International Journal of Pure and Applied Mathematics
Volume 1 No. 3 2002, 239-252

OPTIMAL CONTROL PROBLEM FOR A HYPERBOLIC SYSTEM WITH MIXED CONTROL-STATE CONSTRAINTS INVOLVING OPERATOR OF INFINITE ORDER

W. Kotarski[1], H.A. El-Saify[2 §], G.M. Bahaa[3]

[1]Institute of Informatics
Silesian University
Bedzinska 60, 41-200 Sosnowiec, POLAND
e-mail: kotarski@gate.math.us.edu.pl

[2,3]Dept. of Mathematics
Faculty of Science
Cairo University, Beni Suef, EGYPT
[2]e-mail: elsaify_ah@hotmail.com
[3]e-mail: bahaas99@yahoo.com

Abstract: A distributed control problem for a hyperbolic system with mixed constraints on states and controls involving operator of infinite order is considered. The performance index is more general than the quadratic one and has an integral form. Making use of the Dubovitskii-Milyutin theorem, necessary and sufficient conditions of optimality are derived for the Dirichlet problem. Yet the problem considered there is more general than the one in [7]-[10], [23], [24].

AMS Subject Classification: 49J20, 93C20, 49K20, 35K20
Key Words: infinite order hyperbolic operators, distributed control problems, Dirichlet problem, Dubovitskii-Milyutin theorem, conical approximations, necessary and sufficient conditions of optimality

Received: January 28, 2002

§Correspondence author

0. Introduction

In [7]-[10], [23], [24] the optimal control problems for systems described by hyperbolic operators with an infinite order have been discussed. To obtain optimality conditions the arguments of (see [18]) have been applied. Making use of the Dubovitskii-Milyutin theorem (see [11]), following [12]-[14] Kotarski have obtained necessary and sufficient conditions of optimality for similar systems governed by second order operator with an infinite number of variables. The interest in the study of this class of operators is stimulated by problems in quantum field theory [2]-[5].

In this paper the optimality conditions for distributed control system governed by Dirichlet problem for hyperbolic equations of infinite order are given. The optimality condition is expressed in terms of a set of inequalities.

This paper is organized as follows. In Section 1 we introduce some functional spaces of infinite order. In Section 2, we define the hyperbolic equation with infinite order. In Section 3, we formulate the optimal control problem and we introduce the main results of this paper.

1. Some Functinal Spaces (see [2]-[5])

The object of this section is to give the definition of some function spaces of infinite order and the chains of the constructed spaces which will be used later.

We define the Sobolev space $W^\infty\{a_\alpha, 2\}(R^n)$ (which we shall denoted by $W^\infty\{a_\alpha, 2\}$) of infinite order of periodic functions $\phi(x)$ defined on all boundary Γ of $R^n, n \geq 1$, as follows

$$W^\infty\{a_\alpha, 2\} = \{\phi(x) \in C^\infty(R^n) : \sum_{|\alpha|=0}^{\infty} a_\alpha ||D^\alpha \phi||_2^2 < \infty\},$$

where $a_\alpha \geq 0$ is a numerical sequence, and $||.||_2$ is the canonical norm in the space $L^2(R^n)$ (all functions are assumed to be real valued) and

$$D^\alpha = \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1} \dots (\partial x_n)^{\alpha_n}},$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index for differentiation, $|\alpha| = \sum_{i=1}^n \alpha_i$.

The space $W^{-\infty}\{a_\alpha, 2\}$ is defined as the formal conjugate space to the space $W^{\infty}\{a_\alpha, 2\}$, namely:

$$W^{-\infty}\{a_\alpha, 2\} = \{\psi(x) : \psi(x) = \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha \psi_\alpha(x)\},$$

where $\psi_\alpha \in L^2(R^n)$ and $\sum_{|\alpha|=0}^{\infty} a_\alpha ||\psi_\alpha||_2^2 < \infty$.

The duality pairing of the spaces $W^{\infty}\{a_\alpha, 2\}$ and $W^{-\infty}\{a_\alpha, 2\}$ is postulated by the formula

$$(\phi, \psi) = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_{R^n} \psi_\alpha(x) D^\alpha \phi(x) dx,$$

where

$$\phi \in W^{\infty}\{a_\alpha, 2\}, \qquad \psi \in W^{-\infty}\{a_\alpha, 2\}.$$

From above, $W^{\infty}\{a_\alpha, 2\}$ is everywhere dense in $L^2(R^n)$ with topological inclusions and $W^{-\infty}\{a_\alpha, 2\}$ denotes the topological dual space with respect to $L^2(R^n)$, so we have the following chain:

$$W^{\infty}\{a_\alpha, 2\} \subseteq L^2(R^n) \subseteq W^{-\infty}\{a_\alpha, 2\}.$$

Analogous to the above chain we have:

$$W_0^{\infty}\{a_\alpha, 2\} \subseteq L^2(R^n) \subseteq W_0^{-\infty}\{a_\alpha, 2\},$$

where $W_0^{\infty}\{a_\alpha, 2\}$ is the set of all functions of $W^{\infty}\{a_\alpha, 2\}$ which vanish on the boundary Γ of R^n, i.e.

$$W_0^{\infty}\{a_\alpha, 2\} = \{\phi \in C_0^{\infty}(R^n) : \\ ||\phi||^2 = \sum_{|\alpha|=0}^{\infty} a_\alpha ||D^\alpha \phi||_2^2 < \infty, D^\omega \phi|_\Gamma = 0, |\omega| \leq \alpha - 1\}.$$

We now introduce $L^2(0, T; L^2(R^n))$ which we shall denoted by $L^2(Q)$, where $Q = R^n \times]0, T[$, denotes the space of measurable functions $t \to \phi(t)$ such that

$$||\phi||_{L^2(Q)} = (\int_0^T ||\phi(t)||_2^2 dt)^{\frac{1}{2}} < \infty,$$

endowed with the scalar product $(f, g) = \int_0^T (f(t), g(t))_{L^2(R^n)} dt$, $L^2(Q)$ is a Hilbert space. In the same manner we define the spaces

$L^2(0,T;W^\infty\{a_\alpha,2\})$, $L^2(0,T;W_0^\infty\{a_\alpha,2\})$ and $L^2(0,T;W^{-\infty}\{a_\alpha,2\})$, $L^2(0,T;W_0^{-\infty}\{a_\alpha,2\})$ as its formal conjugate respectivrly.

Finally, we have the following chains:

$$L^2(0,T;W^\infty\{a_\alpha,2\}) \subseteq L^2(Q) \subseteq L^2(0,T;W^{-\infty}\{a_\alpha,2\}),$$

$$L^2(0,T;W_0^\infty\{a_\alpha,2\}) \subseteq L^2(Q) \subseteq L^2(0,T;W_0^{-\infty}\{a_\alpha,2\}).$$

2. Infinite Order Type Equation

The paper [24] is a review article for previous results, that earlier obtained by I.M. Gali, H.A. El-Saify and S.A. El-Zahaby. The results obtained there are for the case of operators with an infinite number of variables which are elliptic, parabolic, hyperbolic or well-posed in the sense of Petrowsky.

Subsequently, J.L. Lions suggested a problem related to this result but in different direction by taking the case of operators of infinite order with finite dimension in the form

$$(A\Phi)(x) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha}\Phi(x,t)\,. \tag{2.1}$$

For this operator the bilinear form $\pi(t;\Phi,\Psi) := (A\Phi,\Psi)_{L^2(R^n)}$ is coercive on $W_0^\infty\{a_\alpha,2\}$.

The operator A is a bounded self-adjoint elliptic operator mapping $W_0^\infty\{a_\alpha,2\}$ onto $W_0^{-\infty}\{a_\alpha,2\}$.

Now let us consider the following evolution equation:-

$$Ay + \frac{\partial^2 y}{\partial t^2} = f, \qquad x\in R^n, \quad t\in(0,T), \tag{2.2}$$

$$y(x,0) = y_1(x), \qquad x\in R^n, \tag{2.3}$$

$$\frac{\partial y}{\partial t}(x,0) = y_2, \qquad x\in R^n, \tag{2.4}$$

$$D^\omega y(x,t) = 0, \qquad x\in\Gamma, \quad t\in(0,T), \tag{2.5}$$

$$|\omega| = 0,1,2,\ldots, \qquad |\omega|\le \alpha-1,$$

where $f\in L^2(0,T;W_0^{-\infty}\{a_\alpha,2\})$, $y_1\in W_0^\infty\{a_\alpha,2\}$, $y_2\in L^2(R^n)$.

Remark 2.1. From [9], [10] and the results of (see [18]) we know that there is the unique solution $(y, \frac{\partial y}{\partial t}) \in L^2(0,T; W_0^{\infty}\{a_\alpha, 2\}) \times L^2(Q)$ to the equations (2.2)-(2.5) and the mapping

$$(f, y_1, y_2) \to (y, \frac{\partial y}{\partial t}) : L^2(0,T; W_0^{-\infty}\{a_\alpha, 2\}) \times W_0^{\infty}\{a_\alpha, 2\} \times L^2(R^n) \\ \to L^2(0,T; W_0^{-\infty}\{a_\alpha, 2\}) \times L^2(Q)$$

is (norm, norm)- continuous. Moreover, the operator $A+\frac{\partial^2}{\partial t^2}$ (see [9], [10]) is the linear bounded which maps $L^2(0,T; W_0^{\infty}\{a_\alpha, 2\})$ onto $L^2(0,T; W_0^{-\infty}\{a_\alpha, 2\})$.

3. Optimal Control Problem for Infinite Order Type Equation: Optimality Conditions

We consider the following optimization problem:

$$Ay + \frac{\partial^2 y}{\partial t^2} = u, \qquad x \in R^n, \quad t \in (0,T), \tag{3.1}$$

$$y(x,0) = y_1(x), \qquad x \in R^n, \tag{3.2}$$

$$\frac{\partial y}{\partial t}(x,0) = y_2, \qquad x \in R^n, \tag{3.3}$$

$$D^{\omega} y(x,t) = 0, \qquad x \in \Gamma, \quad t \in (0,T), \tag{3.4}$$

Let us denote by $Y = L^2(0,T; W_0^{\infty}\{a_\alpha, 2\}) \times L^2(Q)$ the space of states and by $U = L^2(Q)$ the space of controls. The control time T is assumed to be fixed.

The performance functional is given by:

$$I(y,u) = \int_Q F(x,t,y,u)\, dxdt \;\to\; \min, \tag{3.5}$$

where $F : R^n \times (0,T) \times R^1 \times R^1 \to R^1$ satisfies the following conditions:

(A_1) $F(x,t,y,u)$ is continuous with respect to (x,t,y,u),

(A_2) there exists $F_y(x,t,y,u)$, $F_u(x,t,y,u)$ which are continuous with respect to (x,t,y,u),

(A_3) $F(x,t,y,u)$ is strictly convex with respect to the pair (y,u), i.e.:

$$F(x,t,\lambda y_1+(1-\lambda)y_2,\lambda u_1+(1-\lambda)u_2)<\lambda F(x,t,y_1,u_1)+(1-\lambda)F(x,t,y_2,u_2),$$
$$\forall y_1,y_2,u_1,u_2\in R^1,\quad (y_1,u_1)\neq(y_2,u_2),\quad \lambda\in(0,1).$$

The Control-state Constranits.

We assume the following constraints:

on controls $u \in U_{ad}$ is a closed, convex subset in the space$L^2(Q)$ (3.6)

and

on states $y\in Y_{ad}$ is a closed, convex subset in $L^2(0,T;W_0^\infty\{a_\alpha,2\})$ with non-empty interior. (3.7)

Also we assume the following condition: there exists $(\tilde y,\tilde u)$ such as $\tilde y\in$ intY_{ad}, $\tilde u\in U_{ad}$ and $(\tilde y,\tilde u)$ satisfies the equations (3.1)-(3.4) (the so-called Slater condition).

Making use the Dubovitskii-Milyutin Theorem (Theorem 6.1, [11]) we derive the necessary and sufficient optimality conditions to the problem (3.1)-(3.7). The solution of the stated optimal control problem is equivalent to seeking a pair $(y^0,u^0)\in E=Y\times U$ that satisfies Eqs. (3.1)-(3.4) and minimizes the performance functional (3.5) subject to the control constraints (3.6) and the state constraints (3.7). We formulate the necessary and sufficient conditions of optimality in the following theorem.

Theorem 3.1. *Under the assumptions mentioned above, there is a unique solution (y^0,u^0) to problem (3.1)-(3.7). Moreover, there is an adjoint state $p,(p,\frac{\partial p}{\partial t})\in L^2(0,T;W_0^\infty\{a_\alpha,2\})\times L^2(Q)$, which satisfies (in weak sense) the adjoint equation given below, and so the necessary and sufficient conditions of optimality are characterized by the following system of partial differential equations and inequalities:*

State equation.

$$Ay^0+\frac{\partial^2 y^0}{\partial t^2}=u^0,\qquad x\in R^n,\quad t\in(0,T), \tag{3.8}$$

$$y^0(x,0) = y_1(x), \qquad x \in R^n, \tag{3.9}$$

$$\frac{\partial y^0}{\partial t}(x,0) = y_2, \qquad x \in R^n, \tag{3.10}$$

$$D^w y^0(x,t) = 0, \qquad x \in \Gamma, \quad t \in (0,T). \tag{3.11}$$

Adjoint equation.

$$Ap + \frac{\partial^2 p}{\partial t^2} = F_y, \qquad x \in R^n, \quad t \in (0,T), \tag{3.12}$$

$$p(x,T) = 0, \qquad x \in R^n, \tag{3.13}$$

$$\frac{\partial p}{\partial t}(x,T) = 0, \qquad x \in R^n, \tag{3.14}$$

$$D^w p(x,t) = 0, \qquad x \in \Gamma, \quad t \in (0,T). \tag{3.15}$$

Maximum conditions.

$$\int_Q (p + F_u)(u - u^0)dxdt \geq 0 \qquad \forall u \in U_{ad}, \tag{3.16}$$

$$\int_Q (F_y + F_u \mathcal{F})(y - y^0)dxdt \geq 0 \quad \forall y \in Y_{ad}, \tag{3.17}$$

where $\mathcal{F} : \{y \in L^2(0,T; W_0^\infty\{a_\alpha, 2\}); A + \frac{\partial^2}{\partial t^2} \in L(Q)\} \to U$ *is the operator related to the equations (3.1)-(3.4) with zero-initial conditions,* F_y, F_u *are the Frèchet derivatives of* F *with respect to* y, u*, respectively at the point* (y^0, u^0).

Outline of Proof. (see [12]-[15]) We apply the generalized Dubovitskii-Milyutin theorem, we approximate the set representing the inequality conditions by regular admissible cone, the equality constraints by a regular tangent cone, and the performance functional by a regular improvement cone.

Let us denote by Q_1, Q_2, Q_3 the sets in the space $E := Y \times U$ with elements $z := ((y, \frac{\partial y}{\partial t}), u)$ so that

Equality constraint. The set Q_1 representing the equality constraint has the form:

$$Q_1 := \left\{ z \in E; \quad \begin{array}{ll} A(t)y + \dfrac{\partial^2 y}{\partial t^2} = u, & x \in R^n, \ t \in (0,T), \\ y(x,0) = y_1(x), & x \in R^n, \\ \dfrac{\partial y}{\partial t}(x,0) = y_2, & x \in R^n, \\ D^w y(x,t) = 0, & x \in \Gamma, \ t \in (0,T). \end{array} \right.$$

Control constraints. The set $Q_2 = Y \times U_{ad}$ representing the inequality constraints is closed and convex with nonempty interior in the space E has the form:

$$Q_2 := \{z \in E : (y, \frac{\partial y}{\partial t}) \in Y, u \in U_{ad}\}.$$

State constraints.

$$Q_3 := \{z \in E : y \in Y_{ad}, \frac{\partial y}{\partial t} \in L^2(Q), u \in U\}.$$

Thus the optimization problem my be formulated in such a form as

$$I(y,u) \to \min \qquad \text{subject to} \qquad (y,u) \in Q_1 \cap Q_2 \cap Q_3.$$

We approximate the sets Q_1 and Q_2 by the regular tangent cones (RTC), the set Q_3 by the regular admissible cone (RAC) and the performance index by the regular cone of decrease (RFC).

The cone tangent to the set Q_1 at z^0 has the form:

$$RTC(Q_1, z^0) = \{\overline{z} \in E; \qquad P'(z^0)\overline{z} = 0\} =$$

$$= \left\{ \bar{z} \in E; \quad \begin{array}{ll} A\overline{y} + \dfrac{\partial^2 \overline{y}}{\partial t^2} = \overline{u}, & x \in R^n,\ t \in (0,T), \\ \overline{y}(x,0) = 0, & x \in R^n, \\ \dfrac{\partial \overline{y}}{\partial t}(x,0) = 0, & x \in R^n, \\ D^w \overline{y}(x,t) = 0, & x \in \Gamma,\ t \in (0,T), \end{array} \right.$$

where $P'(z^0)\overline{z}$ is the Frèchet differential of the operator

$$P(y, \frac{\partial y}{\partial t}, u) := \left(Ay + \frac{\partial^2 y}{\partial t^2} - u, y(x,0) - y_1, \frac{\partial y}{\partial t}(x,0) - y_2(x) \right)$$

mapping from the space $\mathcal{W} := L^2(0,T; W_0^{\infty}\{a_\alpha, 2\}) \times L^2(Q) \times L^2(Q)$ into the space $\Omega := L^2(0,T; W_0^{-\infty}\{a_\alpha, 2\}) \times W_0^{\infty}\{a_\alpha, 2\} \times L^2(R^n)$. According to Remark 2.1 there exists a unique solution to the equation (3.1)-(3.4) for every u, y_1 and y_2 and it is easy to prove that $P'(z^0)$ is the mapping from the space $\mathcal{W}$ onto Ω, as need in Lusternik theorem (Theorem 9.1, [11]).

The tangent cone $RTC(Q_2, z^0)$ to the set Q_2 at z^0 has the form $Y \times RTC(U_{ad}, u^0)$, where $RTC(U_{ad}, u^0)$ is the tangent cone to the set U_{ad} at the point u^0.

Following [21] it is easy to show that

$$RTC(Q_1 \cap Q_2, z^0) = RTC(Q_1, z^0) \cap RTC(Q_2, z^0).$$

We only need to show the inclusion " $\supset$ ", because we always have " $\subset$ " [16].

It can be easily checked that in the neighborhood V_0 of the point $((y^0, \frac{\partial y^0}{\partial t}), u^0)$ the operator P satisfies the assumptions of the implicit function theorem [6]. Consequently, the set Q_1 can be represented in the neighborhood V_0 in the form

$$\left\{((y, \frac{\partial y}{\partial t}), u) \in E; \quad (y, \frac{\partial y}{\partial t}) = \varphi(u)\right\}, \tag{3.18}$$

where $\varphi : L^2(Q) \to L^2(0,T; W_0^{\infty}\{a_\alpha, 2\}) \times L^2(Q)$ is the operator of class C^1 satisfying the condition $P(\varphi(u), u) = 0$ for u, such as $(\varphi(u), u) \in V_0$.

From this we get

$$RTC(Q_1, z^0) = \{\overline{z} \in E; (\overline{y}, \frac{\partial \overline{y}}{\partial t}) = \varphi_u(u^0)\overline{u}\}. \tag{3.19}$$

Let $((\overline{y}, \frac{\partial \overline{y}}{\partial t}), \overline{u})$ be any element of the set $RTC(Q_1, z^0) \cap RTC(Q_2, z^0)$.

From the definition of the tangent cone we know that there exists an operator $r_u^2 := R^1 \to U$, such as $\frac{r_u^2(\epsilon)}{\epsilon} \to 0$ with $\epsilon \to 0^+$ and

$$\left(\left(y^0, \frac{\partial y^0}{\partial t}\right), u^0\right) + \epsilon\left(\left(\overline{y}, \frac{\partial \overline{y}}{\partial t}\right), \overline{u}\right) + (r_y^2, r_u^2) \in Q_2 \tag{3.20}$$

for a sufficiently small ϵ and with any $r_y^2(\epsilon)$.

From (3.18) it results that for sufficiently small ϵ, we get

$$\left(\varphi(u^0 + \epsilon\overline{u} + r_u^2(\epsilon)), u^0 + \epsilon\overline{u} + r_u^2(\epsilon)\right) \in Q_1.$$

Since φ is a differentiable operator, therefore

$$\varphi(u^0 + \epsilon\overline{u} + r_u^2(\epsilon)) = \varphi(u^0) + \epsilon\varphi_u(u^0)\overline{u} + r_y^1(\epsilon)$$

for some $r_y^1(\epsilon)$ such as $\frac{r_y^1(\epsilon)}{\epsilon} \to 0$ with $\epsilon \to 0^+$.

Taking into account (3.18) and (3.19), we can get

$$\left(\left(y^0, \frac{\partial y^0}{\partial t}\right), u^0\right) + \epsilon\left(\left(\overline{y}, \frac{\partial \overline{y}}{\partial t}\right), \overline{u}\right) + (r^1(\epsilon)_y, r_u^2(\epsilon)) \in Q_1. \tag{3.21}$$

If in (3.20) we have $r_u^2(\epsilon) = r_y^1(\epsilon)$, then it results from (3.20) and (3.21) that $\left(\left(\overline{y}, \frac{\partial \overline{y}}{\partial t}\right), \overline{u}\right)$ is an element of the cone tangent to the set $Q_1 \cap Q_2$ at z^0. It completes the proof of the inclusion " $\supset$ ". From [16] it is known that the tangent cones are closed. Further applying Theorem 3.3 [22] we can prove that the adjoint cones $[RTC(Q_1, z^0)]^*$ and $[RTC(Q_2, z^0)]^*$ are of the same sense [22].

The admissible cone $RTC(Q_3, z^0)$ to the set Q_3at z^0 is

$$RAC(Y_{ad}, y^0) \times L^2(Q) \times U,$$

where $RAC(Y_{ad}, y^0)$ is the admissible cone to the set Y_{ad} at y^0.

Using Theorem 7.5 [11] we find the regular cone of decrease

$$RFC(I, z^0) = \{\overline{z} \in E; I'(z^0)\overline{z} < 0\},$$

where $I'(z^0)\overline{z}$ is the Frèchet differential of the performance functional.

With the assumptions (A1), (A2) this differential exists (compare with the Example 7.2 [11]) and can be written as follows:

$$\int_Q (F_y \overline{y} + F_u \overline{u}) dx dt.$$

If $RFC(I, z^0) \neq \emptyset$, then the adjoint cone consists of the elements of the form (Theorem 10.2 [11]):

$$f_4(\overline{z}) = -\lambda_0 \int_Q (F_y \overline{y} + F_u \overline{u}) dx dt,$$

where $\lambda_0 \geq 0$.

Since $RTC(Q_1, z^0)$ is a subspace of E, then the functionals belonging to $[RTC(Q_1, z^0)]^*$ are (Theorem 10.1 [11]):

$$f_1(\overline{z}) = 0 \qquad \forall \overline{z} \in RTC(Q_1, z^0).$$

The functionals $f_2(\overline{z}) \in [RTC(Q_2, z^0)]^*$ can be expressed as follows

$$f_2(\overline{z}) = f_2^1(\overline{y}, \frac{\partial \overline{y}}{\partial t}) + f_2^2(\overline{u}),$$

where $f_2^1(\overline{y}, \frac{\partial \overline{y}}{\partial t}) = 0 \ \forall (\overline{y}, \frac{\partial \overline{y}}{\partial t}) \in Y$ (Theorem 10.1 [11]), $f_2^2(\overline{u})$ is the support functional to the set U_{ad} at the point u^0 (Theorem 10.5 [11]).

Similarly, the functionals $f_3(\overline{z}) \in [RTC(Q_3, z^0)]^*$ can be expressed as follows:

$$f_3(\overline{z}) = f_3^1(\overline{y}) + f_3^2(\frac{\partial \overline{y}}{\partial t}) + f_3^3(\overline{u}),$$

where $f_3^1(\overline{y})$ is the support functional to the set Y_{ad} at the point y^0 (Theorem 10.5 [11]), $f_3^2(\frac{\partial \overline{y}}{\partial t}) = 0 \quad \forall \frac{\partial \overline{y}}{\partial t} \in L^2(Q)$ and $f_3^3(\overline{u}) = 0 \quad \forall \overline{u} \in U$ (Theorem 10.1 [11]).

Euler-Lagrange equation. Since all assumptions of the Dubovitskii-Milyutin theorem are satisfied and we know suitable cones we are ready now to write down the Euler-Lagrange equation in the following form:

$$f_2^2(\overline{u}) + f_3^1(\overline{y}) = \lambda_0 \int_Q (F_y\overline{y} + F_u\overline{u})dxdt = \frac{1}{2}\lambda_0 \int_Q (F_y\overline{y} + F_u\overline{u})dxdt + \frac{1}{2}\lambda_0 \int_Q (F_y\overline{y} + F_u\overline{u})dxdt \qquad \forall \overline{z} \in RTC(Q_1, z^0). \quad (3.22)$$

We transform $\int_Q F_y\overline{y}dxdt$ introducing the adjoint variable p with the equation (3.12)-(3.15) and taking into account that $(\overline{y}, \frac{\partial \overline{y}}{\partial t})$ is the solution of $P'(z^0)\overline{z} = 0$ for any fixed $\overline{u}$.

We get

$$\int_Q F_y\overline{y}dxdt = \int_Q \left(Ap + \frac{\partial^2 p}{\partial t^2}\right)\overline{y}dxdt = \int_Q pA\overline{y}dxdt$$

$$+ \int_{R^n} \frac{\partial p}{\partial t}\overline{y}\bigg|_0^T dx - \int_{R^n} p\frac{\partial \overline{y}}{\partial t}\bigg|_0^T dx + \int_Q p\frac{\partial^2 \overline{y}}{\partial t^2}\bigg|_0^T dxdt = \int_Q p\overline{u}dxdt.$$

Further $\int_Q F_u\overline{u}dxdt$ can be replaced by $\int_Q F_u\mathcal{F}\overline{y}dxdt$.

Taking the above into account, from (3.22) we obtain

$$f_2^2(\overline{u}) + f_3^1(\overline{y}) = \frac{1}{2}\lambda_0 \int_Q (p + F_u)\overline{u}dxdt + \frac{1}{2}\lambda_0 \int_Q (F_y + F_u\mathcal{F})\overline{y}dxdt. \quad (3.23)$$

A umber λ_0 in (3.23) cannot be equal to zero, because in this case all functionals in the Euler-Lagrange equation would be zero, which is impossible according to the Dubovitskii-Milyutin theorem.

Using the definition of the support functional and dividing both sides of the obtained inequalities by $\frac{1}{2}\lambda_0$, we finally get (3.16), (3.17).

If $RFC(I, z^0) = \emptyset$, then optimality conditions (3.8)-(3.17) are fulfilled with equalities in the maximum conditions (3.16), (3.17).

In order to prove sufficiency of the derived conditions of optimality we make use of the fact that the constraints are convex, the performance functional is continuous and convex, and the Slater condition is satisfied.

Really, there exists a point $(\tilde{y}, \tilde{u}) \in \text{int}Q_2$, such that $(\tilde{y}, \tilde{u}) \in Q_1$. This fact follows immediately from the existence of a nonempty interior in the set Q_2 and from Remark 3.1.

The uniqueness of the solution to the problem (3.1)-(3.7) follows from the strict convexity of the performance functional (3.5) (assumption (A3)). This last remark completes the proof of Theorem 3.1. □

References

[1] Y. Censor, Optimality in multi objective problems, *Appl. Math. Optim.*, **4** (1977), 41-59.

[2] Ju.A. Dubinskii, About one method for solving partial differential equation, *Dokl. Akad. Nauk SSSR*, **258** (1981), 780-784.

[3] Ju.A. Dubinskii, Non-triviality of Sobolev spaces of infinite order for a full Euclidean space and a tours, *Math . USSR. Sb.*, **29** (1976), 393-401.

[4] Ju.A. Dubinskii, Sobolev spaces of infinite order and behavior of solution of some boundary value problems with unbounded increase of the order of the equation, *Math. USSR. Sb.*, **27** (1975), 143-162.

[5] T.D. Dzhuraev, Zh.O. Takhirov, Hyperbolic Stefan problem, *Differential Equations*, **30** (1994), 755-765.

[6] A.D. Ioffe, W.M. Tikhomirov, *Theory of Extermal Problems*, North Holland, Amsterdam (1979).

[7] I.M. Gali, H.A. El-Saify, Optimal control of a system governed by hyperbolic operator with an infinite number of variables, *JMAA*, **85** (1982), 24-30.

[8] I.M. Gali, H.A. El-Saify, Distributed control of a system governed by Dirichlet and Neumann Problems for a self-adjoint elliptic operator with an infinite number of variables, *Journal of Optimization Theory and Applications*, **39** (1983), 293-298.

[9] I.M. Gali, H.A. El-Saify, Control of system governed by infinite order equation of hyperbolic type, In: *Proceeding of the international conference on Functional-Differential systems and related topics*, Poland, **III** (1983), 99-103.

[10] I.M. Gali, H.A. El-Saify, S.A. El-Zahaby, Distributed Control of a system governed by Dirichlet and Neumann Problems for Elliptic equations of infinite order, In: *Proceeding of the international conference on Functional-Differential systems and related topics*, Poland, **III** (1983), 83-87.

[11] I.V. Girsanov, *Lectures on Mathematical Theory of Extremum Problems*, New York, Springer-Verlag (1972).

[12] W. Kotarski, Some problems of optimal and Pareto optimal control for distributed parameter systems, *Reports of Silesian University*, No. 1668, Katowice, Poland (1997), 1-93.

[13] W. Kotarski, Optimal control of a system governed by Petrowsky type equation with an infinite number of variables, *Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math.*, **35** (1996), 73-82.

[14] W. Kotarski, Optimal control of a system governed by a parabolic equation with an infinite number of variables and time delay, *Journal of Optimization Theory and Applications*, **63** (1989), 91-101.

[15] A. Kowalewski, W. Kotarski, On application of Milutin-Dubovicki's theorem to an optimal control problem for systems described by partial differential equations of hyperbolic type with time delay, *Systems Sci.*, **7** (1982), 55-74.

[16] I. Lasiecka, *Conical Approximations of Sets With Application to Optimization*, Technical University, Warsaw, Poland, Ph.D. Thesis (1974), In Polish.

[17] P.J. Laurent, *Approximation et Optimization*, Paris, Hermann (1972).

[18] J.L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, Band **170** (1971).

[19] U. Ledzewicz-Kowalewska, Optimal control problems of hyperbolic systems with operator and non operator equality constraints, *Applicable Analysis*, **27** (1988), 199-215.

[20] F. Tröltzsch, *Optimality Conditions for Parabolic Control Problems and Applications*, Teubner-Texte zür Mathematik, Leipzig, Band **62** (1984).

[21] S. Walczak, One some control problem, *Acta Universitatis Lodziensis, Folia Mathematica*, **1** (1984), 187-196.

[22] S. Walczak, Some properties of cones in normed spaces and their application investigating extremal problems, *J. of Optimization Theory and Applications*, **42** (1984), 561-582.

[23] S.A. El-Zahaby, Necessary conditions for control problems governed by elliptic variational inequalities with an infinite number of variables, *J. Inform. Optim. Sci.*, **15** (1994), 171-178.

[24] S.A. El-Zahaby, Optimal control of systems governed by infinite order operators, *Dienstag Uhr, POT*, **151** (2000).