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# NAVIGATING IN SPACE UNDER CONSTRAINTS

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**Abstract:** Various algorithms have been proposed for the determination of the optimum path(s) in line networks. Moving in space under constraints is a far more complex problem, where research has been relatively scarce. An example would be the determination of the shortest sea course between two harbors. This paper presents a graph-based approach to the problem of the optimum path(s) finding in space; and shows how it can be applied to a variety of spaces and application domains. This approach, although quantitative in nature, it may also support the piecewise qualitative path selection.

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*"When you set out on the voyage to Ithaca, pray that your journey may be long, full of adventures, full of knowledge"* [4]

## 1. Introduction

The determination of the *optimum path* between two physical locations is a very common problem in human navigation and appears very often in applications such as Cartography, Robotics and Geographic Information Systems (GISs). Optimum in this context refers to a minimal accumulation of what amount to incremental travel costs associated with different media. It may be the shortest, fastest, least-expensive, or least-risky path.

When movement is restricted to the chains of a linear network, like road or aircraft corridor networks, a weighted graph can be used as a model and associated algorithms [21], [13], [31], [28], [29] may be applied for the determination of the optimum path.

Moving in space is a far more complex problem, where research has been relatively scarce. Finding the optimum path in space is not only used for travelling or movement. It may also be applied to other planning situations, such as building highways, railways, pipelines and other transport systems. Some representative examples of optimum path finding in space are illustrated in Figure 1. These refer to the determination of the fastest path between two villages (Figure 1a); the shortest sea course between two ports (Figure 1b); the most regular gradient on ground path over a mountainous terrain (Figure 1c); the least-risky path in a hostile environment, for instance, the path with the maximum concealment time vis-a-vis an enemy or an observer (Figure 1d).

Clearly, the space under study has its own peculiarities. For instance, it may be composed of a number of regions or volumes with different travel cost values assigned to them (e.g. walking on grass or sand); it may involve various means of travel (e.g. walking or driving); the direction of movement may introduce a variable cost value (e.g. moving with or against the wind).

The problem of the optimum path finding in space has been examined in the past [38]. The solutions provided are mostly confined to movement on a plane surface (i.e. cross country movement) consisting of zones characterized by different travel cost values and may be classified into raster-based [23], [15], [5], [38], [7], [8], and vector-based [25] approaches. An interesting comparison of algorithms for cross-country movement can be found in [38]. Raster-based algorithms have several advantages since they are easy to implement and perform

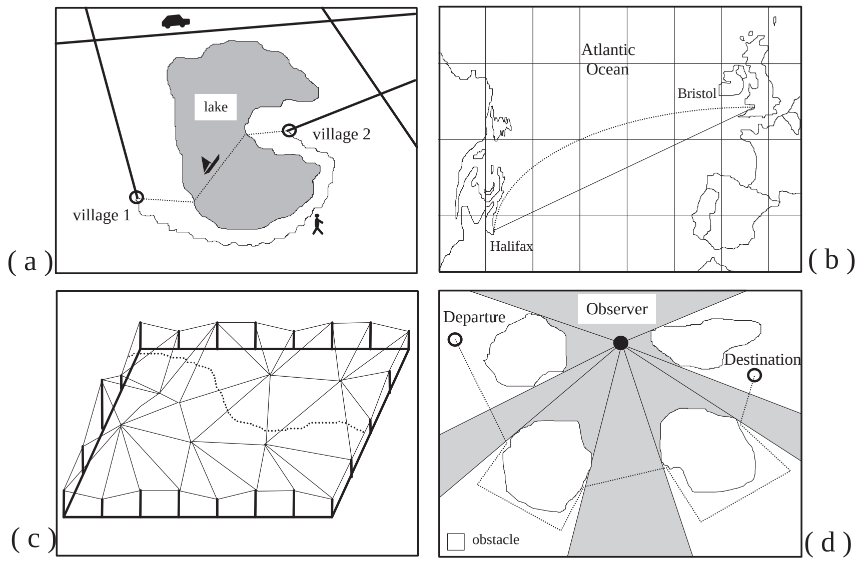



Figure 1: Examples of optimum paths in space.

reasonably well. However, an exact solution can only be found when sampling resolution in the area under consideration tends to infinity. On the other hand vector-based algorithms provide exact solutions to the problem. However, both theory and experiments prove that their performance is not satisfactory.

This paper focuses on a general and effective graph based approach to the determination of the optimum path in space introduced recently by Stefanakis and Kavouras [33]. This approach is quantitative in nature and is based on the degeneration of the space under study into a network, which can be simulated by a weighted graph, so that algorithms of graph theory and artificial intelligence can be easily adopted to indicate the optimum path(s) for the desired trip. What differentiates this approach from other raster-based approaches is that the discretization of the space (Section 2.1) is applied in order to establish the network components only, i.e. nodes and edges. No weight values regarding movement are assigned to grid cells. On the other hand, each network edge is assigned a weight, which indicates the cost of movement along it and is derived by the superimposition of the edge on the vector or raster map of the area under consideration and the travel cost model in use (Section 2.3). This approach can be easily implemented to a wide variety of spaces with different travel cost models associated to them. In addition, although quantitative in essence, this approach can find application in qualitative path selection, which characterizes mostly human navigation [14].

The approach introduced initially by Stefanakis and Kavouras [33], has been extended and refined in the present paper. The objective of the paper is twofold: first to present a general quantitative approach for the determination of optimum path(s) in space; and second to show how this approach can be applied to solve the problem in various cases. Specifically, Section 2 presents the quantitative approach for the determination of the optimum path connecting two physical locations in space. Section 3 gathers together several examples, which show its applicability to a variety of spaces (e.g. 2-D plane, 3-D space, and spherical surface) with a simple cost model (i.e. that of shortest path finding avoiding obstacles). Section 4 examines the possibility of applying the same approach to support route selection when both quantitative and qualitative constraints are involved. Finally, Section 5 concludes the discussion by summarizing the contribution of the paper and giving hints for future research in the area of optimum path finding.

## 2. Determination of the Optimum Path in Space

This Section provides an overview of the approach to the determination of the optimum path in space introduced by Stefanakis and Kavouras [33]. The concept behind this approach is to establish a network connecting a finite number of locations (including departure and destination spots) in space (Figure 2), so that effective algorithms coming from the weighted graph theory [13], [31] and artificial intelligence [28], [29] can be adopted to indicate the optimum path(s) for the desired trip.

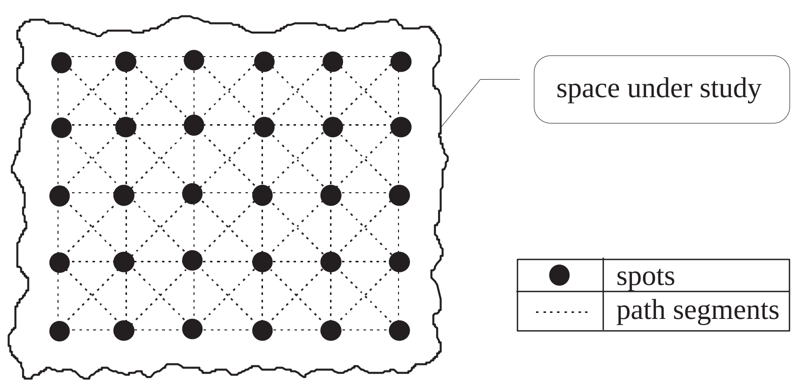



Figure 2: The concept of the new approach on a plane surface.

In summary, the approach consists of four steps:

(a) Determination of a finite number of spots in space.
(b) Establishment of a network connecting these spots.
(c) Formation of the travel cost model.
(d) Determination of the optimum path(s) from the spot of reference (i.e. the departure or destination spot).

These steps are discussed successively in the following Subsections.

### 2.1. Determination of a Finite Number of Spots in Space

The inconvenience that characterizes the movement in space is the infinite number of *spots* (i.e. point locations or nodes) involved in the determination of the optimum path. The proposed solution to overcome this problem is based on the technique of *discretization of space* under study. Discretization [22] is the process of partitioning the continuous space into a finite number of disjoint areas or volumes (cells), whose union results in the space. By representing each of these cells with one spot (e.g. its center point), a finite set of spots is generated to be involved in the process of the determination of the optimum path(s).

A wide variety of *tessellations* (also termed meshes) are available for obtaining the desired partitioning of the space under study [22]. Tessellations may consist of regular or irregular cells. In the former case, space is partitioned by a repeatable pattern of regular polyhedra (regular polygons on plane surface); while in the latter case, space is partitioned by an extending configuration of polyhedra with variable shape and size.

In order to achieve a uniform distribution of spots over the space under study a regular tessellation should be adopted. Some possible tessellations are: rectangular grid, hexagonal grid, and triangular grid for plane surface; cubic blocks and other platonic solids for 3-d space; n-d blocks for n-d space; geographic grid, polyhedral tessellations, and cubic blocks for the spheroid.

### 2.2. Establishment of a Network

After the determination of the spots over the space under study, a *network* should be established to connect adjacent or non-adjacent spots and indicate the possible paths (finite in number) of movement. The proposed scheme for the establishment of the network is based on the tessellation used for the generation of spots (i.e. network nodes) involved in the determination of the optimum path. Specifically, each spot (assigned to a cell) is connected through network edges to the spots of the neighboring cells. Independently on the tessellation adopted to partition space, each cell has three types of neighbor cells: a) *direct,* i.e. neighbors with shared edges; b) *indirect,* i.e. neighbors with common vertices; and c) *remote neighbors.* Remote neighbors are characterized by the level of proximity to the cell of reference. For instance, level-one (level-two) remote neighbors are the cells which are direct or indirect neighbors of the direct or indirect neighbors of the cell of reference (of the level-one remote neighbors of the cell of reference).

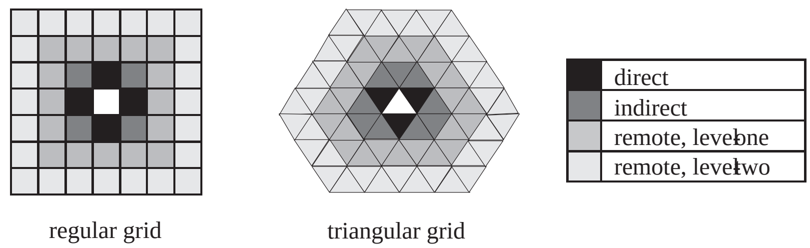



Figure 3: Types of neighbor cells (plane surface).

Figure 3 illustrates an example for the regular and triangular tessellations on the plane surface. Notice that, by increasing the number of neighbors considered, the directions (i.e. degrees of freedom) of movement are augmented. For instance, in a regular tessellation on the plane surface (a chess), the direct neighbors introduce a set of four directions (rook's move is allowed), the indirect neighbors another set of four directions (queen's move is allowed), and the level-one remote neighbors another set of eight directions of movement (queen's+knight's moves are allowed). An exhaustive network would consider all direct, indirect and remote (of any level) neighbors.

### 2.3. Formation of the Travel Cost Model

The *travel cost model* assigns weights to the edges of the network established in the previous step. Its form depends on both the space under study and the application needs. Some representative examples of travel cost models are:

- the model of distance; e.g. the shortest path finding (minimize overall distance)
- the model of time; e.g. the fastest path finding (minimize overall time)
- the model of expenses; e.g. the least expensive path finding (minimize overall expenses)
- the model of risk; e.g. the least risky path finding (minimize overall risk)

In each case the space under study consists of areas (or volumes) that are characterized by a weight, which indicates the cost of movement across them per unit of movement; and depends on the travel cost model in use. Figure 4a, b illustrates an simplified example of a plane surface consisting of areas with variable travel cost values expressed in seconds per meter. Based on those weight values, the edges of the network are assigned a measure, which is equal to the cost of movement from one spot to another (Figure 4c, d). Assuming that the edge $j$ intersects $k$ areas, this measure ($C_{\mathrm{j}}$) is derived from the sum:

$$C_j = \sum_{i=1}^{k} \left[ w_i \cdot d_i \right], \tag{1}$$

where $w_{\mathrm{i}}$ is the weight of area $i$ ($i$= 1, 2, ..., $k$) intersected by $j$ and $d_{\mathrm{i}}$ the length of the edge across that area. For the fast retrieval of areas intersected by a network edge, efficient index structures available in spatial databases [30], such as R-tree variations, can be adopted [34].

The shape of a network edge connecting two spots (network nodes) may vary depending on conventions posed by different configurations and the space under study. Specifically, different line shapes may be considered for forming a network edge (straight, curves, splines, etc.). The most commonly used convention, which is also adopted in this study, is the shortest in length line connecting the two spots. As for the factor of space, in plane surface the shortest line (i.e. network edge) connecting two spots is the straight line connecting them, while in spherical surface the shortest arc of the great circle passing through them. Notice that the line should belong to the space considered. For instance, although the cord is the shortest line connecting two spots lying on a sphere the corresponding arc is considered instead, when movement is restricted on the surface of the sphere (Section 3.3).

Obviously, there are some applications that involve *dynamic travel cost models*. In a dynamic travel cost model the weight values assigned to the areas or volumes of the space under study change over time. These applications are characterized as spatio-temporal and constitute an active area of research nowadays (Frank et al. 2001). Real-world examples that involve dynamic models are the airline routing over Atlantic and navigation of ships and tankers along Aegean sea. In all these applications obstacles and weight values in general depend on the traffic load and meteorological conditions at specific time intervals.

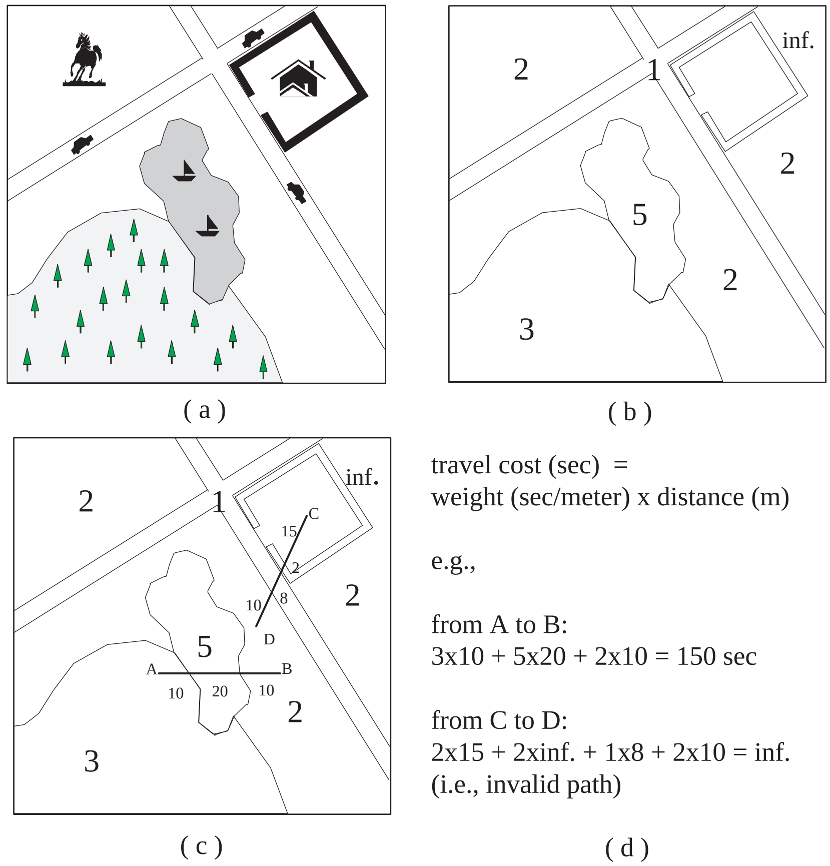



Figure 4: Plane surface of areas with variable travel cost measures.

### 2.4. Determination of the Optimum Path(s)

After the establishment of the network and the formation of the travel cost model, appropriate algorithms from both graph theory [13], [31] and artificial intelligence [28], [29] can be applied for the determination of the optimum path(s) from the node(s) of reference to the node(s) of interest (or the reverse) in the linear network.

A commonly used algorithm available in graph theory is Dijkstra's algorithm [6], which computes the shortest paths from a reference node to all other nodes of a network. Alternative strategies based on the *best-first search* [29] are available in the area of artificial intelligence which minimize significantly the computation time and guarantee both completeness and optimality. A* algorithm [19], [20], [26] is the most representative example of a best-first search algorithm, which derives the optimum path between two nodes (from reference node to destination node) of a network.

## 3. Finding the Shortest Path

The scope of this Section is to show how the four step approach to the determination of the optimum path can be applied in a variety of spaces adopting a simple cost model which is quantitative in nature. Specifically, the problem of shortest path finding is examined in a variety of spaces, such as the plane surface, the 3-D space, and the spherical surface as an approximation of the earth surface. In each case, the space under study is represented by a set of spots, which may be *accessible* or *non-accessible* (i.e. lying on obstacles; usually not considered). For instance, spots lying on the sea are accessible, while those lying on the continents are non-accessible for a ship (Figure 1b). The travel cost between two accessible spots is equal to the length of the shortest line connecting them, if they are *intervisible* [1]. Two spots are intervisible, if the shortest line connecting them passes through no obstacle. If this is not the case (i.e. invalid path), the travel cost between the two spots is computed implicitly passing through intermediate spots.

### 3.1. The Plane Surface

The plane surface is the simplest space to study. Using one of the available two-dimensional tessellations and locating one spot at the center point of each cell can easily determine the finite number of spots. The network is then established by connecting the spots through straight-line segments, which are assigned weight values depending on the structure of the surface and the travel cost model considered.

Figure 5 shows an example on the determination of the shortest path on a plane surface with obstacles (shaded areas). The tessellation used is the regular grid with a resolution of 10 units in both $X$ and $Y$ dimensions, while the established network connects each spot (i.e. network node) with its direct and indirect neighbor spots (Figure 5b) or direct, indirect and level-one remote neighbor spots (Figure 5c) by straight line segments (i.e. eight or sixteen direction of movements are considered). Figures 5b, c depict the accumulated cost values for each network node from the reference node (e.g. Dijkstra's algorithm

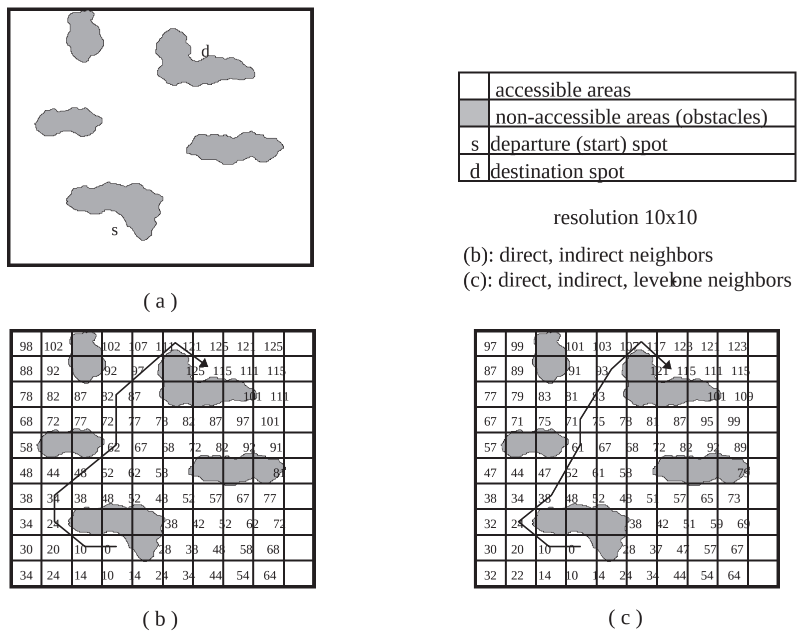


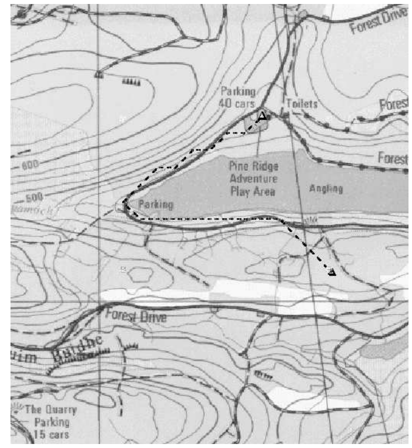

Figure 5: The shortest path on a plane surface (values assigned to the grid cells refer to the accumulated travel costs of the corresponding spots).

product), while solid lines show the shortest paths for the desired movement. Notice that, by increasing the number of neighbors considered, a more accurate optimum path is derived, for the same set of spots (i.e. resolution of the tessellation) on the plane. The effect of both the number of spots and number of neighbors considered is one of the directions for future research (Section 5). Some preliminary work on this topic can be found in [38], [7].

A prototype system for the determination of the optimum path(s) on a plane surface consisting of areas with variable travel cost measures assigned to them has been developed [36] in MapInfo software package [24] on a PC-Windows platform.

Figure 6 illustrates two examples of the graphic outputs generated by the system (statistical results are also provided). Specifically, Figure 6a indicates the shortest sea course between two harbors in the Aegean sea (pixel size is equal to 6 km; total cost: 70,2 km; direct and indirect neighbors considered; time required to compute the path using $A^*$ algorithm: 1 sec in a P120 PC, 32MB RAM); while Figure 6b shows the fastest path from a given location to a parking lot on an orienteering map of Scotland (pixel size is equal to 5 m; total cost: 16 min and 9 sec; direct and indirect neighbors considered; time required to compute the path using $A^*$ algorithm: 30 sec in a P120 PC, 32MB RAM).


(a)




(b)

Figure 6: Real world examples of shortest path finding on plane surface.

### 3.2. The 3-D Space

Moving in the 3-D space is a generalization of movement on a plane surface. The spots are similarly determined using one of the available three dimensional tessellations (Table 1) and the network is established by connecting these spots through straight line segments, which are assigned weight values depending on the structure of the space and the travel cost model considered.

Figure 7 presents an example on the determination of the shortest path in a 3-D space enclosing obstacles (shaded volumes). All obstacles have their basis at level 0, while their height is noted in Figure 7a over the shaded areas. The maximum height of the space is 40 units, while departure and destination spots have an altitude of 5 and 25 units respectively. The tessellation used is the regular grid (i.e. cubic blocks) with a resolution of 10 units in all $X$, $Y$ and $Z$ dimensions. As for the established network, it connects each spot with its direct and indirect neighbors. Figures 7c, b, e, d depict the accumulated cost values (in layers) for each network node from the reference node (i.e. Dijkstra's algorithm product), while solid line shows a shortest path for the desired movement (notice that it jumps across the levels). Figure 7f illustrates a perspective view of the 3-D space and the proposed movement.

The extension of the idea to spaces of higher dimensionality (n-D spaces)

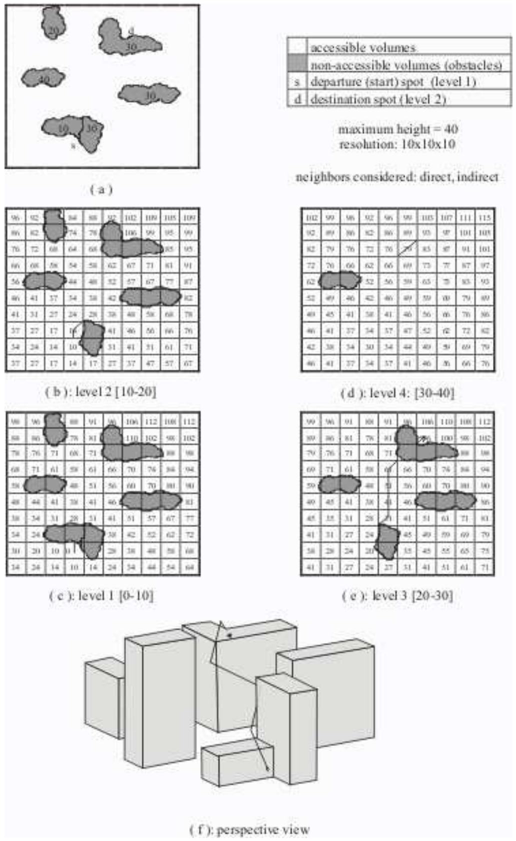


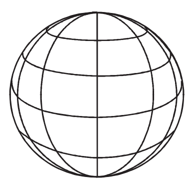

Figure 7: The shortest path in 3-D space (values assigned to the grid cells refer to the accumulated travel costs of the corresponding spots).

is considered as a simple generalization of the concept.

### 3.3. The Spherical Surface

The movement on the surface of a sphere is examined separately due to the peculiarities of the underlying spherical geometry. The problem can be solved by examining the movement on a 3-D space (Section 3.2), where the interior and exterior of the sphere are non-accessible (i.e. obstacles), while movement is allowed inside a spherical ring (it can be thought as a buffer zone of the spherical surface in space). For instance, if cubic blocks are used for the generation of the spots on the three dimensional space, the set of the accessible spots will be located within the cubic blocks which intersect the spherical surface.

A more adaptive scheme for the spherical surface considers the movement on the surface itself. The determination of the spots is based on the available tessellations for the sphere (Table 1). The tessellation, which is closer to cartographers and geo-scientists, in general, is the geographic grid (Figure 8a). Figure 8b illustrates an example for the model of distance on a sphere for a geographic grid of 30 degrees resolution (i.e. 62 spots). The direct and indirect neighbors are considered for each spot on the established network. The accumulated travel cost values are expressed in radians and refer to a spot (on the shaded grid cell) lying on the equator of the sphere (i.e. Dijkstra's algorithm product). Contrary to the plane and 3-D space, the path connecting two spots is not a straight line segment any more, but the shortest arc of the *great circle* (i.e. normal section) passing trough them (Section 2.3). Notice that all cells lying on the top (bottom) row of the matrix are assigned the same value, since they refer to a unique spot lying on the north (south) pole of the sphere.



( a )

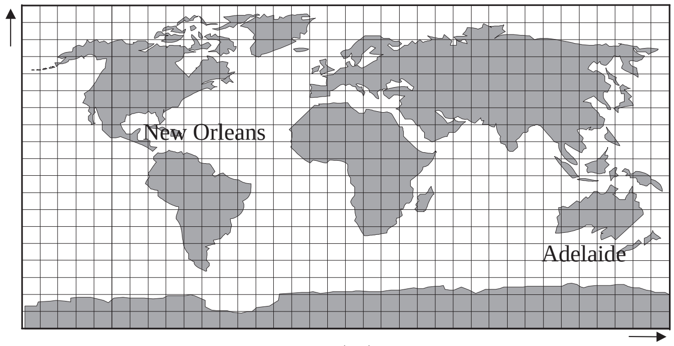
| | 0 | 30 | 60 | 90 | 120 | 150 | 180 | 210 | 240 | 270 | 300 | 330 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 90 | 1.57 | 1.57 | 1.57 | 1.57 | 1.57 | 1.57 | 1.57 | 1.57 | 1.57 | 1.57 | 1.57 | 1.57 |
| 60 | 2.09 | 1.83 | 1.57 | 1.31 | 1.05 | 1.31 | 1.57 | 1.83 | 2.09 | 2.09 | 2.09 | 2.09 |
| 30 | 2.33 | 1.88 | 1.43 | 0.98 | 0.52 | 0.98 | 1.43 | 1.88 | 2.33 | 2.62 | 2.62 | 2.62 |
| 0 | 2.09 | 1.57 | 1.05 | 0.52 | 0 | 0.52 | 1.05 | 1.57 | 2.09 | 2.62 | 3.14 | 2.62 |
| -30 | 2.33 | 1.88 | 1.43 | 0.98 | 0.52 | 0.98 | 1.43 | 1.88 | 2.33 | 2.62 | 2.62 | 2.62 |
| -60 | 2.09 | 1.83 | 1.57 | 1.31 | 1.05 | 1.31 | 1.57 | 1.83 | 2.09 | 2.09 | 2.09 | 2.09 |
| -90 | 1.57 | 1.57 | 1.57 | 1.57 | 1.57 | 1.57 | 1.57 | 1.57 | 1.57 | 1.57 | 1.57 | 1.57 |

( b )

Figure 8: Accumulated cost values on a homogeneous sphere (in rad).

Figure 9 illustrates an example on the determination of the shortest sea course on the earth surface approximated by a sphere. For visualization purposes a Plate Carrée projection [32] has been adopted to represent the continents and oceans of the earth. The solid line shows a shortest path from Adelaide, Australia to New Orleans, USA. The line does not approximate sufficiently the curve of the great circle on the projection plane (as in Figure 1b), due to the low resolution of the grid (10 degrees) and the limited number of neighbors (direct and indirect) considered on the established network.

The major problem faced by this tessellation is that the set of spots generated is non-uniformly distributed over the surface, because of the highly variable shape and size of the geographic grid cells; and as a consequence a variable accuracy on the determination of the optimum path is introduced, which depends




( a )

| | 190 | 200 | 210 | 220 | 230 | 240 | 250 | 260 | 270 | 280 | 290 | 300 | 310 | 320 | 330 | 340 | 350 | 0 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 | 130 | 140 | 150 | 160 | 170 | 180 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 90 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 | 2,74 |
| 80 | 2,58 | 2,61 | 2,64 | 2,67 | 2,70 | 2,73 | 2,76 | 2,79 | | 2,92 | 2,92 | 2,92 | | | | | 2,92 | 2,92 | 2,92 | 2,92 | 2,92 | 2,92 | 2,92 | 2,92 | 2,92 | | | 2,79 | 2,76 | 2,73 | 2,70 | 2,67 | 2,64 | 2,61 | 2,58 | 2,57 |
| 70 | 2,43 | 2,48 | | | | | | | | 3,09 | 3,09 | 3,09 | 3,15 | | | 3,10 | 3,09 | 3,09 | | | | | | | | | | | | | | | | | | 2,40 |
| 60 | 2,29 | 2,34 | 2,40 | 2,46 | | | | | 3,28 | 3,27 | | 3,27 | 3,28 | 3,37 | 3,29 | 3,27 | 3,27 | 3,27 | | | | | | | | | | | | | | | 2,21 | 2,18 | 2,21 | 2,25 |
| 50 | 2,14 | 2,20 | 2,26 | 2,33 | | | | | | | | 3,12 | 3,23 | 3,29 | 3,18 | 3,14 | | | | | | | | | | | | | | | | | | 2,01 | 2,05 | 2,09 |
| 40 | 1,99 | 2,05 | 2,11 | | | | | | | 2,77 | 2,91 | 3,04 | 3,17 | 3,23 | 3,10 | 2,97 | 2,83 | 2,70 | 2,57 | 2,43 | 2,38 | | | | | | | | | | | | | 1,83 | 1,87 | 1,92 |
| 30 | 1,82 | 1,89 | 1,96 | | | | 2,55 | 2,49 | 2,55 | 2,70 | 2,85 | 3,00 | 3,07 | 3,13 | 3,09 | 3,06 | | | | | 2,21 | 2,15 | | | | | | | | | | | 1,61 | 1,67 | 1,70 | 1,76 |
| 20 | 1,65 | 1,72 | 1,79 | 1,86 | | | | 2,31 | 2,48 | 2,64 | 2,80 | 2,97 | 2,90 | 2,96 | 2,91 | | | | | | | 1,97 | 1,90 | | 1,57 | 1,50 | | | 1,29 | | | 1,37 | 1,44 | 1,51 | 1,52 | 1,59 |
| 10 | 1,48 | 1,55 | 1,62 | 1,73 | 1,85 | 1,96 | 2,07 | 2,24 | 2,41 | 2,59 | 2,76 | 2,80 | 2,73 | 2,79 | 2,74 | | | | | | | | 1,73 | 1,60 | 1,46 | 1,33 | 1,26 | | 1,12 | 1,06 | | 1,20 | 1,27 | 1,42 | 1,35 | 1,41 |
| 0 | 1,30 | 1,38 | 1,49 | 1,60 | 1,71 | 1,82 | 1,94 | | | | 2,80 | 2,62 | 2,55 | 2,61 | 2,56 | 2,49 | 2,42 | 2,36 | | | | | | 1,48 | 1,35 | 1,22 | 1,08 | 1,01 | 0,94 | 0,88 | 0,95 | 1,02 | | 1,24 | 1,18 | 1,24 |
| -10 | 1,13 | 1,24 | 1,35 | 1,47 | 1,58 | 1,69 | 1,80 | | | | | | 2,38 | 2,44 | 2,40 | 2,32 | 2,25 | 2,19 | | | | | 1,50 | 1,37 | 1,24 | 1,10 | 0,97 | 0,84 | 0,77 | 0,71 | 0,78 | 0,95 | 1,12 | 1,07 | 1,00 | 1,06 |
| -20 | 1,00 | 1,11 | 1,22 | 1,34 | 1,45 | 1,56 | 1,67 | 1,78 | | | | | 2,20 | 2,26 | 2,25 | 2,16 | 2,07 | 2,01 | | | | | 1,40 | 1,26 | 1,13 | 1,00 | 0,86 | 0,73 | 0,59 | 0,53 | 0,70 | | | | 0,83 | 0,89 |
| -30 | 0,88 | 0,99 | 1,10 | 1,21 | 1,32 | 1,44 | 1,55 | 1,66 | 1,75 | | | | 2,03 | 2,09 | 2,10 | 2,01 | 1,93 | 1,84 | | | | 1,43 | 1,29 | 1,16 | 1,03 | 0,89 | 0,76 | 0,63 | 0,49 | 0,36 | | | | | 0,65 | 0,76 |
| -40 | 0,76 | 0,88 | 0,99 | 1,10 | 1,21 | 1,32 | 1,44 | 1,53 | 1,61 | | | 1,80 | 1,86 | 1,92 | 1,94 | 1,87 | 1,79 | 1,70 | 1,61 | | 1,44 | 1,32 | 1,20 | 1,07 | 0,94 | 0,80 | 0,67 | 0,53 | 0,40 | 0,27 | 0,13 | 0 | | 0,43 | 0,54 | 0,65 |
| -50 | 0,66 | 0,77 | 0,89 | 1,00 | 1,11 | 1,22 | 1,31 | 1,40 | 1,47 | | 1,59 | 1,65 | 1,71 | 1,77 | 1,79 | 1,73 | 1,66 | 1,57 | 1,49 | 1,40 | 1,31 | 1,22 | 1,20 | 1,07 | 0,94 | 0,77 | 0,66 | 0,55 | 0,44 | 0,33 | 0,21 | 0,17 | 0,21 | 0,33 | 0,44 | 0,55 |
| -60 | 0,68 | 0,76 | 0,85 | 0,94 | 1,02 | 1,11 | 1,20 | 1,27 | 1,33 | 1,39 | 1,45 | 1,51 | 1,57 | 1,63 | 1,65 | 1,59 | 1,53 | 1,46 | 1,37 | 1,29 | 1,20 | 1,11 | 1,02 | 0,94 | 0,85 | 0,76 | 0,68 | 0,59 | 0,50 | 0,41 | 0,38 | 0,35 | 0,38 | 0,41 | 0,50 | 0,59 |
| -70 | 0,72 | 0,78 | 0,84 | 0,90 | 0,96 | 1,02 | 1,08 | 1,14 | 1,20 | 1,26 | 1,32 | 1,38 | 1,44 | 1,50 | 1,52 | 1,46 | 1,40 | 1,34 | 1,28 | 1,22 | 1,16 | 1,10 | 1,04 | 0,98 | 0,92 | 0,86 | 0,85 | | | | | | | | 0,60 | 0,66 |
| -80 | 0,84 | | | | | | | | | | 1,44 | 1,47 | 1,50 | 1,53 | | | | | | | | | | | | | | | | | | | | | | |
| -90 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |

( b )

Figure 9: The shortest sea course (in rad; shaded cells in b correspond to non-accessible spots, i.e. land; continents are simplified).

on the region of the sphere studied. Clearly, there is a large concentration of spots around the poles which decreases by approaching the equator.

A more uniform distribution of the spots over the spherical surface can be obtained by adopting tessellations that are based on the recursive decomposition of regular polyhedra. A tessellation, which is based on the recursive subdivision of the octahedron, is considered here. This tessellation has been widely examined in the past as a hierarchical model for the representation of the earth's surface on a Global Geographic Information System [10], [16].

The surface of the earth is approximated by an octahedron whose vertices are located at the poles and at longitudes 0°, 180°, 90° E and 90° W around the equator. That is, the surface is initially projected into eight triangular faces. Each triangle is then recursively subdivided into four by connecting the midpoints of its edges, and so on, until the predefined level of decomposition is reached. The advantage of this tessellation is that the triangular cells derived have a similar shape and size and as a consequence a uniform distribution of the spots over the spherical surface can be obtained.

Algorithms for the conversion of triangular cells to geographic coordinates as well as for the determination of their direct and indirect neighbors are available [18], [17], and can be adopted for locating the spots over the surface and establishing the network over them.

Figure 10 shows a simplified example on the determination of the shortest sea course from Hawaiian Islands to Los Angeles, California. The quadrant 90$^o$ W to 180$^o$ W of the northern hemisphere has been only considered, while the established network is confined to the three direct neighbors of each spot. The tiling corresponds to the fourth level of subdivision and results in 256 triangular cells with an average resolution of 180 km [10].

The same technique applies when a more precise determination of a distance on the earth's surface is needed, although its implementation is more complex. In such a case, the earth's surface (geoid; physical reality) is closely approximated by a "spheroid", i.e. an oblate spheroid, a figure of revolution, on the surface of which, at each point curvature varies in each direction. The shortest line joining two points on a spheroid in not along one of the two normal sections passing through the points, but is a unique curve (falling in between the two normal sections) known generally as the *geodesic curve* or simply the *geodesic* [2]. The *geodesic* has no curvature in the tangent plane and it is locally straight on the surface [37]. Notice that the spheroidal surface may be approximated by the same tessellations used for the sphere.

The problem becomes a bit more complex, when one tries to determine the best route not on a ellipsoidal approximation of the earth's surface but on the *geoid* itself (i.e. an *equipotential surface* of the earth which is partially known and cannot be described as simply as the ellipsoid). This problem is mostly of theoretical and not practical interest, since there are various other major constraints to navigation which outweigh the fine differences caused by the use of models of the geoid. Even in this case, the general approach introduced in this paper can be applied. The ellipsoid can be represented as a discrete 3-D "relief" (terrain model) with respect to the geoid, and the solution provided in paragraph 3.2 can be employed to determine the optimal path on the equipotential surface.

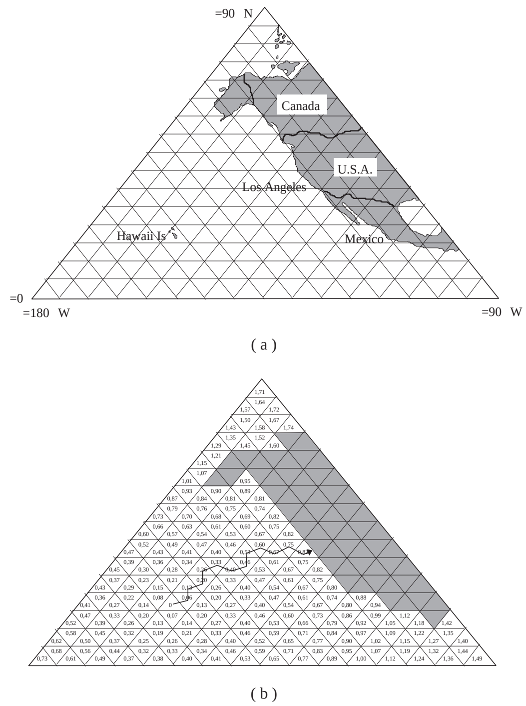



Figure 10: The shortest sea course (in rad; shaded cells in b correspond to non-accessible spots, i.e. land; continents are simplified).

## 4. Piecewise Qualitative Path Selection

Although quantitative in essence, the approach introduced in this paper can probably find application in qualitative path selection, which characterizes mostly human navigation [14]. Such a development lies in the context of more general attempts to qualitative versions of GIS. Route selection based on qualitative criteria is often ridden with subjectivity and contradictions that makes system implementation very difficult.

The spatial approach developed here helps to yield the "best" path in space, while leaving the definition of "bestness" to the user. This is also in accordance with the general practice of spatial science [27]. In this effort, cost models can be based upon various criteria for route selection such as [14]: fewest turns, most scenic/aesthetic, first noticed, longest leg first, many curves, many turns,

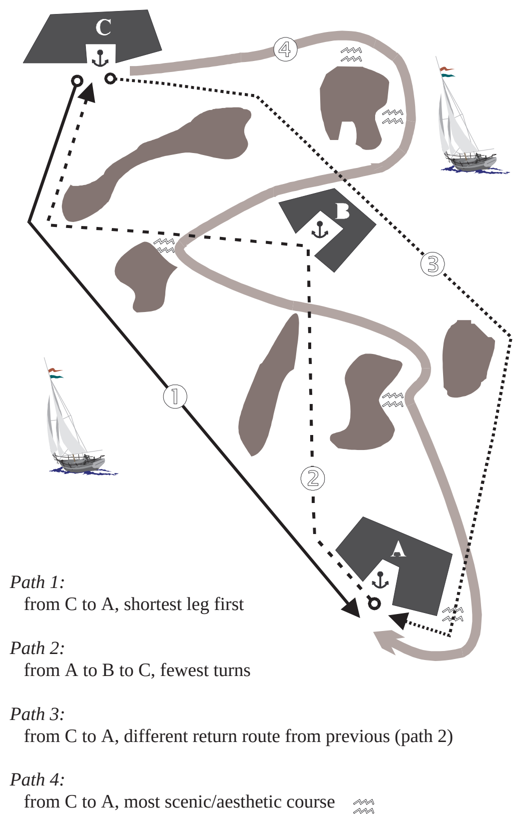



Figure 11: Route selection based on quantitative and qualitative criteria.

different from previous, shortest leg first, and so on. Figure 11 illustrates an example for the optimum sea course finding based on some of these criteria. The proposed approach can contribute to the solution of the problem in two ways:

First, it can be used to formalize quantitatively in a piecewise fashion, any complex qualitative criteria in route selection. Secondly, it can provide an objective solution to route selection in complex spatial cases, for comparison and evaluation of deviations and habits in routes selected by humans using different and less objective criteria.

## 5. Conclusion

A common problem in human navigation is the determination of the optimum path based on both quantitative and qualitative criteria. Although, the problem has been studied extensively for linear networks it is not widely examined for movement in space. However, the problem appears very often in several applications such as Cartography, Robotics and Geographic Information Systems.

The contribution of this paper in the area of optimum path finding in space can be summarized as follows:

(a) An approach to the determination of the optimum path in space has been introduced. The general concept is based on the degeneration of the space under study into a linear network, which can be simulated by a weighted graph, so that algorithms of graph theory and artificial intelligence can be easily adopted to indicate the optimum path(s) for the desired trip.

(b) The approach can be easily implemented to a wide variety of spaces with different travel cost models associated to them.

(c) Although quantitative in nature, the new approach may provide solution to several complex route selection problems, which involve qualitative criteria.

(d) The applicability of the new approach to representative spaces of "real world" problems has been examined. The results are illustrated through several examples.

Future research in the area includes:

(a) An extended analysis on the effects of the values assigned to the parameters of the new approach (i.e. the number of spots considered, how they are distributed in space, types of neighbors taken into account) on the determination of the optimum path in space. Some preliminary findings regarding this topic can be found in [38], [7].

(b) The development and implementation of complex travel cost functions for modeling "real world" problems related to the area of the optimum path finding.

(c) The design and implementation of efficient techniques and data structures to support the incorporation of the new approach in a production system. This involves the adoption of: a) spatial data structures to support the individual steps of the algorithm [30], [34]; b) hierarchical approaches for optimum path finding [38], [3]; and c) techniques for parallel computing [35], [7].

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