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ON SOME PEXIDERIZED QUADRATIC FUNCTIONAL EQUATIONS

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Abstract: In the present paper, we investigate the general solutions of the functional equations (5) and (6) which are pexiderized forms of new quadratic functional equations (3) and (4), respectively.

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1. Introduction

In this paper, we will introduce new functional equations (3) and (4) that have the quadratic property in the sense that the quadratic function $f(x) = cx^2$ ($x \in \mathbf{R}$; $c \in \mathbf{R}$ is a constant) is a solution of each of the following functional equations

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$$f(x+y)+f(x-y)=2f(x)+2f(y), \tag{1}$$

$$f(x+y+z)+f(x)+f(y)+f(z)=f(x+y)+f(y+z)+f(z+x), \tag{2}$$

$$f(x+y-z)+f(x)+f(y)+f(z)=f(x+y)+f(y-z)+f(z-x), \tag{3}$$

$$f(x-y+z)+f(x)+f(y)+f(z)=f(x-y)+f(y-z)+f(z+x). \tag{4}$$

Therefore, we can naturally say that each equation is a quadratic functional equation. In particular, every solution of the 'original' quadratic functional equation (1) is said to be a quadratic function.

It is well-known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function B such that $f(x)=B(x,x)$ for all x (see [1], [2], [3], [4], [5], [6]).

The functional equation (2) was first solved by Pl. Kannappan. In fact, he proved that a functional on a real vector space is a solution of the equation (2) if and only if there exist a symmetric biadditive function B and an additive function A such that $f(x)=B(x,x)+A(x)$ for any x (see [5]). Moreover, the quadratic functional equation (2) was pexiderized and solved by Pl. Kannappan (see [5]).

The functional equation (2) is different from the equations (3) and (4), since every additive function not vanishing everywhere is a solution of (2) but it satisfies neither (3) nor (4).

In Section 2 of this paper, we will show that each of the functional equations (1), (3) and (4) is equivalent to the other. In Sections 3 and 4, we will solve the functional equations

$$f_1(x+y-z)+f_2(x)+f_3(y)+f_4(z)=f_5(x+y)+f_6(y-z)+f_7(z-x), \tag{5}$$

$$f_1(x-y+z)+f_2(x)+f_3(y)+f_4(z)=f_5(x-y)+f_6(y-z)+f_7(z+x) \tag{6}$$

which are pexiderized forms of the equations (3) and (4), respectively, in the class of functions between vector spaces.

2. Solutions of Equations (3) and (4)

We may intuitively expect that both the functional equations (3) and (4) are equivalent to the 'original' quadratic equation (1). Indeed, it is so, as we shall see in the following theorem.

Theorem 1. *If vector spaces X and Y are common domain and range of the f's in the functional equations* (1), (3) *and* (4), *then each of the equations* (1), (3) *and* (4) *is equivalent to the other.*

Proof. First, let us prove the equivalence of the equations (1) and (3). By putting $x = y = z = 0$ in (3), we get $f(0) = 0$. If we set $y = z = 0$ in (3), we see that every solution of the equation (3) is even.

Replacing z by y in (3) and using the evenness of f and $f(0) = 0$, we can transform the equation (3) into the equation (1).

Conversely, assume that a function $f : X \to Y$ is a solution of the functional equation (1). Obviously, it follows from (1) that

$$f(x+y-z) + f(x) = 2f\left(x + \frac{y-z}{2}\right) + 2f\left(\frac{y-z}{2}\right)$$

and

$$f(y) + f(z) = 2f\left(\frac{y+z}{2}\right) + 2f\left(\frac{y-z}{2}\right).$$

Due to (1) and the last two equalities, and by using the fact $f(2x) = 4f(x)$, we obtain

$$\begin{aligned} & f(x+y-z) + f(x) + f(y) + f(z) \\ = \;& 2f\left(x + \frac{y-z}{2}\right) + 2f\left(\frac{y+z}{2}\right) + f(y-z) \\ = \;& f(x+y) + f(y-z) + f(z-x), \end{aligned}$$

since every solution of the equation (1) is even. This implies the equivalence of the functional equations (1) and (3).

It remains to prove the equivalence of the equations (1) and (4). If we set $x = y = z = 0$ in the equation (4), we get $f(0) = 0$. By putting $x = y = 0$ in (4), we see that every solution of the equation (4) is even. By putting $y = z$ in (4) and using the evenness of f and $f(0) = 0$, we can transform the equation (4) into the equation (1).

Now, suppose that a function $f : X \to Y$ satisfies (1) for all $x, y, z \in X$. Then f is even. Hence, we get

$$f(x-y+z) + f(x) = 2f\left(x + \frac{-y+z}{2}\right) + 2f\left(\frac{-y+z}{2}\right)$$

and

$$f(y) + f(z) = 2f\left(\frac{y+z}{2}\right) + 2f\left(\frac{y-z}{2}\right).$$

Thus, it follows from the fact $f(2x) = 4f(x)$ that

$$\begin{aligned} & f(x-y+z)+f(x)+f(y)+f(z) \\ = \;& 2f\left(x+\frac{-y+z}{2}\right)+2f\left(\frac{y+z}{2}\right)+f(y-z) \\ = \;& f(x+z)+f(x-y)+f(y-z). \end{aligned}$$

This means the equivalence of the equations (1) and (4). □

3. General Solutions of Equation (5)

In this section, we will solve the functional equation (5) which is a pexiderized form of the equation (3). Let X and Y be some vector spaces.

Theorem 2. *The functions* $f_i : X \to Y$ $(i = 1, ..., 7)$ *satisfy the functional equation* (5) *if and only if there exist a quadratic function* $Q : X \to Y$, *constants* $c_i \in Y$ $(i = 1, ..., 7)$ *and additive functions* $a_i : X \to Y$ $(i = 1, ..., 4)$ *such that*

$$\begin{aligned} f_1(x) &= Q(x)+a_3(x)-a_4(x)+c_1, \\ f_2(x) &= Q(x)-a_1(x)-a_4(x)+c_2, \\ f_3(x) &= Q(x)-a_1(x)-a_2(x)+a_3(x)-a_4(x)+c_3, \\ f_4(x) &= Q(x)+a_2(x)+a_4(x)+c_4, \\ f_5(x) &= Q(x)-a_1(x)+a_3(x)-a_4(x)+c_5, \\ f_6(x) &= Q(x)-a_2(x)+a_3(x)-a_4(x)+c_6, \\ f_7(x) &= Q(x)+a_4(x)+c_7, \end{aligned} \tag{7}$$

where the following equation

$$c_1+c_2+c_3+c_4 = c_5+c_6+c_7 \tag{8}$$

must be satisfied by the c_i*'s.*

Proof. Define $c_i = f_i(0)$ for $i = 1, ..., 7$. Putting $x = y = z = 0$ in (5) yields the relation (8). For $i = 1, ..., 7$ define $\tilde{f}_i(x) = f_i(x) - c_i$ for all $x \in X$.

By $h_i(x)$ and $g_i(x)$ denote the even and odd part of $\tilde{f}_i(x)$, respectively:

$$h_i(x) = \frac{\tilde{f}_i(x)+\tilde{f}_i(-x)}{2} \quad \text{and} \quad g_i(x) = \frac{\tilde{f}_i(x)-\tilde{f}_i(-x)}{2}.$$

Then we have $\tilde{f}_i(0) = h_i(0) = g_i(0) = 0$ for $i = 1, ..., 7$. According to (5) and (8), it is not difficult to show that the h_i's or the g_i's themselves satisfy the equation (5).

If we put $y = z = 0$, $x = z = 0$, $x = y = 0$, $x = 0$ and $y = z$, $y = 0$ and $z = x$, or $z = 0$ and $x = -y$ in the equation (5) in which the h_i's substitute the corresponding f_i's, and if we replace the remaining variable by x, we get the following equalities:

$$\begin{aligned} h_1(x) + h_2(x) &= h_5(x) + h_7(x), \\ h_1(x) + h_3(x) &= h_5(x) + h_6(x), \\ h_1(x) + h_4(x) &= h_6(x) + h_7(x), \\ h_3(x) + h_4(x) &= h_5(x) + h_7(x), \\ h_2(x) + h_4(x) &= h_5(x) + h_6(x), \\ h_2(x) + h_3(x) &= h_6(x) + h_7(x), \end{aligned}$$

respectively. From these equations we can easily deduce the relation

$$h_1(x) = h_2(x) = h_3(x) = h_4(x) = h_5(x) = h_6(x) = h_7(x).$$

The last relations, together with the equation (5) written with h_i's instead of f_i's, guarantee the existence of a quadratic function $Q : X \to Y$ such that

$$h_1(x) = h_2(x) = h_3(x) = h_4(x) = h_5(x) = h_6(x) = h_7(x) = Q(x) \tag{9}$$

for all x in X.

We now consider the functional equation (5) in which the f_i's are replaced by the corresponding g_i's.

If we set $z = 0$ in (5), we obtain the Pexider equation

$$(g_1 - g_5)(x + y) = (-g_2 - g_7)(x) + (-g_3 + g_6)(y)$$

which implies the existence of an additive function $a_1 : X \to Y$ such that

$$g_1 - g_5 = -g_2 - g_7 = -g_3 + g_6 = a_1. \tag{10}$$

If we put $x = 0$ in (5), if we replace z by $-z$, and if we consider the oddness of g_i, we have the Pexider equation

$$(g_1 - g_6)(y + z) = (-g_3 + g_5)(y) + (g_4 - g_7)(z),$$

and this relation says that there exists an additive function $a_2 : X \to Y$ satisfying

$$g_1 - g_6 = -g_3 + g_5 = g_4 - g_7 = a_2. \tag{11}$$

In a similar way, if we set $y = 0$ in (5), we see the existence of an additive function $a_3 : X \to Y$ such that

$$g_1 + g_7 = -g_2 + g_5 = g_4 + g_6 = a_3. \tag{12}$$

Combining (10), (11) and (12), we have

$$\begin{aligned} g_1 &= a_3 - g_7, \\ g_2 &= -a_1 - g_7, \\ g_3 &= -a_1 - a_2 + a_3 - g_7, \\ g_4 &= a_2 + g_7, \\ g_5 &= -a_1 + a_3 - g_7, \\ g_6 &= -a_2 + a_3 - g_7. \end{aligned} \tag{13}$$

Applying these relations to (5), we get

$$\begin{aligned} & g_7(x+y-z) + g_7(x) + g_7(y) - g_7(z) \\ &\quad = \ g_7(x+y) + g_7(y-z) - g_7(z-x). \end{aligned} \tag{14}$$

When $x = y = z$ in (14), we have

$$g_7(2x) = 2g_7(x).$$

If we put $y = z$ in (14) and if we replace x and y in the resulting equation by $(1/2)(x+y)$ and $(1/2)(x-y)$, respectively, we see that g_7 itself is an additive function. Let

$$g_7 = a_4, \tag{15}$$

where $a_4 : X \to Y$ is an additive function.

Hence, it follows from (9), (13) and (15) that the relations in (7) holds true.

Conversely, if there exist a quadratic function $Q : X \to Y$, constants $c_i \in Y$ $(i = 1, ..., 7)$ with (8) and additive functions $a_i : X \to Y$ $(i = 1, ..., 4)$ such that the f_i's are of the forms in (7), it is obvious that the f_i's satisfy the functional equation (5). □

4. General Solutions of Equation (6)

Similarly as we did in the last theorem, we can solve the functional equation (6) in the class of functions between vector spaces. Let X and Y be given vector spaces.

Theorem 3. *The functions* $f_i : X \to Y$ $(i = 1, ..., 7)$ *satisfy the functional equation* (6) *if and only if there exist a quadratic function* $Q : X \to Y$, *constants* $c_i \in Y$ $(i = 1, ..., 7)$ *and additive functions* $a_i : X \to Y$ $(i = 1, ..., 4)$ *such that*

$$\begin{aligned} f_1(x) &= Q(x) + a_3(x) + a_4(x) + c_1, \\ f_2(x) &= Q(x) - a_1(x) + a_4(x) + c_2, \\ f_3(x) &= Q(x) + a_1(x) + a_2(x) - a_3(x) - a_4(x) + c_3, \\ f_4(x) &= Q(x) - a_2(x) + a_4(x) + c_4, \\ f_5(x) &= Q(x) - a_1(x) + a_3(x) + a_4(x) + c_5, \\ f_6(x) &= Q(x) + a_2(x) - a_3(x) - a_4(x) + c_6, \\ f_7(x) &= Q(x) + a_4(x) + c_7, \end{aligned} \tag{16}$$

where the c_i*'s satisfy the relation* (8).

Proof. We will use the same notations as in the proof of Theorem 2. As in the proof of Theorem 2, we can easily see that the c_i's satisfy the relation (8), that $h_i(0) = g_i(0) = 0$ holds for $i = 1, ..., 7$, and that the h_i's or the g_i's satisfy the equation (6).

First, we consider the functional equation (6) in which the h_i's substitute the corresponding f_i's.

If we put $y = z = 0$, $x = z = 0$, $x = y = 0$, $x = 0$ and $y = z$, $y = 0$ and $z = -x$, or $z = 0$ and $x = y$ in (6), and if we replace the remaining variable by x, we get

$$h_1(x) = h_2(x) = h_3(x) = h_4(x) = h_5(x) = h_6(x) = h_7(x).$$

Theorem 1, together with (6) and these equations, implies that there exists a quadratic function $Q : X \to Y$ with $Q(x) = h_i(x)$ for $i = 1, ..., 7$.

Now, let the equation (6) be written with g_i's instead of f_i's.

By letting $z = 0$, $x = 0$ and $y = 0$ in (6) and considering the oddness of g_i, we conclude that there exist additive functions $a_i : X \to Y$ $(i = 1, 2, 3)$ such that

$$\begin{aligned} g_1 - g_5 &= -g_2 + g_7 = g_3 - g_6 = a_1, \\ g_1 + g_6 &= g_3 + g_5 = -g_4 + g_7 = a_2, \\ g_1 - g_7 &= -g_2 + g_5 = -g_4 - g_6 = a_3. \end{aligned}$$

From these equations we get

$$\begin{aligned} g_1 &= a_3 + g_7, \\ g_2 &= -a_1 + g_7, \\ g_3 &= a_1 + a_2 - a_3 - g_7, \\ g_4 &= -a_2 + g_7, \\ g_5 &= -a_1 + a_3 + g_7, \\ g_6 &= a_2 - a_3 - g_7. \end{aligned} \tag{17}$$

Applying these relations to (6), we obtain

$$\begin{aligned} & g_7(x - y + z) + g_7(x) - g_7(y) + g_7(z) \\ &= \ g_7(x - y) - g_7(y - z) + g_7(z + x). \end{aligned} \tag{18}$$

Putting $x = y = z$ in (18) yields

$$g_7(2x) = 2g_7(x).$$

By setting $y = z$ in (18) and replacing x and y in the resultant by $(1/2)(x + y)$ and $(1/2)(x - y)$, it follows that g_7 is additive. Put

$$g_7 = a_4,$$

where $a_4 : X \to Y$ is an additive function.

By (17) and the fact $h_i(x) = Q(x)$ $(i = 1, ..., 7)$, we conclude that each relation in (16) is true.

Conversely, if there exist a quadratic function $Q : X \to Y$, constants $c_i \in Y$ $(i = 1, ..., 7)$ with (8) and additive functions $a_i : X \to Y$ $(i = 1, ..., 4)$ such that each of the relations in (16) holds true, then it is obvious that the f_i's satisfy the functional equation (6). □

Remark. As we have seen in Theorems 2 and 3, the pexiderized functional equations (5) and (6), which are pexiderized forms of the equations (3) and (4), respectively, have different general solutions, even though the 'original' functional equations (3) and (4) have the same general solutions in the class of functions between vector spaces as we see in Theorem 1.

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