International Journal of Pure and Applied Mathematics
Volume 1 No. 1 2002, 49-66

BIORTHOGONALITY, PERMANENCE AND APPROXIMATION METHODS FOR DIFFERENTIAL EQUATIONS

M.K. El-Daou[1 §], E.L. Ortiz[2]
[1]Applied Sciences Dept.
College of Technological Studies, KUWAIT
e-mail: modaou@paaet.edu.kw
[2]Dept. of Mathematics,
Imperial College, London, U.K.
e-mail: e.ortiz@ic.ac.uk

Abstract: We introduce sequences of linear functionals uniquely associated with each of several methods for the approximate solution of differential equations and discuss the classification of the latter in terms of these functionals. We use Davis's [4] notion of *permanence* to discuss the structure of these methods further. We show that the Tau method, formulated in terms of sequences of canonical polynomials (see Ortiz [10]), is a permanent technique which makes no direct reference to a Hilbert structure. We also show that, for a given operator, the linear functionals associated with it are biorthogonal to the canonical polynomials. Furthermore, we make precise here a duality which exists between the Tau method and collocation, using the linear functionals associated with each of these methods. We relate them to spectral or Fourier expansion techniques.

AMS Subject Classification: 65L60, 65J10
Key Words: interpolation, collocation, Tau method, canonical polynomials, orthogonal polynomials

Received: January 6, 2002

[§]Correspondence author

1. Introduction

In *Interpolation and Approximation*, Davis [4] gave interpolation processes an abstract framework. He used the concept of biorthogonality to classify interpolation methods in terms of linear functionals uniquely associated with each of them. More recently, Brezinski [1]-[2] gave a comprehensive discussion of biorthogonality with interesting applications, which include differential equations.

For Langrange's method, the addition of a further interpolation point makes it necessary to recompute all previously determined coefficients, while for Newton's method it is only necessary to add just one more term, leaving all previously computed coefficients unaltered. Davis called *permanent* those interpolation methods which have this last property.

In this paper we attempt a similar analysis for methods for the approximate solution of differential equations. We use the abstract framework introduced in Davis [4] to discuss the notion of permanence in the context of numerical methods for differential equations. The notions of permanence and biorthonormality are used further in the discussion of structural relations between different numerical methods. We concentrate our discussion on three *polynomial* methods: the Tau method, collocation and Fourier spectral methods, that is, techniques based, on one way or other, on the use of truncated polynomial expansions. In a series of papers aimed at a discussion across numerical methods El-Daou, Ortiz and Samara [5], El-Daou and Ortiz ([6]-[7]-[8]) have shown that it is possible to simulate analytically a large class of polynomial and discrete numerical approximation methods using the recursive formulation of the Tau method as an analytical tool. Through these equivalence results, the techniques developed in this paper apply to a much wider class of numerical techniques which includes Galerkin's method and discrete variable techniques.

We identify the functionals which define a Tau method approximate solution of a differential equation defined by a differential operator D and show that they are biorthogonal to the sequence of canonical polynomials associated with D (see Ortiz [10]). We also give a precise meaning to the duality between the Tau method and collocation showing that these functionals, which we call *Lanczos's functionals*, are dual through an interpolation process in the image of D, to the *point evaluation functionals* which define the collocation method. The latter are realized by δ-functions defined in a space of distributions, the former are functionals which detect individual coefficients in an expansion.

Furthermore, we show that the Tau method, which makes no reference to a Hilbert structure, is the only *permanent* method among those considered in this paper. The connection between this last property and the notion of canonical polynomials is also discussed.

Taking into account the possibility of simulating other methods in terms of the Tau method, our results indicate that a large family of methods for the numerical solution of differential equations can be formulated in a recursive form. This has some interest in relation to modern computer architectures.

2. Permanence Representations

In this section we review the concept of *permanence*, and start reviewing some basic results and definitions on generalized interpolation due to Davis [4].

Let X be a real (or complex) vector space and let $\mathcal{L}(X)$ indicate a space of linear functionals acting on elements of X. The vector space of linear operators acting on the elements of X will be denoted by $\mathcal{A}(X)$. Given $\{x_k, x_{k+1}, \ldots, x_n\} \subset X$ and $\{L_k, L_{k+1}, \ldots, L_n\} \subset \mathcal{L}(X)$, $k \leq n \in \mathbf{N}$, we set

$$\underline{x_k^n} := (x_k, x_{k+1}, \ldots, x_n), \quad \underline{x} := \underline{x_0^\infty},$$

$$\underline{L_k^n} := (L_k, L_{k+1}, \ldots, L_n), \quad \underline{L} := \underline{L_0^\infty}$$

and

$$G[\underline{L_k^n}, \underline{x_k^n}] := \det\,[L_i(x_j)]_{i,j=k}^n\,. \tag{1}$$

Definition 1. We say that $\underline{x_0^n}$ and $\underline{L_0^n}$ define a biorthonormal system if $L_i(x_j) = \delta_{ij}$ for all $i, j = 0, 1, \ldots, n$.

We require some preliminary results (Lemma 1, Theorems 1 and 2), proofs of which are given in Davis [4].

Lemma 1. *If $\underline{x_0^n}$ and $\underline{L_0^n}$ are independent then*

$$G[\underline{L_0^n}, \underline{x_0^n}] \neq 0. \tag{2}$$

Conversely, if either $\underline{x_0^n}$ or $\underline{L_0^n}$ is independent and (2) holds then the other set is also independent.

Theorem 1. (Newton's Interpolation Formula) *Let X be an infinite dimensional linear space. Let $\{x_j; j \in \mathbf{N}\}$ be a sequence of elements of X such that for each $n \geq 1$, $\underline{x_0^n}$ is independent. Assume further that $\{L_j; j \in \mathbf{N}\}$ is a* *sequence of linear functionals in* $\mathcal{L}(X)$ *such that for each* $n \geq 1$, $G[\underline{L_0^n}, \underline{x_0^n}] \neq 0$. *Then two uniquely determined sequences* $\{\hat{x}_j; j \in \mathbf{N}\} \subset X$ *and* $\{\hat{L}_j; j \in \mathbf{N}\} \subset \mathcal{L}(X)$ *can be constructed:*

$$\hat{x}_j = \frac{1}{G[\underline{L_0^{j-1}}, \underline{x_0^{j-1}}]} \begin{vmatrix} L_0(x_0) & L_0(x_1) & \dots & L_0(x_j) \\ L_1(x_0) & L_1(x_1) & \dots & L_1(x_j) \\ \vdots & \vdots & \ddots & \vdots \\ L_{j-1}(x_0) & L_{j-1}(x_1) & \dots & L_{j-1}(x_j) \\ x_0 & x_1 & \dots & x_j \end{vmatrix} \tag{3}$$

$$\hat{L}_j = \frac{1}{G[\underline{L_0^{j}}, \underline{x_0^{j}}]} \begin{vmatrix} L_0(x_0) & L_1(x_0) & \dots & L_j(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ L_0(x_{j-1}) & L_1(x_{j-1}) & \dots & L_j(x_{j-1}) \\ L_0 & L_1 & \dots & L_j \end{vmatrix}$$

such that

$$\hat{L}_i(\hat{x}_j) = \delta_{ij} \ \ \forall i, j \geq 0.$$

Furthermore, for all $n \in \mathbf{N}$, *if* $x \in span\{\underline{x_0^n}\}$ *then*

$$x = \sum_{j=0}^{n} \hat{L}_j(x)\hat{x}_j. \tag{4}$$

Theorem 2. (Lagrange's Interpolation Formula) *Let*

$$X := span\{\underline{x_0^n}\}$$

and let $\underline{L_0^n}$ *be independent in* $\mathcal{L}(X)$. *Then n+1 uniquely determined independent elements* $\{\hat{x}_0, \hat{x}_1, \dots, \hat{x}_n\}$ *can be constructed*

$$\hat{x}_i = a_{i0}x_0 + a_{i1}x_1 + \cdots + a_{in}x_n; \ i = 0, 1, ..., n, \tag{5}$$

such that

$$L_i(\hat{x}_j) = \delta_{ij}.$$

Furthermore, (i) if $x \in span\{\underline{x_0^n}\}$ *then*

$$x = \sum_{k=0}^{n} L_k(x)\hat{x}_k\,, \tag{6}$$

and (ii) if $\{\omega_0, \omega_1, ..., \omega_n\}$ *are arbitrary numbers from* $\mathbf{R}$ *then*

$$x := \frac{1}{G[\underline{L_0^n}, \underline{x_0^n}]} \begin{vmatrix} 0 & x_0 & x_1 & \dots & x_n \\ \omega_0 & L_0(x_0) & L_0(x_1) & \dots & L_0(x_n) \\ \omega_1 & L_1(x_0) & L_1(x_1) & \dots & L_1(x_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_n & L_n(x_0) & L_n(x_1) & \dots & L_n(x_n) \end{vmatrix}$$

satisfies uniquely $L_i(x) = \omega_i$ *for all* $i \in \{0, 1, ..., n\}$.

Comparing (4) and (6) we notice that if wish to add one more term to the latter, say x_{n+1}, we must recompute $\{\hat{x}_0, \hat{x}_1, ..., \hat{x}_n\}$ using (5); this is because each $\hat{x}_i$ depends on *all the basis elements* $\{x_0, x_1, ..., x_n\}$. In the case of (4) each $\hat{x}_i$ depends only on *the first* i *basis elements* $\{x_0, x_1, ..., x_i\}$ as we see form (3); hence adding more terms does not require changing any of those already computed. Consequently, Newton's generalized representation of the element of X is said to be *permanent*.

We shall now define two types of representations:

Definition 2. Let $X := \text{span}\{\underline{x_0^n}\}$ and assume that $\underline{\hat{x}_0^n}$ is another basis of X. We say that $\underline{\hat{x}_0^n}$ defines permanent representations for the elements of X if the transformation matrix from $\underline{x_0^n}$ to $\underline{\hat{x}_0^n}$, denoted by $\text{M}(\underline{x_0^n}, \underline{\hat{x}_0^n})$, is lower triangular; equivalently, for all $i = 0, 1, ..., n$, $\hat{x}_i$ depends linearly only on $x_0, x_1, ..., x_i$.

On the other hand, if $\text{M}(\underline{x_0^n}, \underline{\hat{x}_0^n})$ is not lower triangular, then $\underline{\hat{x}_0^n}$ defines nonpermanent representations for the elements of X.

In the next section we shall examine the permanence properties of three different approximation methods for the numerical solution of differential equations: the Tau method, the collocation technique and orthogonal expansion methods. Let us introduce some definitions and notations:

Let $A \in \mathcal{A}(X)$, the space of linear operators acting on the elements of $X := \text{span}\{\underline{x}\}$ with $\underline{x} := (x_0, x_1, x_2, ...)$. Then, for all $i \in \mathbf{N}$ we can write

$$A(x_i) = a_{i0}x_0 + a_{i1}x_1 + a_{i2}x_2 + \dots \quad (a_{ij} \in \mathbf{R}). \tag{7}$$

- The operator A will be called *banded-from-above* if the matrix $\mathbf{M}(A, \underline{x})$ given by

$$\mathbf{M}(A,\underline{x}) := (a_{ij})_{i,j\in\mathbf{N}} = \begin{pmatrix} a_{00} & a_{01} & \cdots \\ a_{10} & a_{11} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \tag{8}$$

is banded from above, that is there exists an integer $k \in \mathbf{Z}$ such that

$$a_{ij} = 0 \;\; \forall (j-i) \geq k+1; \;\; i,j \geq 0. \tag{9}$$

$\mathcal{A}_b(X)$ will denote the linear subspace of $\mathcal{A}(X)$ which contains all the banded-from-above operators.

- Let h be an *integer-valued function* acting on the elements of $\mathcal{A}_b(X)$ defined by

$$h(A) := \inf\{k;\; k \text{ satisfies (9)}\}. \tag{10}$$

Following Ortiz [10], $h(A)$ will be called the *height* of A.

- Let us set:

$$W(A) := \{i \in \mathbf{N};\; a_{i\,h(A)+i} \neq 0\}, \quad W^c(A) := \mathbf{N} - W(A). \tag{11}$$

We say that $W(A)$ is *disconnected* in $\mathbf{N}$ if there exists a finite integer $p \geq 2$ and t subsets $\{W_1, W_2, ..., W_p\} \subset \mathbf{N}$ such that

$$W(A) = \bigcup_{i=1}^{p} W_i \;\; \text{and} \;\; \sup(W_i) < m_i < \inf(W_{i+1}), \tag{12}$$

for some $m_i \in \mathbf{N}$, $i \in \{1, 2, \ldots, p\}$. Otherwise $W(A)$ is called *connected.*

- For all $k \in \mathbf{N}$, let $\pi_k \in \mathcal{L}(X)$ be defined by

$$\pi_k : \sum_{i\in\mathbf{N}} \lambda_i x_i \in X \longmapsto \lambda_k \in \mathbf{R}\,, \quad (k \in \mathbf{N}). \tag{13}$$

which we shall call *a k-canonical mapping* of X.

3. Permanence of Methods

Definition 3. An approximation method is permanent (resp. nonpermanent) if the approximations it defines have permanent (resp. nonpermanent) representations.

3.1. Permanent Approximation Methods

i) Truncated orthogonal expansion. We shall use this simple method to illustrate the concept of permanence. Let $\mathcal{L}^2[-1,1]$ be the Hilbert space of square integrable functions on $[-1,1]$. Assume that $\underline{V} = \{V_i(x), i \in \mathbf{N}\}$ is an orthonormal polynomial basis of $\mathcal{L}^2[-1,1]$ with weight $w(x)$. Then any function $y(x) \in \mathcal{L}^2[-1,1]$ has a unique convergent expansion of the form

$$y(x) = \sum_{k=0}^{\infty} c_k V_k(x)\,, \qquad (c_k \in \mathbf{R})\,, \tag{14}$$

where

$$c_k = \pi_k(y) := \int_{-1}^{1} y(t)V_k(t)w(t)dt\,, \qquad (k \geq 1)\,. \tag{15}$$

If the series (14) is truncated after the term $c_n V_n(x)$, we obtain a polynomial approximation $y_n(x)$ of $y(x)$, of degree n, given by

$$y_n(x) := \sum_{k=0}^{n} c_k V_k(x). \tag{16}$$

$y_n(x)$ will be called *a truncated orthogonal V-expansion of $y(x)$ of order n.* We now prove that:

Theorem 3. *The method of truncated orthogonal V-expansion is permanent in $\mathcal{L}^2[-1,1]$ with a biorthonormal system defined by $\{V_k; k \in \mathbf{N}\} \subset X$ and $\{\pi_k; k \in \mathbf{N}\} \subset \mathcal{L}(X)$ where $X := span\{V_k; k \in \mathbf{N}\}$.*

Proof. X is an infinite dimensional space generated by $\{V_k; k \in \mathbf{N}\}$ where $\{V_k; k = 0,1,...,n\}$ are independent for all $n \in \mathbf{N}$. The orthonormality of $\{V_k; k \in \mathbf{N}\}$ implies that the functionals $\{\pi_k; k = 0,1,...,n\}$ are independent for all $n \in \mathbf{N}$. So, by Lemma 1

$$G[\underline{\pi_0^{j-1}}, \underline{V_0^{j-1}}] = \pi_0(V_0)\pi_1(V_1)...\pi_{j-1}(V_{j-1}) = 1.$$

From Theorem 1 we can generate two sequences $\{\widetilde{V}_k; k \in \mathbf{N}\}$ and $\{\widetilde{\pi}_k; k \in \mathbf{N}\}$ given by

$$\widetilde{V}_j = \frac{1}{G[\underline{\pi_0^{j-1}}, \underline{V_0^{j-1}}]} \begin{vmatrix} \pi_0(V_0) & 0 & \dots & 0 \\ 0 & \pi_1(V_1) & 0 & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \dots & \pi_{j-1}(V_{j-1}) & 0 \\ V_0 & V_1 & \dots & V_j \end{vmatrix} \tag{17}$$

and

$$\widetilde{\pi}_j = \frac{1}{G[\underline{\pi_0^j}, \underline{V_0^j}]} \begin{vmatrix} \pi_0(V_0) & 0 & \dots & 0 \\ 0 & \pi_1(V_1) & 0 & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \dots & \pi_{j-1}(V_{j-1}) & 0 \\ \pi_0 & \pi_1 & \dots & \pi_j \end{vmatrix} \tag{18}$$

such that

$$\widetilde{\pi}_i(\widetilde{V}_j) = \delta_{ij}.$$

Moreover,

$$u = \sum_{k=0}^{n} \widetilde{\pi}_k(u)\widetilde{V}_k \tag{19}$$

for all $u(x) \in \text{span}\{V_0, V_1, \dots, V_n\}$ and $n \in \mathbf{N}$. From (17) and (18) we get

$$\tilde{V}_j = V_j \text{ and } \tilde{\pi}_j = \frac{1}{\pi_j(V_j)}\pi_j \equiv \pi_j.$$

Hence (19) becomes

$$u = \sum_{k=0}^{n} \pi_k(u)V_k,$$

which is the truncated orthogonal V-expansion (16). □

Let us now consider some approximation methods for the solution of differential equations and discuss their permanence properties. In particular we shall identify the biorthonormal systems which characterize each one of the methods we referred to in the introduction. We begin with the Tau method:

ii) Biorthonormality and canonical polynomials. Let $\mathcal{A}_b(X) := \mathcal{D}$ be the class of differential operators with polynomial coefficients acting on $X := \mathcal{C}^\nu[-1,1]$, the space of ν–times continuously differential functions. Let $D \in \mathcal{D}$ be a differential operator of order $\nu \in \mathbf{N}$ and $\{B_i, i = 1, 2, ..., \nu\}$ be ν linear functionals defined on $\mathcal{C}^\nu[-1,1]$.

Assume that $y(x)$ is the exact solution of the boundary value problem:

$$Dy(x) = f(x), \qquad x \in [-1,1], \tag{20}$$

$$B_i(y) = \gamma_i, \qquad i = 1, 2, ..., \nu, \quad \gamma_i \in \mathbf{R}, \tag{21}$$

where $f(x) := \sum_{i=0}^{d_f} f_i x^i$, ($f_i \in \mathbf{R}$) is a polynomial of degree $d_f \in \mathbf{N}$. Ortiz has shown in [10] that every linear differential operator $D \in \mathcal{D}$ is uniquely associated with a sequence $\underline{Q} := \{Q_n(x);\ n \in \mathbf{N}-W(D)\}$ of *canonical polynomials* defined by

$$DQ_n = x^n + r_n(x)\,,$$

where $r_n(x) \in R_W := \text{span}\{x^i; i \notin W(D)\}$. He has also given a self starting, recursive formula for the generation of the elements of $\underline{Q}$.

In this section we show that the canonical polynomials $\{Q_m(x);\ m \in \mathbf{N} - W(D)\}$ are biorthonormal to linear functionals related to the given differential operator and to the canonical mappings of $\mathcal{C}^\nu[-1,1]$. We also show that each $Q_m(x)$ has a determinant form.

For any operator $D \in \mathcal{D}$ the set $W(D)$, defined by (11), has a finite complement in $\mathbf{N}$ (Ortiz [10]). This means that $W(D)$ is not necessarily connected in $\mathbf{N}$, but has always an infinite connected component in it. Let us consider first the case of connected $W(D)$ and thereafter we generalize our results.

Definition 4. Let $D \in \mathcal{D}$ and let π_k be a k-canonical mapping defined by (13). We call $L_k := \pi_k D$ a k-Lanczos functional.

We now show that Lanczos functionals and canonical polynomials are biorthonormal sequences. For simplicity, $W(D)$ and $h(D)$ will be indicated by W and h respectively.

Lemma 2. *Assume that $D \in \mathcal{D}$ and that $W := W(D)$ is connected; let then $\kappa := \inf[W]$ and let $X_\kappa := span\{x^n; n \geq \kappa\}$. For all $k \geq \kappa$ let π_k be the k-canonical mapping of X_κ and L_k be the k-Lanczos functional. Then for all m and $n \geq \kappa$ we have*

$$L_{n+h}\left(Q_{m+h}(x)\right) = \delta_{mn} \tag{22}$$

where $Q_i(x)$ is the i^{th} canonical polynomial of D.

Proof. Let m and $n \geq \kappa$. From the definitions of L_n and $Q_m(x)$ it follows that

$$L_{n+h}\left(Q_{m+h}(x)\right) = \pi_{n+h}[DQ_{m+h}] = \pi_{n+h}\left[x^{m+h} + r_{m+h}(x)\right]\,,$$

when $m = n$, we have

$$\pi_{n+h}\left[x^{m+h} + r_{m+h}(x)\right] = 1\,,$$

because $r_{m+h}(x) \in R_W := \text{span}\{x^i; i \notin W\}$; i.e $r_{m+h}(x)$ has no nonzero coefficient along x^{n+h} for any $n \in W$.

On the other hand, if $n < m$ and $L_{n+h}\left(Q_{m+h}(x)\right) \neq 0$, we can find a nonzero $\alpha \in \mathbf{R}$ such that

$$\pi_{n+h}\left[x^{m+h} + r_{m+h}(x)\right] = \alpha.$$

This implies that $r_{m+h}(x)$ can be written as

$$r_{m+h}(x) = \alpha x^{n+h} + \hat{r}_{m+h}(x); \quad \hat{r}_{m+h}(x) \in R_W,$$

which contradicts the facts that $r_{m+h}(x) \in R_W$ and $n \in W$. □

Let $P_m(x) := Dx^m$ be the *generating polynomial* associated with the operator D (see Ortiz [10]) and let $P_{mk} := \pi_k[P_m]$ be the k-th coefficient of $P_m(x)$. We now give our main result concerning canonical polynomials $\underline{Q}$:

Theorem 4. *Suppose that the assumptions of Lemma 2 hold true. Then there exist uniquely determined sequences* $\{Q^*_{h+m}(x); m \geq \kappa\} \subset X_\kappa$ *and* $\{L^*_{h+m}; m \geq \kappa\}$ *given by*

$$Q^*_{h+m} = \frac{1}{G[\underline{L^{h+m-1}_{h+\kappa}}, \underline{x^{m-1}_{\kappa}}]} \times \begin{vmatrix} P_{\kappa\ \kappa+h} & P_{\kappa+1\ \kappa+h} & \ldots & P_{m\ \kappa+h} \\ 0 & P_{\kappa+1\ \kappa+h+1} & \ldots & P_{m\ \kappa+h+1} \\ 0 & 0 & & \ldots \\ \vdots & \vdots & \ddots & \ldots \\ 0 & \ldots & P_{m-1\ m+h-1} & P_{m\ m+h-1} \\ x^\kappa & x^{\kappa+1} & \ldots & x^m \end{vmatrix} \tag{23}$$

and

$$L^*_{h+m} = \frac{L_{h+m}}{P_{m\ h+m}},$$

such that

$$Q_{h+m}(x) = \frac{1}{P_{m\ m+h}} Q^*_{h+m}(x), \quad \forall m \geq \kappa. \tag{24}$$

Further, for all $y \in$span$\{x^i; i = \kappa, \kappa+1, ..., \kappa+m\}$ *we have*

$$y = \sum_{i=\kappa}^{\kappa+m} L^*_{h+i}(y) Q^*_{h+i} = \sum_{i=\kappa}^{\kappa+m} L_{h+i}(y) Q_{h+i}.$$

Proof. Let us show that, for any $m \in \mathbf{N}$, SX $:= \{x^{\kappa+i}; 0 \le i \le m-\kappa\}$ and SL $:= \{L_{h+\kappa+i}; 0 \le i \le m-\kappa\}$ are independent. Given i and $j \ge 0$, we write

$$\begin{aligned} L_{h+\kappa+i}[x^{\kappa+j}] &= \pi_{h+\kappa+i}[Dx^{\kappa+j}] = \pi_{h+\kappa+i}[P_{\kappa+j}(x)] \\ &= \pi_{h+\kappa+i}\left[P_{\kappa+j\ 0} + P_{\kappa+j\ 1}x + ... + P_{\kappa+j\ \kappa+j+h}x^{\kappa+j+h}\right] \\ &= \begin{cases} P_{\kappa+j\ \kappa+j+h}\,, & \text{if} \quad i=j\,, \\ 0\,, & \text{if } i>j\,, \end{cases}. \end{aligned}$$

which implies that

$$G[\underline{L_{h+\kappa}^{h+m-1}}, \underline{x_{\kappa}^{m-1}}] = P_{\kappa\ \kappa+h} \times P_{\kappa+1\ \kappa+h+1} \times P_{m-1\ m+h-1} \neq 0$$

and, therefore, by Lemma 1, sequences SX and SL are independent and satisfy the conditions of Theorem 1. From the latter two sequences can be found, $\{Q^*_{h+i}; i \ge \kappa\} \subset X_\kappa$ given by (23) and $\{L^*_{h+i}; i \ge \kappa\} \subset \mathcal{L}(X_\kappa)$ given by

$$\begin{aligned} L^*_{h+m} &= \frac{1}{G[\underline{L_{h+\kappa}^{h+m}}, \underline{x_{\kappa}^{m}}]} \begin{vmatrix} P_{\kappa\ h+\kappa} & 0 & \dots & 0 \\ P_{\kappa+1\ h+\kappa} & P_{\kappa+1\ h+\kappa+1} & 0 & \vdots \\ \vdots & \vdots & \ddots & \\ P_{m-1\ h+m-1} & & P_{m-1\ h+m-1} & 0 \\ L_{h+\kappa} & L_{h+\kappa+1} & \dots & L_{h+m} \end{vmatrix} \\ &= \frac{L_{h+m}}{P_{m\ h+m}}, \end{aligned} \tag{25}$$

such that

$$L^*_{m+h}(Q^*_{n+h}) = L_{m+h}\left(\frac{Q^*_{n+h}}{P_{m\ m+h}}\right) = \delta_{mn}\,, \quad \forall m,n \ge \kappa. \tag{26}$$

On account of (22) it follows that

$$Q_{m+h}(x) = \frac{1}{P_{m\ m+h}} Q^*_{m+h}(x)\,, \quad \forall m \ge \kappa. \qquad \square$$

Corollary 5. *Let $D \in \mathcal{D}$. Suppose that W is disconnected in the sense of (12), $\kappa := \inf(W)$. Then there exists a uniquely determined sequence $\{Q^*_{h+m}(x); m \in W\} \subset span\{x^m; m \in W\}$ given by*

$$Q^*_{h+m} = \frac{1}{G_W[\underline{L_{h+\kappa}^{h+m-1}}, \underline{x_{\kappa}^{m-1}}]} det \begin{bmatrix} \mathbf{M}_W^{h+m} \\ \underline{x_W^{\kappa+m}} \end{bmatrix}, \tag{27}$$

where

$$
\begin{aligned}
G_W[\underline{L_{h+k}^{h+m}}, \underline{x_k^m}] &:= \det\left[L_{h+i}(x^j)\right]^{m}_{\substack{i,j=k\\ i,j\in W}}, \\
\mathbf{M}_W^{h+m} &:= [P_{i\ h+j}]^T_{\substack{i=\kappa,\kappa+1,\dots,m\\ j=\kappa,\kappa+1,\dots,m-1\\ i,j\in W}}, \\
\underline{x_W^{\kappa+m}} &:= (x^i)_{\substack{i=\kappa,\kappa+1,\dots,m\\ i\in W}},
\end{aligned}
$$

such that the elements of the sequence of canonical polynomials associated with D are given by

$$Q_{h+m}(x) = \frac{1}{P_{m\ h+m}} Q^*_{h+m}(x)\,, \quad \forall m \in W.$$

In the Tau method (Ortiz [10]) an approximate polynomial solution of $y(x)$ is a special element $y_n(x)$ generated in terms of the canonical polynomials basis; it is defined by a transformation which leaves the coefficients of the right hand side polynomial terms invariant (see Ortiz [11]). Therefore:

Corollary 6. *The Tau method is a permanent approximation method in $\mathcal{C}^\nu[-1,1]$.*

Let us consider an illustrative numerical example on the generation of the sequence of canonical polynomials using the results of this section, in particular Corollary 5.

Example 1. Let us consider the differential operator $D \in \mathcal{D}$ given by

$$Dy(x) := (2+2x^3)y''(x) - \frac{43}{5}(1+x^2)y' + 3xy(x)\,.$$

Let us generate the canonical polynomials of D. Using Corollary 5, we have

$$P_k(x) := Dx^k = \left(-2k+2k^2\right)x^{k-2} - \frac{43k}{5}x^{k-1} + \left(3 - \frac{53k}{5} + 2k^2\right)x^{k+1}.$$

Clearly $h = 1$, $W = \{0,1,2,3,4\} \cup \{k; k \geq 6\}$; $\kappa := \inf(W) = 0$ and

$$P_{k\,j} = \begin{cases} -2k + 2k^2\,, & \text{if j=k-2,} \\ -43k/5\,, & \text{if j=k-1,} \\ 5 - 53k/5 + 2k^2\,, & \text{if j=k+1,} \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Replacing in (27) we find $Q^*_{m+1}(x)$ for all $m \in W$. For example, when $m = 1$,

$$Q^*_2(x) = \frac{1}{G_W[\underline{L^1_1}, \underline{x^0_0}]} \begin{vmatrix} 3 & 0 \\ 1 & x \end{vmatrix}\,, \qquad \text{where} \quad G_W[\underline{L^1_1}, \underline{x^0_0}] := 3$$

which gives $Q_2(x) = \frac{-5x}{28}$. Similarly, when $m = 6$

$$Q^*_7(x) = \frac{1}{G_W[\underline{L^6_1}, \underline{x^5_0}]} \begin{vmatrix} 3 & 0 & \frac{-86}{5} & 12 & 0 & 0 \\ 0 & \frac{-28}{5} & 0 & \frac{-129}{5} & 24 & 0 \\ 0 & 0 & \frac{-51}{5} & 0 & \frac{-172}{5} & 0 \\ 0 & 0 & 0 & \frac{-54}{5} & 0 & 60 \\ 0 & 0 & 0 & 0 & \frac{-37}{5} & \frac{-258}{5} \\ 1 & x & x^2 & x^3 & x^4 & x^6 \end{vmatrix}\,,$$

where

$$G_W[\underline{L^6_1}, \underline{x^5_0}] := \begin{vmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & \frac{-28}{5} & 0 & 0 & 0 \\ \frac{-86}{5} & 0 & \frac{-51}{5} & 0 & 0 \\ 12 & \frac{-129}{5} & 0 & \frac{-54}{5} & 0 \\ 0 & 24 & \frac{-172}{5} & 0 & \frac{-37}{5} \end{vmatrix}$$

giving

$$Q_7(x) = \frac{3187336}{322677} - \frac{431075x}{88578} + \frac{73960x^2}{35853} + \frac{250x^3}{513} - \frac{430x^4}{703} + \frac{5x^6}{57}\,.$$

After some algebraic manipulation it follows that the determinant satisfies the same recursive relation as canonical polynomials, namely $\forall k \geq 0$,

$$Q_{k+1}(x) = \left(x^k - (2k^2 - 2k)Q_{k-2}(x) + \frac{43k}{5}Q_{k-1}(x)\right) \Big/ (3 - \frac{53k}{5} + 2k^2).$$

3.2. Nonpermanent Approximation Methods

i) Approximate orthogonal V-expansion methods. It is obvious that the evaluation of the coefficients $\{c_k; k \geq 0\}$ of an orthogonal expansion by means of (15) is more complex when $y(x) \in \mathcal{C}^\nu[-1,1]$ is *implicitly* defined by a differential equation. In such case we can, however, find *approximations* of those coefficients. One of such approaches is the "Chebyshev" method of Lanczos [9] (see also Clenshaw [3]), now widely used for the numerical treatment of differential equations. The main idea of such techniques when applied to problem (20)-(21), for a given $m \in \mathbf{N}$, is to find a polynomial $y^*(x)$ of degree $\leq m+\nu-1$ which approximates the exact solution $y(x)$ and is expressed in terms of orthogonal polynomials, namely:

$$y^*(x) = \sum_{k=0}^{m+\nu-1} c_k V_k(x). \tag{28}$$

Its coefficients $\{c_k, k = 0, 1, \ldots, m+\nu-1\}$ must satisfy two different groups of conditions:

(i) $(\pi_k D)(y^*) = \pi_k(f)$ for all $k = 0, 1, ..., m-1$.

(ii) $y^*(x)$ must satisfy the boundary conditions (21).

Let

$$L_i = \pi_i D; \;\; 0 \leq i \leq m-1\,, \tag{29}$$

$$L_i = B_{i-m+1}; \; N \leq i \leq m+\nu-1\,, \tag{30}$$

$$w_i = \begin{cases} \pi_i(f)\,, & \text{when } \; i = 0, 1, ..., m-1\,, \\ \gamma_{i-N+1}\,, & \text{when } \; i = m, m+1, ..., m+\nu-1. \end{cases} \tag{31}$$

Then conditions (i)-(ii) yield the algebraic system:

$$\mathbf{M}\underline{\underline{c}} = \underline{\underline{w}}\,, \tag{32}$$

where

$$\mathbf{M} := \begin{bmatrix} L_0(V_0) & L_0(V_1) & ... & L_0(V_{m+\nu-1}) \\ L_1(V_0) & L_1(V_1) & ... & L_1(V_{m+\nu-1}) \\ \vdots & \ddots & .... & \vdots \\ L_m(V_0) & L_m(V_1) & ... & L_m(V_{m+\nu-1}) \end{bmatrix},$$

$$\underline{\underline{c}} := \begin{bmatrix} c_0 & c_1 & \dots & c_{m+\nu-1} \end{bmatrix}^T \text{ and } \underline{\underline{w}} := \begin{bmatrix} w_0 & w_1 & \dots & w_{m+\nu-1} \end{bmatrix}^T .$$

Hence

$$\underline{\underline{c}} = \mathbf{M}^{-1}\underline{\underline{w}}.$$

Let us show that such procedure is nonpermanent.

Lemma 7. *The set of linear functionals $\{L_i; 0 \leq i \leq m+\nu-1\}$ given by (30) is independent in $\mathcal{L}(X_{m+\nu-1})$.*

Proof. Clearly $\{L_i,\ 0 \leq i \leq m-1\}$ are independent. Suppose that $\{L_i,\ 0 \leq i \leq m-1\} \cup \{B_1\}$ are not independent. Then there exists a set of numbers $\{\alpha_0, \alpha_1, ..., \alpha_{m-1}\}$ such that:

$$B_1(y) = \alpha_0 L_0(y) + \alpha_1 L_1(y) + ... + \alpha_{m-1} L_{m-1}(y) \text{ for all } y(x)$$

and, in particular:

$$\sum_{i=0}^{m+\nu-1} c_i B_1(V_i) = \alpha_0 \sum_{i=0}^{m+\nu-1} c_i L_0(V_i) + ... + \alpha_{m-1} \sum_{i=0}^{m+\nu-1} c_i L_{m-1}(V_i)\,,$$

which means that

$$\mathbf{M}_{m+1} = \alpha_0 \mathbf{M}_1 + ... + \alpha_{m-1} \mathbf{M}_m\,,$$

i.e the (m+1)-th row of matrix $\mathbf{M}$ is a linear combination of the first m rows. This conrtradicts the assumption that $\mathbf{M}$ is invertible and thus $\{B_1,\ L_i,\ 0 \leq i \leq m-1\}$ are independent. □

Corollary 8. *Let $X = span\{V_0, V_1, \dots, V_{m+\nu-1}\}$ and let $\{L_i;\ 0 \leq i \leq m+\nu-1\}$ be defined as in (30). Then:*

a) The approximate orthogonal V-expansion method is nonpermanent in $\mathcal{C}^\nu[-1,1]$ and associated with a biorthonormal system defined by the two sequences

$$\{V_0, V_1, ..., V_{m+\nu-1}\} \subset X_{m+\nu-1} \text{ and } \{L_0, L_1, ..., L_{m+\nu-1}\} \subset \mathcal{L}(X_{m+\nu-1}).$$

b) Further, the approximate solution $y_{m+\nu-1}$ is given for all $m \in \mathbf{N}$ by

$$y_{m+\nu-1}$$

$$= \frac{-1}{G} \begin{vmatrix} 0 & V_0 & V_1 & \dots & V_{m+\nu-1} \\ w_0 & L_0(V_0) & L_0(V_1) & \dots & L_0(V_{m+\nu-1}) \\ w_1 & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{m+\nu-1} & L_{m+\nu-1}(V_0) & L_{m+\nu-1}(V_1) & \dots & L_{m+\nu}(V_{m+\nu-1}) \end{vmatrix},$$

where $G := det\,|L_i(V_j)|_{i,j=0}^{m+\nu-1}$ *and the* w_i*'s are given by (31).*

ii) Collocation from the stand point of biorthonormality. Let $\{z_i, i = 0, ..., m-1\}$ be m fixed distinct points of $[-1,1]$. Let $\pi_k^* \in \mathcal{L}(X)$ be defined by

$$\pi_k^*(g) := g(z_k) \text{ for all } 0 \le k \le m-1,$$

which we call *a k-point evaluation mapping* of X.

The main idea of collocation, when applied to problem (20)-(21), is to find a polynomial $Y(x)$

$$Y(x) = \sum_{j=0}^{m+\nu-1} c_j x^j \tag{33}$$

of degree $\le m+\nu-1$ which approximates the exact solution $y(x)$; it has coefficients $\{c_j, j = 0, 1, \dots, m+\nu-1\}$ fixed by two groups of conditions:

(i) $(\pi_k^* Dy)(Y) = \pi_k^*(f)$ for all $k = 0, 1, ..., m-1$;

(ii) $Y(x)$ must satisfy the boundary conditions (21).

Let

$$L_i = \pi_i^* D, \quad 0 \le i \le m-1, \tag{34}$$
$$L_i = B_{i-m+1}, \quad N \le i \le m+\nu-1, \tag{35}$$

$$w_i = \begin{cases} \pi_i^*(f), & \text{when } i = 0, 1, ..., m-1, \\ \gamma_{i-m+1}, & \text{when } i = m, m+1, ..., m+\nu-1. \end{cases} \tag{36}$$

These conditions yield an algebraic system similar to (32) with $L_k := \pi_k^* D$ and V_k replaced by x^k. Using the arguments of the pervious section we can show that Lemma 7 holds true for $L_i := \pi_i^* D$. A direct application of Theorem 2 gives the following result:

Corollary 9. *Let* $X = span\{1, x, \ldots, x^{m+\nu-1}\}$ *and let*

$$L_i = \pi_i^* D\,,\ \ (0 \le i \le m-1) \text{ and } L_j = B_{j-m+1}\,,\ \ (N \le j \le m+\nu-1)\,.$$

a) The collocation method is a nonpermanent method associated with a biorthonormal system consisting of sequences

$$\{1, x, \ldots, x^{m+\nu-1}\} \subset X \text{ and } \{L_0, L_1, ..., L_{m+\nu-1}\} \subset \mathcal{L}(X)\,.$$

b) The collocation polynomial solution at $z_0, z_1, ..., z_{m-1}$ *of problem (20)–(21) is uniquely determined by*

$$Y = \frac{-1}{G}\begin{vmatrix} 0 & 1 & x & \ldots & x^{m+\nu-1} \\ \omega_0 & L_0(1) & L_0(x) & \ldots & L_0(x^{m+\nu-1}) \\ \omega_1 & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \omega_{m+\nu-1} & L_{m+\nu-1}(1) & L_{m+\nu-1}(x) & \ldots & L_{m+\nu}(x^{m+\nu-1}) \end{vmatrix}, \tag{37}$$

where $G := det\left|L_i(x^j)\right|_{i,j=0}^{m+\nu-1}$ *and*

$$w_i = \begin{cases} \pi_i^*(f)\,, & \text{when } \ i = 0, 1, ..., m-1\,, \\ \gamma_{i-m+1}\,, & \text{when } \ i = m, m+1, ..., m+\nu-1\,. \end{cases}$$

References

[1] C. Brezinski, *Biorthogonality and its Applications to Numerical Analysis*, Marcel Dekker, Inc., New York (1992).

[2] C. Brezinski, *Projection Methods for Systems of Equations*, Studies in Computational Mathematics **7**, Elsevier Science (1997).

[3] C. W. Clenshaw, The numerical solution of linear differential equations in Chebyshev series, *Proc. Camb. Phil. Soc.*, **53** (1957), 134-149.

[4] P. Davis, *Interpolation and Approximation*, Dover, Inc., New York (1975).

[5] M. K. El-Daou, E. L. Ortiz and H. Samara, A unified approach to the Tau method and Chebyshev series expansions techniques, *Comput. Maths. Appls.*, **25** (1993), 73-82.

[6] M. K. El-Daou and E. L. Ortiz, A recursive formulation of Galerkin's Method in terms of the Tau method: bounded and unbounded domains, *Comput. Maths. Appls.*, **35** (1998), 83-94.

[7] M. K. El-Daou and E. L. Ortiz, The Tau method as an analytic tool in the discussion of equivalence results across numerical methods, *Computing*, **60** (1998), 365-376.

[8] M. K. El-Daou and E. L. Ortiz, A recursive formulation of colocation in terms of canonical polynomials, *Computing*, **52** (1994), 177-202.

[9] C. Lanczos, Trigonometric interpolation of empirical and analytical functions, *J. Math. Phys.*, **17** (1938), 123-199.

[10] E. L. Ortiz, The Tau method, *SIAM J. Numer. Anal.*, **6** (1969), 80-92.

[11] E. L. Ortiz, Canonical polynomials in the Lanczos Tau method, In: *Studies in Numerical Analysis* (Ed. B.K.P. Scaife), Academic Press, New York (1974), 73-93.