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FUZZY POINT BCK/BCI-ALGEBRAS

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Abstract: Using the notion of fuzzy points, the notions of fuzzy point BCK/BCI-algebra, quasi subalgebra, and quasi ideal are established. Some characterizations of fuzzy subalgebras are provided by using such concepts.

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1. Introduction

The fundamental concept of a fuzzy set, introduced by Zadeh [2] in 1965, provides a natural foundation for treating mathematically the fuzzy phenomena which exist pervasively in our real world and for building new branches of fuzzy

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mathematics. In the area of fuzzy BCK/BCI-algebra, several researchers have been carried out since 1991. In [1], Lele et al. used the notion of fuzzy point to study some properties of BCK-algebras. In this paper, using the notion of fuzzy points, we will construct the concept of fuzzy point BCK/BCI-algebra, quasi subalgebra, and quasi ideal. We will give characterizations of fuzzy subalgebras by using such concepts.

2. Preliminaries

We first recall some basic concepts which are used to present the paper.

An algebra $(X, *, 0)$ of type $(2, 0)$ is said to be a *BCI-algebra* if it satisfies:

- $((x * y) * (x * z)) * (z * y) = 0$,
- $(x * (x * y)) * y = 0$,
- $x * x = 0$,
- $x * y = 0$ and $y * x = 0$ imply $x = y$

for all $x, y, z \in X$. If a BCI-algebra X satisfies the equality

- $0 * x = 0$ for all $x \in X$,

then we say that X is a *BCK-algebra*. A subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. A fuzzy set μ in a BCK/BCI-algebra X is called a *fuzzy subalgebra* of X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$. A fuzzy set μ in a set X is called a *fuzzy point* if it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at x is p $(0 < p \leq 1)$ we denote this fuzzy point by x_p, where the point x is called its *support*.

3. Fuzzy Point BCK/BCI-algebras

Let $FP(X)$ denote the set of all fuzzy points in a BCI-algebra X and define a binary operation $\odot$ on $FP(X)$ by $x_p \odot y_q = (x * y)_{\min\{p,q\}}$ for all $x_p, y_q \in FP(X)$. We can easily check the followings:

(p1) $((x_p \odot y_q) \odot (x_p \odot z_r)) \odot (z_r \odot y_q) = 0_{\min\{p,q,r\}}$, (p2) $(x_p \odot (x_p \odot y_q)) \odot y_q = 0_{\min\{p,q\}}$,

(p3) $x_p \odot x_p = 0_p$

for all $x_p, y_q, z_r \in FP(X)$, but the following does not hold:

(p4) $x_p \odot y_q = 0_{\min\{p,q\}}$ and $y_q \odot x_p = 0_{\min\{p,q\}}$ imply $x_p = y_q$.

For example, let $X = \{0, a, b, c\}$ be a BCI-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	c	0	a
a	a	0	a	c
b	b	c	0	a
c	c	a	c	0

Consider two fuzzy points $a_{0.3}$ and $a_{0.5}$ in X. Then $a_{0.3} \odot a_{0.5} = 0_{0.3}$ and $a_{0.5} \odot a_{0.3} = 0_{0.3}$, but $a_{0.3} \neq a_{0.5}$. Hence $FP(X)$ may not be a BCI-algebra under the operation $\odot$.

Moreover, if X is a BCK-algebra, then $FP(X)$ satisfies the following equality:

(p5) $0_p \odot x_q = 0_{\min\{p,q\}}$.

We now call $FP(X)$ the *quasi BCK/BCI-algebra*.

Definition 3.1. A subset S of the quasi BCK/BCI-algebra $FP(X)$ is called a *quasi subalgebra* if $x_p \odot y_q \in S$ whenever $x_p, y_q \in S$.

We wish to make a BCK/BCI-algebra by using fuzzy points. In order to do this, we first compress the set of all fuzzy points in X. Let $FP_q(X)$ denote the set of all fuzzy points in X with the value q $(0 < q \leq 1)$. It is easily to check that $(FP_q(X), \odot, 0_q)$ is a BCK/BCI-algebra, which is called a *fuzzy point BCK/BCI-algebra*.

Definition 3.2. A subset S of a fuzzy point BCK/BCI-algebra $FP_q(X)$ is called a *fuzzy point subalgebra* if $x_q \odot y_q \in S$ whenever $x_q, y_q \in S$.

Example 3.3. For the BCI-algebra $X = \{0, a, b, c\}$ mentioned in above, it is routine to check that $(FP_{0.3}(X), \odot, 0_{0.3})$ is a fuzzy point BCI-algebra, and that $S = \{0_{0.3}, b_{0.3}\}$ is a fuzzy point subalgebra of $FP_{0.3}(X)$.

Theorem 3.4. *$FP_q(X)$ is a quasi subalgebra of $FP(X)$ for every $q \in (0, 1]$.*

Proof. Straightforward. □

For a fuzzy set μ in a BCK/BCI-algebra X, denote

$$FP(\mu) := \{x_p \in FP(X) \mid \mu(x) \geq p,\ p \in (0, 1]\},$$
$$FP_q(\mu) := \{x_q \in FP_q(X) \mid \mu(x) \geq q\} \text{ for } q \in (0, 1].$$

Theorem 3.5. *Let μ be a fuzzy set in a BCK/BCI-algebra X. Then the following are equivalent:*

(i) *μ is a fuzzy subalgebra of X.*

(ii) *$FP_q(\mu)$ is a fuzzy point subalgebra of $FP_q(X)$ for every $q \in (0, 1]$.*

(iii) *$U(\mu; r)$ is a subalgebra of X when it is non-empty for every $r \in (0, 1]$.*

(iv) *$FP(\mu)$ is a quasi subalgebra of $FP(X)$.*

Proof. (i) ⇒ (ii) Assume that μ is a fuzzy subalgebra of X and let $x_q, y_q \in FP_q(\mu)$ where $q \in (0, 1]$. Then $\mu(x) \geq q$ and $\mu(y) \geq q$. It follows that

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\} \geq q$$

so that $x_q \odot y_q = (x * y)_q \in FP_q(\mu)$. Hence $FP_q(\mu)$ is a fuzzy point subalgebra of $FP_q(X)$.

(ii) ⇒ (iii) Suppose that $FP_q(\mu)$ is a fuzzy point subalgebra of $FP_q(X)$ for every $q \in (0, 1]$. Let $x, y \in U(\mu; r)$, where $r \in (0, 1]$. Then $\mu(x) \geq r$ and $\mu(y) \geq r$, and so $x_r, y_r \in FP_r(\mu)$. It follows that $(x * y)_r = x_r \odot y_r \in FP_r(\mu)$ so that $\mu(x * y) \geq r$, i.e. $x * y \in U(\mu; r)$. Therefore $U(\mu; r)$ is a subalgebra of X.

(iii) ⇒ (iv) Suppose $U(\mu; r)$ $(\neq \emptyset)$ is a subalgebra of X for every $r \in (0, 1]$. Let $x_p, y_q \in FP(\mu)$ and let $t = \min\{p, q\}$. Then $\mu(x) \geq p \geq t$ and $\mu(y) \geq q \geq t$, and thus $x, y \in U(\mu; t)$. It follows that $x * y \in U(\mu; t)$ because $U(\mu; t)$ is a subalgebra of X. Thus $\mu(x*y) \geq t$, which implies that $x_p \odot y_q = (x*y)_{\min\{p,q\}} = (x * y)_t \in FP(\mu)$. Hence $FP(\mu)$ is a quasi subalgebra of $FP(X)$.

(iv) ⇒ (i) Assume that $FP(\mu)$ is a quasi subalgebra of $FP(X)$. For any $x, y \in X$, we have $x_{\mu(x)}, y_{\mu(y)} \in FP(\mu)$ which imply that

$$(x * y)_{\min\{\mu(x),\mu(y)\}} = x_{\mu(x)} \odot y_{\mu(y)} \in FP(\mu),$$

that is, $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$. Consequently, μ is a fuzzy subalgebra of X. □

Proposition 3.6. *Let μ be a fuzzy set in a BCK/BCI-algebra X. If $FP(\mu)$ is a quasi subalgebra of $FP(X)$, then $0_p \in FP(\mu)$ for all $p \in \mathrm{Im}(\mu)$.*

Proof. Let $p \in \mathrm{Im}(\mu)$. Then there exists $x \in X$ such that $\mu(x) = p$. Hence $x_p \in FP(\mu)$, and so $0_p = (x * x)_p = x_p \odot x_p \in FP(\mu)$. □

Combining Theorem 3.5 and Proposition 3.6, we have the following corollary.

Corollary 3.7. *If μ is a fuzzy subalgebra of a BCK/BCI-algebra X, then $0_p \in FP(\mu)$ for all $p \in \mathrm{Im}(\mu)$.*

Proposition 3.8. *If $FP_q(\mu)$ is a fuzzy point subalgebra of $FP_q(X)$, then $0_q \in FP_q(\mu)$.*

Proof. For every $x_q \in FP_q(\mu)$, we have $0_q = (x*x)_q = x_q \odot x_q \in FP_q(\mu)$. □

By means of Theorem 3.5 and Proposition 3.8, we get the following corollary.

Corollary 3.9. *If μ is a fuzzy subalgebra of a BCK/BCI-algebra X, then $0_q \in FP_q(\mu)$ for all $q \in (0, 1]$.*

Proposition 3.10. *Let μ be a fuzzy set in a BCK/BCI-algebra X and let $p, q \in (0, 1]$ with $p \geq q$. If $x_p \in FP(\mu)$, then $x_q \in FP(\mu)$.*

Proof. Straightforward. □

Definition 3.11. For a fuzzy set μ in a BCK/BCI-algebra X, the set $FP(\mu)$ is called a *quasi ideal* of $FP(X)$ if

- $0_p \in FP(\mu)$ for all $p \in \mathrm{Im}(\mu)$,
- $x_p \odot y_q \in FP(\mu)$ and $y_q \in FP(\mu)$ imply that $x_{\min\{p,q\}} \in FP(\mu)$.

Theorem 3.12. *If μ is a fuzzy ideal of a BCK/BCI-algbera X, then $FP(\mu)$ is a quasi ideal of $FP(X)$.*

Proof. Since $\mu(0) \geq \mu(x)$ for all $x \in X$, we have $\mu(0) \geq p$ for all $p \in \mathrm{Im}(\mu)$. Hence $0_p \in FP(\mu)$. Let $x_p, y_q \in FP(X)$ be such that $x_p \odot y_q \in FP(\mu)$ and

$y_q \in FP(\mu)$. Then $\mu(x * y) \geq \min\{p, q\}$ and $\mu(y) \geq q$. Since μ is a fuzzy ideal of X, it follows that

$$\mu(x) \geq \min\{\mu(x * y), \mu(y)\} \geq \min\{p, q\}$$

so that $x_{\min\{p,q\}} \in FP(\mu)$. This completes the proof. □

Proposition 3.13. *Let μ be a fuzzy set in a BCK-algebra X. If the set*

$$FPIm(\mu) := \{x_p \in FP(X) \mid \mu(x) \geq p,\, p \in \mathrm{Im}(\mu)\}$$

is a quasi ideal of $FP(X)$, then it is a quasi subalgebra of $FP(X)$.

Proof. Let $x_p, y_q \in FPIm(\mu)$. Note that

$$(x_p \odot y_q) \odot x_p = ((x * y) * x)_{\min\{p,q\}} = 0_{\min\{p,q\}} \in FPIm(\mu).$$

Using Definition 3.11, it follows that $x_p \odot y_q \in FPIm(\mu)$. This completes the proof. □

Lemma 3.14. *For any fuzzy set μ in a BCK-algebra X, if $FPIm(\mu)$ is a quasi subalgebra of $FP(X)$, then μ is a fuzzy subalgebra of X.*

Proof. Let $x, y \in X$ be such that $\mu(x) = p$ and $\mu(y) = q$. Then $x_p, y_q \in FPIm(\mu)$, and so $(x * y)_{\min\{p,q\}} = x_p \odot y_q \in FPIm(\mu)$. It follows that

$$\mu(x * y) \geq \min\{p, q\} = \min\{\mu(x), \mu(y)\}.$$

Hence μ is a fuzzy subalgebra of X. □

Theorem 3.15. *Let μ be a fuzzy set in a BCK-algebra X. If the set $FPIm(\mu)$ is a quasi ideal of $FP(X)$, then μ is a fuzzy ideal of X.*

Proof. Assume that $FPIm(\mu)$ is a quasi ideal of $FP(X)$. Then it is a quasi subalgebra of $FP(X)$. Hence μ is a fuzzy subalgebra of X, and so

$$\mu(0) = \mu(x * x) \geq \min\{\mu(x), \mu(x)\} = \mu(x)$$

for all $x \in X$. Let $x, y \in X$ and let $p = \min\{\mu(x * y), \mu(y)\}$. Then $x_p \odot y_p = (x * y)_p \in FPIm(\mu)$ and $y_p \in FPIm(\mu)$. It follows from Definition 3.11 that $x_p \in FPIm(\mu)$ so that $\mu(x) \geq p = \min\{\mu(x * y), \mu(y)\}$. This completes the proof. □

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