type PDEs;
- invent a constructive method of computing isolated solutions of PDEs;
- work out a good model for the solution space of an exterior differential system (particularly, which topology can be regarded as a 'dual' for the structure of exterior differential algebra, in the sense, that continuos morphisms of solution spaces would induce corresponding inverse morphism of exterior differential algebras).