International Journal of Pure and Applied Mathematics
Volume 1 No. 2 2002, 159-163

A RESULT ON STRICTLY SEPARATING FUNCTIONALS

Ömer Gök
Yıldız Technical University
Faculty of Arts and Sciences
Mathematics Dept., Davutpaşa Campus
İstanbul, TURKEY
e-mail: gok@yildiz.edu.tr

Abstract: Let $\mathcal{S}$ be a linear subspace of Banach space operators which is closed in the weak operator topology. In this paper, we show that $\mathcal{S}$ is reflexive if a disjoint pair of separating functional and strictly separating functional exists.

AMS Subject Classification: 47D15
Key Words: strictly separating functionals, separating vectors

1. Introduction

In this paper, we prove similar results of E.A. Azoff et al [1] and L. Ding [2], [3]. There are several results in the literature concerning the relationship between reflexivity and strictly separating vectors, e.g. L. Ding [2] and E.A. Azoff et al [1]. The importance of separating vectors in algebraic reflexivity is demonstrated in D.R. Larson [6]. In Hadwin [4], [5] the author studied very

Received: January 2, 2002

general versions of reflexivity and hyperreflexivity. In Hadwin [4] the author proved theorems concerning the reflexivity and hyperreflexivity of graphs.

We suppose X, Y are Banach spaces. By $L(X,Y)$ we denote the set of all continuous linear operators acting between X and Y and suppose $\mathcal{S}$ is a weak-operator closed linear subspace of $L(X,Y)$. By X' we denote the topological dual of X. We say that a vector e in X is a separating vector for $\mathcal{S}$ if the mapping $S \to Se$ is 1-1 on $\mathcal{S}$. We say that a linear functional y' on Y is a separating functional for $\mathcal{S}$ if the mapping $S \to y' \circ S$ is 1-1 on $\mathcal{S}$. We say that a vector x in X is strictly separating for $\mathcal{S}$ if x is a separating vector for $\mathcal{S}$ and $\mathcal{S}x$ is norm closed. Similarly, a continuous linear functional y' on Y is strictly separating for $\mathcal{S}$ if y' is separating for $\mathcal{S}$ and $\{y' \circ S : S \in \mathcal{S}\} = \{S'y' : S \in \mathcal{S}\}$ is a norm closed subspace of X'. We consider continuous rank-one tensors $x \otimes y'$ with $x \in X$ and y' a continuous linear functional on Y. If $\phi : \mathcal{S} \to \mathcal{S}e$ is defined by $\phi(S) = Se$, then ϕ is continuous, 1-1 and onto, and it follows from the closed graph theorem that ϕ^{-1} is continuous. Therefore, given a ψ in $L(X,Y)'$, choose $y' \in Y'$ so that $y' \mid \mathcal{S}e = \psi \circ \phi^{-1}$; clearly ψ agrees with $e \otimes y'$ on $\mathcal{S}$. If $v \in X$ and $y' \in Y'$, we define the rank-one-tensor $v \otimes y'$ in $L(X,Y)'$ by $(v \otimes y')(T) = y'(Tv)$.

We define the sets $\mathcal{S}' = \{b' \in L(Y', X') : b \in \mathcal{S}\}$ and $\mathcal{S}'y' = \{b'y' : b \in \mathcal{S}, \quad y' \in Y'\} = \{y' \circ b : b \in \mathcal{S}, \quad y' \in Y'\}$. If $\theta : \mathcal{S}' \to \mathcal{S}'y'$ is defined by $\theta(b') = b'y', b \in \mathcal{S}$, then θ is continuous, one-one and onto, it follows from the closed graph theorem (or open mapping theorem) that θ^{-1} is continuous.

Definition. Let $\mathcal{S}$ be a linear subspace of $L(X,Y)$.

(i) $Ref\mathcal{S} = \{b \in L(X,Y) : bx \in [\mathcal{S}x] \quad \text{for each} \quad x \in X\}$, where $[\cdot]$ denotes the norm closure.

(ii) $\mathcal{S}$ is said to be reflexive if $\mathcal{S} = Ref\mathcal{S}$.

If $X = Y$ and $\mathcal{S}$ is a unital operator algebra, $Ref\mathcal{S}$ coincides with $AlgLat\mathcal{S}$, where $AlgLat\mathcal{S}$ is the algebra of continuous linear operators that leave invariant all $\mathcal{S}$-invariant subspaces.

Definition. Let $\mathcal{S}$ be a linear subspace of $L(X,Y)$ and suppose $y' \in Y'$.

(i) The evaluation map $f_{y'} : \mathcal{S} \to X'$ is defined by $f_{y'}(b) = y' \circ b = b'y'$.

(ii) The functional $y' \in Y'$ separates $\mathcal{S}$ if $f_{y'}$ is injective on $\mathcal{S}$.

The following is the main result of the paper.

Theorem. *Let $\mathcal{S}$ be a subspace of $L(X,Y)$ which is closed in the weak operator topology. Suppose $\mathcal{S}$ admits a separating functional $x' \in Y'$ and a* *strictly separating functional* $y' \in Y'$ *satisfying* $\mathcal{S}'x' \cap \mathcal{S}'y' = \{0\}$. *Then,* $\mathcal{S}$ *is reflexive.*

Throughout, $\mathcal{S}$ will denote a weak operator-closed linear subspace of $L(X,Y)$ and we write U for its collection of strictly separating functionals.

Definition. Given $b \in Ref\mathcal{S}$ and $y' \in U$, we write $g_b(y')$ for the unique operator in $\mathcal{S}'$ satisfying $g_b(y')y' = b'y' = y' \circ b$.

Thus g_b is an operator-valued function defined on the collection of strictly separating functionals.

Proposition 1. *Let* $\mathcal{S}$ *be a weak operator-closed linear subspace of* $L(X,Y)$ *admitting a strictly separating functional* $y' \in Y'$. *Then, the following are equivalent.*

(i) $\mathcal{S}$ *is reflexive.*

(ii) For each $b \in Ref\mathcal{S}$, *the function* g_b *is constant in a neighborhood of* y'.

Proof. Since g_b is constant when $b \in \mathcal{S}$, it is clear that (i) implies (ii). For the converse, assume (ii) and suppose $b \in Ref\mathcal{S}$. Set $s = g_b(y')$. By hypothesis, we have $sz' = b'z'$ for all z' in some neighborhood of y' whence $b' = s \in \mathcal{S}'$ and $b \in \mathcal{S}$.

Proposition 2. *For each* $b \in Ref\mathcal{S}$ *the function* g_b *is continuous on its domain.*

Proof. Fix b and suppose $\{x'_n\}$ is a sequence in U converging to $x'_0 \in U$. For each n, set $s'_n = g_b(x'_n)$. It is a matter of definition and continuity of b that $\lim_n s'_n x'_n = \lim_n b'x'_n = b'x'_0 = s'_0 x'_0$. By the Uniform Boundedness Principle, the $\{s'_n\}$ are uniformly bounded, hence $\lim_n s'_n x'_0 = s'_0 x'_0$. Since x'_0 is a strictly separating functional for $\mathcal{S}$, we get $\lim_n s'_n = s'_0$ as desired.

Definition. Suppose $b \in Ref\mathcal{S}$, x' is a strictly separating functional for $\mathcal{S}$, and y' is any functional in Y'. Write V for the set of complex numbers λ such that $x' + \lambda y' \in U$ and define the operator-valued function $h_{x',y'} : V \to \mathcal{S}'$ by setting $h_{x',y'}(\lambda) = g_b(x' + \lambda y')$.

In other words,

$$(A) \qquad h_{x',y'}(\lambda)(x' + \lambda y') = (x' + \lambda y')\circ b = b'(x' + \lambda y').$$

By Proposition 2, $h_{x',y'}$ is continuous throughout its domain.

Proposition 3. *The function $h_{x',y'}$ is analytic throughout its domain and*

$h'_{x',y'}(0)x' = (b' - h_{x',y'}(0))y'$, *where $x', y' \in Y'$ and $b \in Ref(\mathcal{S})$.*

Proof. Suppose $\lambda, \lambda_0 \in V$.

$$\frac{h_{x',y'}(\lambda) - h_{x',y'}(\lambda_0)}{\lambda - \lambda_0}(x' + \lambda_0 y') = b'y' - h_{x',y'}(\lambda)y'.$$

By the continuity of $h_{x',y'}$, we conclude that

$$(B) \qquad \lim_{\lambda \to \lambda_0} \frac{h_{x',y'}(\lambda) - h_{x',y'}(\lambda_0)}{\lambda - \lambda_0}(x' + \lambda_0 y') = b'y' - h_{x',y'}(\lambda_0)y'$$

converges in X'. Since $x' + \lambda_0 y'$ is a strictly separating functional for $\mathcal{S}$ and all of the difference quotients involved belong to $\mathcal{S}'(x' + \lambda y')$,

$$\lim_{\lambda \to \lambda_0} \frac{h_{x',y'}(\lambda) - h_{x',y'}(\lambda_0)}{\lambda - \lambda_0}$$

converges in weak operator topology to a member of $\mathcal{S}$. The proof ends by substituting $\lambda_0 = 0$ in equation (B).

Proposition 4. *Suppose $c \in Ref\mathcal{S}$ satisfies $x' \circ c = 0$ for some strictly separating functional $x' \in Y'$. Then, the range of c' (adjoint of c) is contained in $\mathcal{S}'x'$.*

Proof. Fix $z \in Y'$. The hypothesis $x' \circ c = 0$ means that $h_{x',z}(0)x' = (x' + 0z) \circ c = x' \circ c = 0$. It follows from Proposition 3 that $z \circ c = c'z = h'_{x',z}(0)x'$. On the other hand, $h'_{x',z}(0)x'$ is a limit of difference quotients belonging to $\mathcal{S}'x'$, hence $z \circ c = c'z \in \mathcal{S}'x'$ as expected.

Theorem 5. *Let $\mathcal{S}$ be a subspace of $L(X,Y)$ which is closed in the weak operator topology. Suppose $\mathcal{S}$ admits a separating functional $z' \in Y'$ and a strictly separating functional $y' \in Y'$ satisfying $\mathcal{S}'z' \cap \mathcal{S}'y' = \{0\}$. Then $\mathcal{S}$ is reflexive.*

Proof. Replacing z' by $\epsilon z' + y'$ for sufficiently small ϵ if necessary, we may assume that z', y' both strictly separating functionals are for $\mathcal{S}$.

Let $b \in Ref(\mathcal{S})$. Then $c' = b' - g_b(z')$, where c' is the adjoint of c. This implies that $z' \circ c = c'z' = b'z' - b'z' = 0$. c belongs to $Ref(\mathcal{S})$ and satisfies $z' \circ c = 0$, whence $range(c') \subset \mathcal{S}'z'$ by Proposition 4. In particular, $y' \circ c = c'y' \in \mathcal{S}'z' \cap \mathcal{S}'y'$ so $y' \circ c = 0$. Thus a second application of Proposition 4 yields $range(c') \subset \mathcal{S}'z' \cap \mathcal{S}'y'$. This means $c' = 0$, so $b' = g_b(z') \in \mathcal{S}'$ and $b \in \mathcal{S}$ as desired.

Definition. Two subspaces M, N of a Banach space are strongly disjoint if the distance between their unit spheres is strictly positive.

Corollary 6. *Let $\mathcal{S}$ be a subspace of $L(X,Y)$ which is closed in the weak operator topology. Suppose $\mathcal{S}$ admits strictly separating functionals $x', y' \in Y'$ such that $\mathcal{S}'x'$ and $\mathcal{S}'y'$ are strongly disjoint subspaces of X'. Then $\mathcal{S}$ is reflexive.*

Proof. It is clear that the strong disjointness implies that $\mathcal{S}'x' \cap \mathcal{S}y' = \{0\}$. By the Theorem 5, $\mathcal{S}$ is reflexive.

References

[1] E.A. Azoff, L. Ding and W.R. Wogen, Separating versus strictly separating vectors, *Proc. Amer. Math. Soc.*, **124** (1996), 3135-3142.

[2] L. Ding, On strictly separating vectors and reflexivity, *Operator Theoery and Integral Equations*, **19** (1994), 373-380.

[3] L. Ding, On a pattern of reflexive operator spaces, *Proc. Amer. Math. Soc.*, **124** (1996), 3101-3108.

[4] D. Hadwin, A general view of reflexivity, *Trans. Amer. Math. Soc.*, **344** (1994), 325-360.

[5] D. Hadwin, Compressions, graphs, and hyperreflexivity, *J. Funct. Anal.*, **145** (1997), 1-23.

[6] D. R. Larson, Reflexivity, algebraic reflexivity, and linear interpolation, *Amer. J. Math.*, **110** (1988), 283-299.