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GENERALIZED NEWMARK SCHEMES FOR SINGULAR SECOND ORDER INITIAL-VALUE PROBLEMS

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Abstract: An important class of singular second order initial-value problems is $y'' + (\alpha/x)\, y' + f(x,y) = 0$, $0 < x < x_f$, $y(0) = a$, $y'(0) = 0$, with $\alpha = 1$ (cylindrical symmetry) or $\alpha = 2$ (spherical symmetry). This class includes the well-known singular equations of Emden and Liouville which have found applications in electrohydrodynamics, thermal explosions and stellar stability. For *regular* second order initial-value problems (with $\alpha = 0$), a well-known *singly-implicit* one-step integration scheme is due to Newmark. In the present paper, we describe Newmark-like singly-implicit one-step integration schemes for problems with $\alpha = 1$ and 2. The second order convergence of the obtained *generalized Newmark schemes* is justified mathematically and demonstrated computationally through problems of practical interest.

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1. Introduction

An important class of singular second order initial-value problems which occur in many applied problems is described by

$$y'' + \frac{\alpha}{x} y' + f\left(x, y\right) = 0, \quad 0 < x < x_f \ , \quad y\left(0\right) = a, \ y'\left(0\right) = 0, \tag{1.1}$$

where a is a finite constant, $\alpha = 1$ for problems with cylindrical symmetry and $\alpha = 2$ for problems with spherical symmetry. This class includes the well-known singular equations of Emden and Liouville. These singular second order differential equations have found applications in electrohydrodynamics (Keller [1]) in the theory of thermal explosions (Chambré [2]; see also Ames [3]) and in the study of stellar stability (Chandrasekhar [4]). Corresponding singular two-point boundary-value problems also occur frequently in the study of generalized axially symmetric potentials after the separation of variables has been employed (Parter [5]). Therefore the interest in singular initial-value problems (1.1) is also due to the fact that shooting techniques for the solution of singular two-point boundary-value problems require the solution of initial-value problem.

For *regular* problems (1.1) with $\alpha = 0$, a well-known and widely used scheme is due to Newmark (see, e.g. Fried [6]):

$$y_{k+1} = y_k + hy'_k - \frac{h^2}{4}\left[f_k + f_{k+1}\right], \tag{1.2a}$$

$$y'_{k+1} = y'_k - \frac{h}{2}\left[f_k + f_{k+1}\right]. \tag{1.2b}$$

Here, and in the following, for a positive integers N, we let $h = x_f/N$, $x_k = kh$, $k = 0(1)N$, and set $y_k = y\left(x_k\right)$, etc. Newmark is a one-step *singly-implicit* scheme in which the computation of the displacement y through (1.2a) is *implicit*, while the computation of the velocity y' through (1.2b) is *explicit*. Newmark is a second order P-stable scheme.

For singular problems (1.1) with spherical symmetry ($\alpha = 2$), Chawla et al [7] had shown that, interestingly, the already existing explicit Nyström methods for regular problems (with $\alpha = 0$), when applied to the singular problem (1.1), give their proper respective orders of convergence.

In the present paper, we obtain Newmark-like one-step *singly-implicit* second order schemes for the integration of singular second order initial-value problems (1.1) for the two important case of cylindrical symmetry ($\alpha = 1$) and spherical symmetry ($\alpha = 2$). The second order convergence of the obtained *generalized Newmark schemes* is justified mathematically and demonstrated computationally through problems of practical interest.

2. A Generalized Newmark Scheme for (1.1) with Cylindrical Symmetry

For the case of cylindrical symmetry, we consider the differential equation in (1.1) written in the form:

$$\left(xy'\right)' + xf = 0. \tag{2.1}$$

2.1. Derivation of the Scheme

We now set $v = xy'$, then $v' = -xf$. Integrating $v' = -xf$ from x_k to x_{k+1} we obtain

$$v_{k+1} = v_k - \psi_{k+1}, \tag{2.2}$$

where we have set

$$\phi\left(x\right) = xf\left(x, y\left(x\right)\right),\ \ \psi\left(x\right) = \int_{x_k}^{x} \phi\left(t\right) dt.$$

By applying the trapezoidal rule, we have

$$\psi_{k+1} = \int_{x_k}^{x_{k+1}} \phi\left(t\right) dt = \frac{h}{2}\left[\phi_k + \phi_{k+1}\right] - \frac{h^3}{12}\phi''\left(\xi_k\right), \quad \xi_k \in \left(x_k, x_{k+1}\right), \tag{2.3}$$

and then from (2.2) we obtain

$$v_{k+1} = v_k - \frac{h}{2}\left[x_k f_k + x_{k+1} f_{k+1}\right] + t_k^{(1)\prime}, \tag{2.4}$$

where $t_k^{(1)\prime} = \left(h^3/12\right)\phi''\left(\xi_k\right).$

Again, integrating $v' = -xf$ from x_k to x, dividing by x and then integrating from x_k to x_{k+1} , we have

$$y_{k+1} = y_k + J_k v_k - \int_{x_k}^{x_{k+1}} \frac{1}{x}\psi\left(x\right) dx, \tag{2.5}$$

where we have set $J_k = \ln\left(x_{k+1}/x_k\right).$ By linearly interpolating $\psi\left(x\right)$ at x_k and x_{k+1}, and noting that $\psi_k = 0$, we have

$$\int_{x_k}^{x_{k+1}} \frac{1}{x}\psi\left(x\right) dx = \frac{1}{h}a_k\psi_{k+1} + \frac{1}{2}b_k\psi''\left(\zeta_k\right), \quad \zeta_k \in \left(x_k, x_{k+1}\right), \tag{2.6}$$

where

$$a_k = h - x_k \ln\left(\frac{x_{k+1}}{x_k}\right),\ b_k = -\frac{1}{2}\left(x_{k+1}^2 - x_k^2\right) + x_k x_{k+1} \ln\left(\frac{x_{k+1}}{x_k}\right).$$

In (2.5), substituting the integral from (2.6) and ψ_{k+1} from (2.3), we obtain

$$y_{k+1} = y_k + J_k v_k - \frac{1}{2} a_k \left[x_k f_k + x_{k+1} f_{k+1}\right] + t_k\,(1)\,, \tag{2.7}$$

where

$$t_k^{(1)} = \frac{h^2}{12} a_k \phi''\left(\xi_k\right) - \frac{1}{2} b_k \psi''\left(\zeta_k\right).$$

Now, since $v_0 = 0$ and $\lim\limits_{x_k \to 0} x_k J_k = 0$, $a_0 = h$, from (2.7) and (2.4) we obtain the following scheme. For $k = 0$:

$$\begin{aligned} y_1 &= y_0 - \frac{h}{2} x_1 f_1, \\ v_1 &= -\frac{h}{2} x_1 f_1, \end{aligned} \tag{2.8a}$$

and for $k \geq 1$:

$$\begin{aligned} y_{k+1} &= y_k + J_k v_k - \frac{1}{2} a_k \left[x_k f_k + x_{k+1} f_{k+1}\right], \\ v_{k+1} &= v_k - \frac{h}{2}\left[x_k f_k + x_{k+1} f_{k+1}\right]. \end{aligned} \tag{2.8b}$$

We call the scheme described by (2.8) a *generalized Newmark (GN1)* scheme for singular second order initial value problems (1.1) with *cylindrical symmetry.*

2.2. Convergence of the GN1 Scheme

In the following, we assume that f is sufficiently differentiable and $|\partial f/\partial y| \leq K$, constant. We first note that $t_k^{(1)\prime} = O\left(h^3\right)$. Again, since for $h \to 0^+$, $a_k \sim h^2/\left(2x_k\right)$ and $b_k \sim -h^3/\left(6x_k\right)$, therefore $t_k^{(1)} = O\left(h^3\right)$.

With $\widetilde{y}_k$and $\widetilde{v}_k$denoting numerical approximations for y_k and v_k provided by the generalized Newmark scheme (2.8), let $e_k = y_k - \widetilde{y}_k$, $d_k = v_k - \widetilde{v}_k$. From (2.4), (2.7) and (2.8) we obtain the following equations for the errors for $k = 0(1)N-1$,

$$e_{k+1} = e_k + J_k d_k - \frac{1}{2} a_k \left[x_k g_k e_k + x_{k+1} g_{k+1} e_{k+1}\right] + t_k^{(1)}, \tag{2.9a}$$

$$d_{k+1} = d_k - \frac{1}{2}h\left[x_k g_k e_k + x_{k+1} g_{k+1} e_{k+1}\right] + t_k^{(1)\prime}, \qquad (2.9b)$$

where $e_0 = d_0 = 0$, and g_k are certain evaluations of $\partial f / \partial y$.

We now introduce the following $(N+1)$ −dimensional vectors and matrices. Let

$$\mathbf{e} = (e_0, e_1, ..., e_N), \qquad \mathbf{d} = (d_0, d_1, ..., d_N),$$
$$\mathbf{t}^{(1)} = \left(0, t_0^{(1)}, ..., t_{N-1}^{(1)}\right), \qquad \mathbf{t}^{(1)\prime} = \left(0, t_0^{(1)\prime}, ..., t_{N-1}^{(1)\prime}\right)$$

and let

$$C = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & \cdot & & \\ & & & \cdot & \\ & & & -1 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & & & & \\ 0 & 0 & & & \\ & J_1 & \cdot & & \\ & & \cdot & \cdot & \\ & & & J_{N-1} & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & & & & \\ & a_0 & & & \\ & & a_1 & & \\ & & & \cdot & \\ & & & & a_{N-1} \end{bmatrix},$$

$$G = \begin{bmatrix} 0 & & & & \\ x_1 g_1 & & & & \\ x_1 g_1 & x_2 g_2 & & & \\ & \cdot & & \cdot & \\ & & & x_{N-1} g_{N-1} & x_N g_N \end{bmatrix}.$$

Then, the error equations in (2.9a) and (2.9b) can be written in matrix form as

$$C\mathbf{e} = J\mathbf{d} - \frac{1}{2}AG\mathbf{e} + \mathbf{t}^{(1)}, \qquad (2.10a)$$

$$C\mathbf{d} = -\frac{h}{2}G\mathbf{e} + \mathbf{t}^{(1)\prime}. \qquad (2.10b)$$

Elimination of $\mathbf{d}$, gives

$$\mathbf{e} = -\frac{1}{2}P^{(1)}\mathbf{e} + \mathbf{r}^{(1)}, \qquad (2.11)$$

where

$$P^{(1)} = C^{-1}M^{(1)}, \quad \mathbf{r}^{(1)} = C^{-1}\tau^{(1)},$$

and

$$M^{(1)} = hJC^{-1}G + AG, \;\; \tau^{(1)} = JC^{-1}\mathbf{t}^{(1)\prime} + \mathbf{t}^{(1)}.$$

Now, it can be seen that $M^{(1)} = \left(m_{ij}^{(1)}\right)$ is a lower triangular matrix with zeros in the first column and

$$\begin{aligned} m_{i,i}^{(1)} &= a_{i-1}x_i g_i, \;\; i = 1(1)N, \\ m_{i,i-1}^{(1)} &= \left(hJ_{i-1} + a_{i-1}\right)x_{i-1}g_{i-1}, \;\; i = 2(1)N, \\ m_{i,j}^{(1)} &= 2hJ_{i-1}x_j g_j, \;\; j = 1(1)i-2, \;\; i = 3(1)N. \end{aligned}$$

Since for $h \to 0^+$, $a_k \sim h^2/\left(2x_k\right)$, $J_k \sim h/x_k$ for $M^{(1)}$ it follows that $\max\left|m_{i,j}^{(1)}\right| \lesssim \frac{3}{2}Kh^2$. Since $Nh = x_f$, for $P^{(1)} = \left(p_{i,j}^{(1)}\right)$, it follows that

$$\max\left|p_{i,j}^{(1)}\right| \lesssim \frac{3}{2}x_f Kh. \tag{2.12}$$

Next, it is easy to see that

$$\left\|JC^{-1}\right\|_\infty = \max\left[2J_1, 3J_2, ..., NJ_{N-1}\right],$$

so that for $h \to 0^+$, $\left\|JC^{-1}\right\|_\infty = 2$. It follows that for h sufficiently small, $\left\|\tau^{(1)}\right\|_\infty = O\left(h^3\right)$, and then $\left\|\mathbf{r}^{(1)}\right\|_\infty = O\left(h^2\right)$.

Now, from (2.11), with the help of (2.12), obtain

$$\begin{aligned} |e_i| &\le \frac{1}{2}\max\left|p_{i,j}^{(1)}\right|\sum_{j=0}^{i}|e_j| + \left|r_i^{(1)}\right| \\ &\le \left(\frac{3}{4}x_f K\right)h\sum_{j=0}^{i}|e_j| + \left|r_i^{(1)}\right|, \;\; i = 0(1)N. \end{aligned} \tag{2.13}$$

Since $e_0 = 0$, with the help of Gronwall's lemma (see [8]) convergence of order two follows from (2.13).

3. A Generalized Newmark Scheme for (1.1) with Spherical Symmetry

For the case of spherical symmetry, we consider the differential equation in (1.1) written in the form:

$$(xy)'' + xf = 0. \tag{3.1}$$

3.1. Derivation of the Scheme

We now set $w = (xy)'$, then $w' = -xf$. Integrating $w' = -xf$ from x_k to x_{k+1},we have

$$w_{k+1} = w_k - \psi_{k+1}. \tag{3.2}$$

By the trapezoidal formula (2.3), we obtain

$$w_{k+1} = w_k - \frac{h}{2}\left[x_k f_k + x_{k+1} f_{k+1}\right] + t_k^{(1)\prime}. \tag{3.3}$$

Again, integrating $w' = -xf$ from x_k to x, and then from x_k to x_{k+1}, we have

$$x_{k+1} y_{k+1} = x_k y_k + h w_k - \int_{x_k}^{x_{k+1}} \psi(t)\, dt. \tag{3.4}$$

Evaluating the integral in (3.4) by the trapezoidal rule and noting that $\psi_k = 0$, we have

$$x_{k+1} y_{k+1} = x_k y_k + h w_k - \frac{h}{2}\psi_{k+1} + \frac{h^3}{12}\phi'^{(\eta_k)}, \qquad \eta_k \in (x_k, x_{k+1}). \tag{3.5}$$

Finally, substituting ψ_{k+1}, we obtain

$$x_{k+1} y_{k+1} = x_k y_k + h w_k - \frac{h^2}{4}\left[x_k f_k + x_{k+1} f_{k+1}\right] + t_k^{(2)}, \tag{3.6}$$

where

$$t_k^{(2)} = \frac{h^3}{12}\left[\phi'(\eta_k) + \frac{h}{2}\phi''(\xi_k)\right].$$

Now, since $w_0 = a$, from (3.6) and (3.3) follows the following scheme. For $k \geq 0$

$$\begin{aligned} x_{k+1} y_{k+1} =& x_k y_k + h w_k - \frac{h^2}{4}\left[x_k f_k + x_{k+1} f_{k+1}\right], \\ w_{k+1} =& w_k - \frac{h}{2}\left[x_k f_k + x_{k+1} f_{k+1}\right]. \end{aligned} \tag{3.7}$$

We call the scheme described by (3.7) a *generalized Newmark (GN2)* scheme for singular second order initial-value problems (1.1).

It is interesting to note here that the scheme (3.7) can also be obtained by applying the Newmark scheme (1.2) directly to the differential equation written in the form (3.1). This situation for singular problems with *spherical symmetry* has already been noted by Chawla et al [7]. However, for either case of derivation of the scheme (3.7), it remains to be justified mathematically that it does, indeed, provide second order convergence. This is shown in the following section.

3.2. Convergence of the GN2 Scheme

First, note that both $t_k^{(2)}, t_k^{(1)\prime} = O\left(h^3\right)$. Now, for numerical approximations $\widetilde{y}_k, \widetilde{w}_k$ provided by the scheme (3.7), we again denote the errors by $e_k = y_k - \widetilde{y}_k, d_k = w_k - \widetilde{w}_k$. Then, from (3.3), (3.6) and (3.7), we obtain the error equations

$$x_{k+1}e_{k+1} = x_k e_k + h d_k - \frac{h^2}{4}\left[x_k g_k e_k + x_{k+1} g_{k+1} e_{k+1}\right] + t_k^{(2)}, \tag{3.8a}$$

$$d_{k+1} = d_k - \frac{h}{2}\left[x_k g_k e_k + x_{k+1} g_{k+1} e_{k+1}\right] + t^{(1)\prime}, \tag{3.8b}$$

where again g_k are certain evaluations of $\partial f/\partial y$.

Let

$$\mathbf{t}^{(2)} = \begin{bmatrix} 0 \\ t_0^{(2)} \\ \cdot \\ \cdot \\ t_{N-1}^{(2)} \end{bmatrix}, \quad A = \begin{bmatrix} 1 & & & & \\ 0 & x_1 & & & \\ & -x_1 & x_2 & & \\ & & \cdot & \cdot & \\ & & & -x_{N-1} & x_N \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \cdot & & \\ & & \cdot & \cdot & \\ & & & 1 & 0 \end{bmatrix}.$$

Since $e_0 = d_0 = 0$, the error equations in (3.8) can be expressed in matrix form as

$$A\mathbf{e} = hB\mathbf{d} - \frac{h^2}{4}G\mathbf{e} + \mathbf{t}^{(2)}, \tag{3.9a}$$

$$C\mathbf{d} = -\frac{h}{2}G\mathbf{e} + \mathbf{t}^{(1)\prime}. \tag{3.9b}$$

Eliminating $\mathbf{d}$ we have

$$\mathbf{e} = -\frac{h^2}{4}P^{(2)}\mathbf{e} + \mathbf{r}^{(2)}, \tag{3.10}$$

where

$$P^{(2)} = A^{-1}M^{(2)}, \ \mathbf{r}^{(2)} = A^{-1}\tau^{(2)},$$

$$M^{(2)} = 2BC^{-1}G + G, \ \tau^{(2)} = hBC^{-1}\mathbf{t}^{(1)\prime} + \mathbf{t}^{(2)}.$$

Now, it can be seen that

$$M^{(2)} = \begin{bmatrix} 0 & & & & & \\ 0 & x_1 g_1 & & & & \\ 0 & 3x_1 g_1 & x_2 g_2 & & & \\ 0 & 4x_1 g_1 & 3x_2 g_2 & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ 0 & 4x_1 g_1 & \cdot & 4x_{N-2} g_{N-2} & 3x_{N-1} g_{N-1} & x_N g_N \end{bmatrix}.$$

It is now easy to see that if $M^{(2)} = \left(m_{i,j}^{(2)} \right)$, then

$$\max \left| m_{i,j}^{(2)} \right| \leq 4 x_f K.$$

Since

$$A^{-1} = \begin{bmatrix} 1 & & & & \\ 0 & 1/x_1 & & & \\ 0 & 1/x_2 & 1/x_2 & & \\ \cdot & \cdot & \cdot & \cdot & \\ 0 & 1/x_N & & 1/x_N & 1/x_N \end{bmatrix},$$

it follows that if $A = \left(a_{i,j}^{-1} \right)$, $\max \left| a_{i,j}^{-1} \right| = 1/h$. Therefore, if $P^{(2)} = \left(p_{i,j}^{2} \right)$, then

$$\max \left| p_{i,j}^{(2)} \right| \leq 4 x_f K / h. \tag{3.11}$$

Again, it is easy to see that $\left\| BC^{-1} \right\|_\infty = x_f / h$, and therefore $\left\| \tau^{(2)} \right\|_\infty = O\left(h^3 \right)$. Since $\left\| A^{-1} \right\|_\infty = 1/h$, it follows that $\left\| \mathbf{r}^{(2)} \right\|_\infty = O\left(h^2 \right)$.

Now, from (3.10), with the help of (3.11), we have

$$\begin{aligned} |e_i| &\leq \frac{h^2}{4} \max \left| p_{i,j}^{2} \right| \sum_{j=0}^{i} |e_j| + \left| r_i^{(2)} \right| \\ &\leq h x_f K \sum_{j=0}^{i} |e_j| + \left| r_i^{(2)} \right|. \end{aligned} \tag{3.12}$$

Since $e_0 = 0$, again with the help of Gronwall's lemma, convergence of order two of the scheme follows from (3.12).

4. Numerical Experiments

To illustrate the computational performance of the obtained singly-implicit one-step generalized Newmark schemes GN1 and GN2 for singular problems (1.1), and to verify their second order of convergence, in the following we consider four examples of practical interest.

Problem 1. J.B. Keller [5] encountered an equation of the form (1.1) with $\alpha = 1$ and $f(x, y) = -e^y$ in an electrohydrodynamics problem. Chambré [2] (see Ames [3]) had developed an exact solution for the differential equation (1.1) with $f(x, y) = \nu e^y$. We consider the corresponding initial value problem:

$$\begin{aligned} &y'' + \frac{1}{x}y' + \nu e^y = 0, \quad 0 < x < 1, \\ &y(0) = 2\ln(\beta + 1), \; y'(0) = 0, \end{aligned} \tag{4.1}$$

with $\nu = -1$ for which case a unique solution, which is nonpositive, is given by

$$y(x) = 2\ln\left(\frac{\beta + 1}{\beta x^2 + 1}\right), \quad \text{where } \beta = 2\sqrt{6} - 5.$$

We computed the solution of the problem (4.1) by the generalized Newmark scheme GN1; the corresponding error norms $\|\mathbf{e}\|_\infty$ are shown in Table 1. The numerical approximations do confirm second order convergence of the GN1 scheme.

N	$\|\mathbf{e}\|_\infty$	order
20	1.3(-3)	
40	3.7(-4)	1.8
80	1.0(-4)	1.9
160	2.7(-5)	1.9

Table 1: Error-norms and order

Problem 2. We consider the problem:

$$y'' + \frac{1}{x}y' + \cos y = 0, \quad 0 < x < 20, \quad y(0) = 1, \; y'(0) = 0. \tag{4.2}$$

We computed the solution of the problem (4.2) by the generalized Newmark scheme GN1. The errors in computing the solution at $x_f = 20$ with various step lengths are shown in Table 2 and the numerical results verify the second order convergence of the GN1 scheme.

h	error	order
0.2	5.8(-3)	
0.1	1.2(-3)	2.27
0.05	2.2(-4)	2.45
0.025	2.8(-5)	2.97

Table 2: Errors at $x_f = 20$

The approximations obtained with $h = 0.1$ are displayed in Figure 1 versus the 'exact' solution (computed using $N = 1600$). It is clear from the figure that the GN1 scheme provides quite satisfactory approximations for the true solution over this long range of integration.

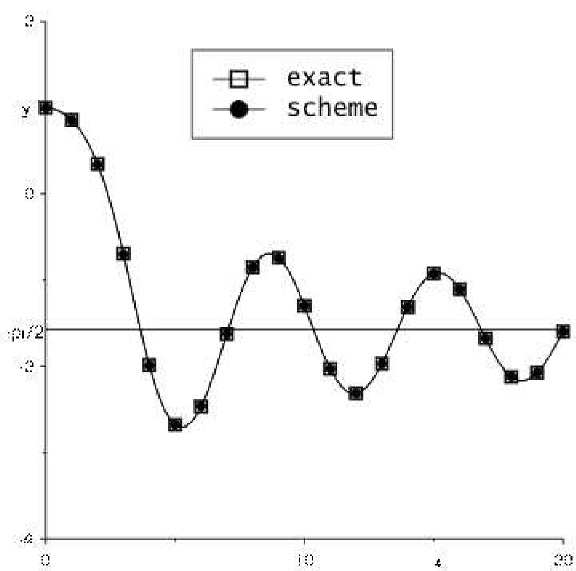

Figure 1: Problem 2

Problem 3. Chandrasekhar [4] derived an equation of the form (1.1) with $\alpha = 2$ and $f(x, y) = y^\gamma$, γ a physical constant, in connection with the equilibrium of isothermal gas spheres (Ames [3], p. 106). For the case $\gamma = 5$ the solution is known and we treat the associated initial value problem:

$$y'' + \frac{2}{x}y' + y^5 = 0, \quad 0 < x < 1,\ y(0) = 1,\ y'(0) = 0. \tag{4.3}$$

whose exact solution is $y(x) = \left(1 + x^2/3\right)^{-1/2}$.

We computed the solution of the problem (4.3) by the generalized Newmark scheme GN2 for various step lengths h. The corresponding error norms

$\|\mathbf{e}\|_\infty$ are shown in Table 3 and the numerical results verify the second order of convergence of the GN2 scheme.

Problem 4. We consider Emden's equation:

$$y'' + \frac{2}{x}y' + \sin y = 0, \quad 0 < x < 20, \ y(0) = 1, \ y'(0) = 0. \tag{4.4}$$

N	$\|\mathbf{e}\|_\infty$	order
4	1.4(-2)	
8	3.8(-3)	1.9
16	9.7(-4)	2.0
32	2.4(-4)	2.0

Table 3: Error-norms and order

We computed the solution of the problem (4.4) by the generalized Newmark scheme GN2. The errors in computing the solution at $x_f = 20$ with various step lengths are shown in Table 4 and the results verify second order convergence of the GN2 scheme.

h	error	order
0.2	1.95(-3)	
0.1	4.67(-4)	2.06
0.05	1.10(-4)	2.08
0.025	2.20(-5)	2.33

Table 4: Errors at $x_f = 20$

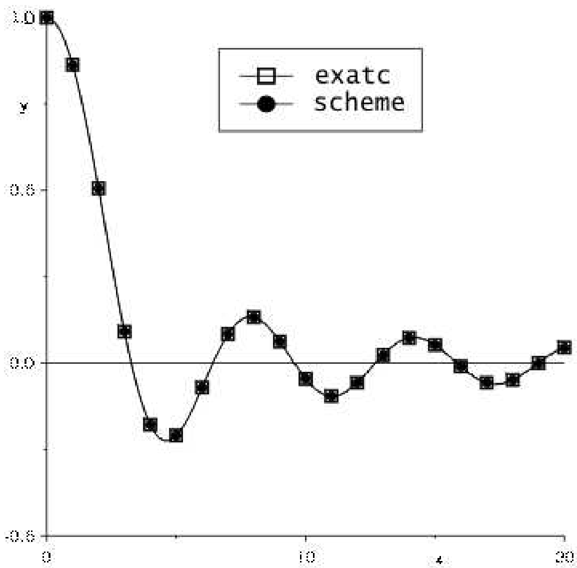

Figure 2: Problem 4

The approximations obtained with $h = 0.1$ are displayed in Figure 2 versus the 'exact' solution (computed using $N = 1600$). It is clear from the figure that the GN2 scheme provides quite satisfactory approximations for the true solution over this long range of integration.

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