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A MULTISCALING APPROACH TO THE NARROW GAS SLIDER BEARING IN SLIP FLOW

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Abstract: The method of multiple scales is used to obtain a uniformly valid expansion for the steady state pressure distribution of a finite rectangular isothermal gas slider bearing when the ratio of width to length is small, i.e. the bearing has narrow geometry.

Specific results are given for a variety of simple bearing geometries.

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Key Words: multiscaling, slider bearing, perturbations

1. Introduction

In this report, we are concerned with the task of calculating the steady state pressure distribution in a rectangular isothermal gas-lubricated slider bearing. Of particular interest here is the situation where the bearing has narrow geometry (i.e. $\epsilon \ll 1$, where ϵ is the ratio of transverse to longitudinal di-

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mensions of the bearing). For such bearings, the pressure field is determined as a solution of the so-called Reynolds equation, a nonlinear elliptic partial differential equation.

As we will see below, when cast in dimensionless form, this Reynolds equation involves two parameters - the ϵ described above, and the bearing number Λ, which depends on the physical properties of the gas lubricant, and the relative motion of the two sliding surfaces comprising the bearing.

For small ϵ (i.e. narrow geometries), a perturbation solution of the Reynolds equation based on the limit $\epsilon \to 0$ is indicated. For the purposes of this analysis, we will assume that the bearing number Λ, is $O(1)$ on the scale of $\epsilon \to 0$. Such circumstances are known to occur in a variety of applications of interest in engineering.

For $\epsilon \to 0$ the boundary value problem determining the pressure field in the bearing is known to be a singular perturbation problem, so appropriate methods need to be used in its solution. One such well-established method, the *method of matched expansions*, has been applied to this problem by Shepherd and DiPrima [2], to yield a useful approximation to the pressure field, valid for small ϵ . This method involves the approximations of the pressure on subdomains of the bearing and the linking of these approximations by a process called matching (Ch. 4, [3]). An alternative method, to be used here, is the *method of multiple scales*, (Ch. 6, [3]) which aims to detect the dependence of the pressure on so-called "slow" and "fast" variables; and by recasting the Reynolds equation in terms of these variables, to generate an approximation to the pressure in a single process. This technique, while apparently more involved computationally, has the advantage that it can display properties of the solution that are not so apparent in the results obtained by matching.

2. Governing Equations

Consider a finite rectangular isothermal gas slider bearing of transverse dimension B and longitudinal dimension L (see Figure 1).We choose coordinates X and Z parallel to L and B, respectively, so that the boundaries of the bearing are defined by $0 \le X \le L$, $-B/2 \le Z \le B/2$. If the two bearing surfaces are the XZ plane and the surface $Y = H(X,Z)$, and these move with constant relative speed U in the X -direction, the steady state pressure $P(X,Z)$

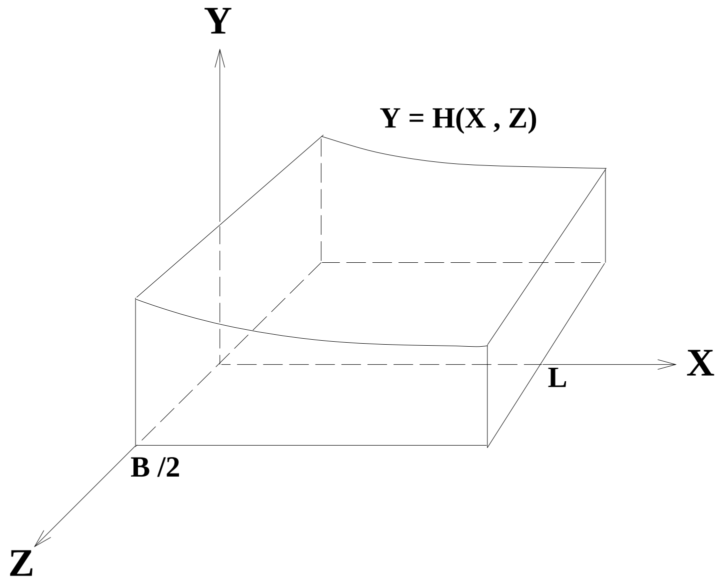

Figure 1: Geometry for the narrow gas slider

satisfies the nonlinear modified Reynolds equation

$$\frac{\partial}{\partial X}\left[H^3 P\left(1+\frac{6a\lambda_a}{PH}\right)\frac{\partial P}{\partial X}\right]+\frac{\partial}{\partial Z}\left[H^3 P\left(1+\frac{6a\lambda_a}{PH}\right)\frac{\partial P}{\partial Z}\right]$$

$$=6\mu U\frac{\partial}{\partial X}(PH)\,, \quad (1)$$

where μ is the viscosity of the gas in the bearing gap, and the positive combination $6a\lambda_a$ is a measure of the local molecular mean free path in the lubricant gas.

At the edges of the bearing the pressure returns to its ambient value P_a, and thus satisfies the boundary conditions

$$P(X,\pm B/2)=P(0,Z)=P(L,Z)=P_a. \quad (2)$$

By adopting dimensionless variables p, h, x and z defined by

$$p=\frac{P}{P_a},\ h=\frac{H}{H_0},\ x=\frac{X}{L},\ z=\frac{Z}{B}$$

respectively, and defining the dimensionless parameters Λ, K and ϵ by

$$\Lambda=\frac{6\mu UL}{P_a H_0^2},\quad K=\frac{6a\lambda_a}{H_0 P_a},\quad \epsilon=\frac{B}{L}$$

respectively, we can write the above problem in dimensionless form as

$$\epsilon^2 \frac{\partial}{\partial x}\left[h^3 p\left(1+\frac{K}{hp}\right)\frac{\partial p}{\partial x}\right]+\frac{\partial}{\partial z}\left[h^3 p\left(1+\frac{K}{hp}\right)\frac{\partial p}{\partial z}\right]=\epsilon^2\Lambda\frac{\partial}{\partial x}(hp) \tag{3}$$

on $0<x<1,\ \ -1/2<z<1/2$ together with the boundary conditions

$$p(x,\pm 1/2)=1, \quad 0\le x\le 1, \tag{4}$$

$$p(0,z)=p(1,z)=1, \quad -1/2\le z\le 1/2. \tag{5}$$

In the above: H_0 is a representative value of $H(X,Z)$, Λ is the bearing number, effectively a measure of the gas flow speed, and K is a form of the Knudsen number, a measure of the degree of slip in the flow. Clearly, for non-slip flow, $K=0$. The parameter ϵ is a measure of the narrowness of the bearing.

For bearings having narrow geometries, we have $0<\epsilon\ll 1$, so we may analyse the above problem by perturbation techniques based on the limit $\epsilon\to 0$. We assume that $h(x,z)$ is sufficiently differentiable in both variables for the analysis to be valid; and in order that the right hand side of (3) be $O(\epsilon^2)$, we assume that $\Lambda=O(1)$ on the scale of $\epsilon\to 0$.

3. Multiscaled Form Of The Equations

The analysis of the above problem by the method of matched expansions (see Aliu and Shepherd [1]) leads us to expect that the pressure field $p(x,z,\epsilon)$ may be viewed as depending, not only on the "slow" variables x and z, but also on two "fast" variables ξ and η, where

$$\xi=\frac{g(x,z)}{\epsilon} \tag{6}$$

and

$$\eta=\frac{1-r(x,z)}{\epsilon}. \tag{7}$$

In the above, we expect the functions g and r to be positive with other appropriate properties that will become evident in the following analysis.

Under the transformation

$$(x,z)\to(x,z,\xi,\eta)$$

above, the derivatives $\frac{\partial}{\partial x}$ $\frac{\partial}{\partial z}$ are converted to

$$\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial x} + \frac{g_x}{\epsilon}\frac{\partial}{\partial \xi} - \frac{r_x}{\epsilon}\frac{\partial}{\partial \eta}, \tag{8}$$

$$\frac{\partial}{\partial z} \equiv \frac{\partial}{\partial z} + \frac{g_z}{\epsilon}\frac{\partial}{\partial \xi} - \frac{r_z}{\epsilon}\frac{\partial}{\partial \eta}, \tag{9}$$

where subscripts denote partial derivatives in the usual way.

With this choice we note, as above, that the solution $p(x, z, \epsilon)$ of (3), (4) and (5) may be viewed as a function $P(x, z, \xi, \eta, \epsilon)$ of the four variables x, z, ξ, η, with ϵ as a parameter.

If the transformations (8) and (9) are applied to the equation (3), we obtain the following partial differential equation for $P(x, z, \xi, \eta, \epsilon)$

$$\begin{aligned}
&\epsilon^2\frac{\partial}{\partial x}[h^3P\frac{\partial P}{\partial x}] + \epsilon\frac{\partial}{\partial x}[h^3g_xP\frac{\partial P}{\partial \xi}] - \epsilon\frac{\partial}{\partial x}[h^3r_xP\frac{\partial P}{\partial \eta}] \\
&+\epsilon h^3g_x\frac{\partial}{\partial \xi}[P\frac{\partial P}{\partial x}] - \epsilon h^3r_x\frac{\partial}{\partial \eta}[P\frac{\partial P}{\partial x}] + h^3g_x^2\frac{\partial}{\partial \xi}[P\frac{\partial P}{\partial \xi}] \\
&-h^3g_xr_x\frac{\partial}{\partial \xi}[P\frac{\partial P}{\partial \eta}] - h^3g_xr_x\frac{\partial}{\partial \eta}[P\frac{\partial P}{\partial \xi}] + h^3r_x^2\frac{\partial}{\partial \eta}[P\frac{\partial P}{\partial \eta}] \\
&+\epsilon^2K\frac{\partial}{\partial x}[h^2\frac{\partial P}{\partial x}] + \epsilon K\frac{\partial}{\partial x}[h^2g_x\frac{\partial P}{\partial \xi}] - \epsilon K\frac{\partial}{\partial x}[h^2r_x\frac{\partial P}{\partial \eta}] \\
&+\epsilon Kh^2g_x\frac{\partial^2 P}{\partial \xi \partial} - \epsilon Kh^2r_x\frac{\partial^2 P}{\partial \eta \partial x} + Kh^2g_x^2\frac{\partial^2 P}{\partial \xi^2}] \\
&+Kh^2r_x^2\frac{\partial^2 P}{\partial \eta^2} - 2Kh^2g_xr_x\frac{\partial^2 P}{\partial \xi \partial \eta} + \frac{\partial}{\partial z}[h^3P\frac{\partial P}{\partial z}] \\
&+\epsilon^{-1}\frac{\partial}{\partial z}[h^3g_zP\frac{\partial P}{\partial \xi}] - \epsilon^{-1}\frac{\partial}{\partial z}[h^3r_zP\frac{\partial P}{\partial \eta}] + \epsilon^{-1}h^3g_z\frac{\partial}{\partial \xi}[P\frac{\partial P}{\partial z}]
\end{aligned}$$

$$
\begin{aligned}
&-\epsilon^{-1}h^3 r_z \frac{\partial}{\partial \eta}[P\frac{\partial P}{\partial z}] + \epsilon^{-2}h^3 g_z^2 \frac{\partial}{\partial \xi}[P\frac{\partial P}{\partial \xi}] + \epsilon^{-2}h^3 r_z^2 \frac{\partial}{\partial \eta}[P\frac{\partial P}{\partial \eta}] \\
&-\epsilon^{-2}h^3 g_z r_z \frac{\partial}{\partial \xi}[P\frac{\partial P}{\partial \eta}] - \epsilon^{-2}h^3 g_z r_z \frac{\partial}{\partial \eta}[P\frac{\partial P}{\partial \xi}] + K\frac{\partial}{\partial z}[h^2\frac{\partial P}{\partial z}] \\
&+\epsilon^{-1}K\frac{\partial}{\partial z}[h^2 g_z\frac{\partial P}{\partial \xi}] - \epsilon^{-1}K\frac{\partial}{\partial z}[h^2 r_z\frac{\partial P}{\partial \eta}] + \epsilon^{-1}Kh^2 g_z\frac{\partial^2 P}{\partial \xi \partial z} \\
&-\epsilon^{-1}Kh^2 r_z \frac{\partial^2 P}{\partial \eta \partial z} + \epsilon^{-2}Kh^2 g_z^2 \frac{\partial^2 P}{\partial \xi^2} + \epsilon^{-2}Kh^2 r_z^2 \frac{\partial^2 P}{\partial \eta^2} \\
&-2\epsilon^{-2}Kh^2 g_z r_z \frac{\partial^2 P}{\partial \xi^2} = \epsilon^2 \Lambda \frac{\partial}{\partial x}[hP] + \epsilon \Lambda h g_x \frac{\partial P}{\partial \xi} - \epsilon \Lambda h r_x \frac{\partial P}{\partial \eta}
\end{aligned}
\tag{10}
$$

Equation (10), is the multiscaled form of the equation (3). Note, that in (10), the function h is regarded as depending on x and z only.

We also note that the partial differential equation (10) is assumed to be valid, not only on $0 \le x \le 1$, $-1/2 \le y \le 1/2$, but also on $0 \le \xi,\ \eta < \infty$.

4. Perturbation Analysis

We assume that for small ϵ, the function $P(x, z, \xi, \eta, \epsilon)$ has a Poincaré expansion in ϵ of the form

$$
\begin{aligned}
P(x, z, \xi, \eta, \epsilon) = P_0(x, z, \xi, \eta) + \\
\epsilon P_1(x, z, \xi, \eta) + \epsilon^2 P_2(x, z, \xi, \eta) + \cdots,
\end{aligned}
\tag{11}
$$

valid uniformly on $0 \le x \le 1;\ -1/2 \le z \le 1/2;\ 0 \le \xi, \eta < \infty$.
Following the approach of the multiple scaling method (Ch. 6, [3]), we assume that for each $P_n(x, z, \xi, \eta)$

$$
\left| \frac{P_{n+1}(x, z, \xi, \eta)}{P_n(x, z, \xi, \eta)} \right| < +\infty, \quad n = 0, 1, 2, \ldots. \tag{12}
$$

uniformly on $0 \le x \le 1$, $-1/2 \le z \le 1/2$, $0 \le \xi,\ \eta < \infty$.
Substituting the expansion (11) into the partial differential equation (10) and equating like powers of ϵ to zero yields a sequence of partial differential equations for the coefficient functions $P_0,\ P_1, \ldots$.

From the $O(\epsilon^{-2})$ terms we obtain the relationship

$$h^3 g_z^2 \frac{\partial}{\partial \xi}(P_0 \frac{\partial P_0}{\partial \xi}) + h^3 r_z^2 \frac{\partial}{\partial \eta}(P_0 \frac{\partial P_0}{\partial \eta}) - 2h^3 g_z r_z \frac{\partial}{\partial \xi}(P_0 \frac{\partial P_0}{\partial \eta})$$

$$+Kh^2 g_z^2 \frac{\partial^2 P}{\partial \xi^2} + Kh^2 r_z^2 \frac{\partial^2 P}{\partial \eta^2} - 2Kh^2 g_z r_z \frac{\partial^2 P}{\partial \xi \partial \eta} = 0.$$

From this we find that (see the Appendix)

$$g = g(x), \qquad r = r(x). \tag{13}$$

From the $O(1)$ terms we obtain

$$\left.\begin{aligned} & h^3 g_x^2 \frac{\partial^2}{\partial \xi^2}(P_0^2) - 2h^3 g_x r_x \frac{\partial^2}{\partial \xi \partial \eta}(P_0^2) h^3 r_x^2 \frac{\partial^2}{\partial \eta^2}(P_0^2) \\ & + \frac{\partial}{\partial z}[h^3 \frac{\partial}{\partial z}(P_0^2)] + 2Kh^2 g_x^2 \frac{\partial^2 P_0}{\partial \xi^2} + 2Kh^2 r_x^2 \frac{\partial^2 P_0}{\partial \eta^2} \\ & -2Kh^2 g_x r_x \frac{\partial^2 P_0}{\partial \xi \eta} + 2K \frac{\partial}{\partial z}[h^2 \frac{\partial P_0}{\partial z}] = 0 \end{aligned}\right\}. \tag{14}$$

From the $O(\epsilon)$ terms;

$$\left.\begin{aligned} L[P_1] \equiv\; & h^3 g_x^2 \frac{\partial^2}{\partial \xi^2}(P_0 P_1) - 2h^3 g_x r_x \frac{\partial^2}{\partial \xi \partial \eta}(P_0 P_1) \\ & + h^3 r_x^2 \frac{\partial^2}{\partial \eta^2}(P_0 P_1) + \frac{\partial}{\partial z}[h^3 \frac{\partial}{\partial z}(P_0 P_1)] + Kh^2 g_x^2 \frac{\partial^2 P_1}{\partial \xi^2} \\ & + Kh^2 r_x^2 \frac{\partial^2 P_1}{\partial \eta^2} - 2Kh^2 g_x r_x \frac{\partial^2 P_1}{\partial \xi \partial \eta} + K \frac{\partial}{\partial z}[h^2 \frac{\partial P_1}{\partial z}] \\ & = \Lambda h g_x \frac{\partial P_0}{\partial \xi} - \Lambda h r_x \frac{\partial P_0}{\partial \eta} - \frac{\partial}{\partial x}(h^3 g_x P_0 \frac{\partial P_0}{\partial \xi}) \\ & + \frac{\partial}{\partial x}(h^3 r_x P_0 \frac{\partial P_0}{\partial \eta}) - h^3 g_x \frac{\partial}{\partial \xi}(P_0 \frac{\partial P_0}{\partial x}) \\ & + h^3 r_x \frac{\partial}{\partial \eta}(P_0 \frac{\partial P_0}{\partial x}) - Kh^2 g_x \frac{\partial^2 P_0}{\partial \xi \partial x} + Kh^2 r_x \frac{\partial^2 P_0}{\partial \eta \partial x} \\ & - K \frac{\partial}{\partial x}[h^2 g_x \frac{\partial P_0}{\partial \xi}] + K \frac{\partial}{\partial x}[h^2 r_x \frac{\partial P_0}{\partial \eta}] \end{aligned}\right\}. \tag{15}$$

From the $O(\epsilon^2)$ terms we obtain

$$\left.\begin{array}{l} L[P_2] = \Lambda\frac{\partial}{\partial x}(hP_0) + \Lambda h g_x \frac{\partial P_1}{\partial \xi} - \Lambda h r_x \frac{\partial P_1}{\partial \eta} - \frac{\partial}{\partial x}(h^3 P_0 \frac{\partial P_0}{\partial x}) \\ -\frac{\partial}{\partial x}[h^3 g_x \frac{\partial}{\partial \xi}(P_0 P_1)] + \frac{\partial}{\partial x}[h^3 r_x \frac{\partial}{\partial \eta}(P_0 P_1)] - h^3 g_x \frac{\partial}{\partial \xi}[\frac{\partial}{\partial x}(P_0 P_1)] \\ +h^3 r_x \frac{\partial}{\partial \eta}[\frac{\partial}{\partial x}(P_0 P_1)] - h^3 g_x^2 \frac{\partial^2}{\partial \xi^2}(P_1^2) + 2h^3 g_x r_x \frac{\partial^2}{\partial \xi \partial \eta}(P_1^2) \\ +h^3 r_x^2 \frac{\partial^2}{\partial \eta^2}(P_1^2) - K\frac{\partial}{\partial x}[h^2 g_x \frac{\partial P_1}{\partial \xi}] + K\frac{\partial}{\partial x}[h^2 r_x \frac{\partial P_1}{\partial \eta}] \\ -Kh^2 g_x \frac{\partial^2 P_1}{\partial \xi \partial x} + Kh^2 r_x \frac{\partial^2 P_1}{\partial \eta \partial x} - \frac{\partial}{\partial z}[h^3 \frac{\partial}{\partial z}(P_1^2)] \end{array}\right\}. \quad (16)$$

Finally, from the $O(\epsilon^3)$ terms we find

$$\left.\begin{array}{l} L[P_3] = \Lambda\frac{\partial}{\partial x}(hP_1) + \Lambda h g_x \frac{\partial P_2}{\partial \xi} - \Lambda h r_x \frac{\partial P_2}{\partial \eta} - \frac{\partial}{\partial x}[h^3 \frac{\partial}{\partial x}(P_0 P_1)] \\ -\frac{\partial}{\partial x}[h^3 g_x \frac{\partial}{\partial \xi}(P_1^2 + P_0 P_2)] + \frac{\partial}{\partial x}[h^3 r_x \frac{\partial}{\partial \eta}(P_1^2 + P_0 P_2)] \\ -h^3 g_x \frac{\partial^2}{\partial \xi \partial x}[P_1^2 + P_0 P_2] + h^3 r_x \frac{\partial^2}{\partial \eta \partial x}[P_1^2 + P_0 P_2] \\ -h^3 g_x^2 \frac{\partial^2}{\partial \xi^2}[P_1 P_2] + 2h^3 g_x r_x \frac{\partial^2}{\partial \xi \partial \eta}[P_1 P_2] - h^3 r_x^2 \frac{\partial^2}{\partial \eta^2}[P_1 P_2] \\ -\frac{\partial}{\partial z}[h^3 \frac{\partial}{\partial z}(P_1 P_2)] - K\frac{\partial}{\partial x}[h^2 g_x \frac{\partial P_2}{\partial \xi}] + K\frac{\partial}{\partial x}[h^2 r_x \frac{\partial P_2}{\partial \eta}] \\ -Kh^2 g_x \frac{\partial^2 P_2}{\partial \xi \partial x} + Kh^2 r_x \frac{\partial^2 P_2}{\partial \eta \partial x} \end{array}\right\}. \quad (17)$$

Applying the change of variable

$$\theta = \frac{1}{2}(\xi - \eta) \quad (18)$$

and noting from the expected symmetry of the boundary layers at the leading ($x = 0$) and trailing ($x = 1$) edges we can choose

$$g_x = r_x, \quad (19)$$

we find that (14) is transformed into

$$h^2 g_x^2 \frac{\partial}{\partial \theta}\left[(hP_0 + K)\frac{\partial P_0}{\partial \theta}\right] + \frac{\partial}{\partial z}\left[h^2 (hP_0 + K)\frac{\partial P_0}{\partial z}\right] = 0. \quad (20)$$

Equation (20) together with the matching results of [2] leads us to choose

$$P_0 \quad \equiv \quad 1. \tag{21}$$

In the same manner, applying (18) and (21) to (15), we find

$$M[P_1] \equiv h^2 g_x^2 (h+K) \frac{\partial^2 P_1}{\partial \theta^2} + \frac{\partial}{\partial z}\left[h^2 (h+K) \frac{\partial P_1}{\partial z}\right] = 0, \tag{22}$$

and consequently choose

$$P_1 \quad \equiv \quad 0. \tag{23}$$

The equations for P_2 and P_3 then reduce to

$$M[P_2] = \Lambda \frac{\partial h}{\partial x} \tag{24}$$

and

$$M[P_3] = \Lambda h g_x \frac{\partial P_2}{\partial \theta} - \frac{\partial}{\partial x}\left[h^2 g_x (h+K) \frac{\partial P_2}{\partial \theta}\right] - h^2 (h+K) g_x \frac{\partial^2 P_2}{\partial \theta \partial x}, \tag{25}$$

respectively.

Following [2] we may remove the nonhomogeneity in (24) by defining

$$P_2(x,z,\theta) = \hat{p}_2(x,z) + v_2(x,z,\theta). \tag{26}$$

Substituting (26) into (24) we obtain:

$$\frac{\partial}{\partial z}\left[h^2 (h+K) \frac{\partial \hat{p}_2(x,z)}{\partial z}\right] = \Lambda \frac{\partial h}{\partial x}, \tag{27}$$

and

$$M[v_2] = 0. \tag{28}$$

Recalling the form (11) of the expansion for P and the choices (21) and (23) for P_0 and P_1, we see that the function P_2 must satisfy the condition $P_2 = 0$ at $z = \pm\frac{1}{2}$ for all x and all $0 \leq \xi,\, \eta < \infty$ - consequently all $-\infty < \theta < \infty$. Thus, in (26) we expect that

$$\hat{p}_2(x, \pm\frac{1}{2}) = 0 \tag{29}$$

and, noting that p_2 satisfies the ordinary differential equation (27), obtain the solution

$$\hat{p}_2(x,z) = \Lambda[F_1(x,z) - \frac{F_1(x,1/2)}{F_2(x,1/2)}F_2(x,z)], \tag{30}$$

where

$$F_1(x,z) = \int_{-1/2}^{z} h^{-2}(x,s)\,[h(x,s)+K]^{-1}\int_0^s \frac{\partial h}{\partial x}(x,t)dtds \tag{31}$$

and

$$F_2(x,z) = \int_{-1/2}^{z} h^{-2}(x,s)\,[h(x,s)+K]^{-1}\,ds. \tag{32}$$

This is in complete agreement with the findings of [2]. The problem for $v_2 = v_2(x,z,\theta)$ can be solved in a standard manner by the method of eigenfunction expansions. Thus, v_2 is given by

$$v_2(x,z,\theta) = \sum_{n=1}^{\infty}[A_{1n}(x)e^{-\theta\frac{\sqrt{\lambda_n}}{g_x}} + A_{2n}(x)e^{\theta\frac{\sqrt{\lambda_n}}{g_x}}]\psi_n(x,z), \tag{33}$$

where $\psi_n(x,z)$ are the normalized eigenfunctions, $\lambda_n(x)$ the eigenvalues of the regular Sturm-Liouville system

$$\frac{\partial}{\partial z}\left[h^2(h+K)\frac{\partial\psi_n}{\partial z}\right] + \lambda_n(x)h^2(h+K)\psi_n = 0, \tag{34}$$

$$\psi_n(x,\pm\frac{1}{2}) = 0. \tag{35}$$

If we substitute (26) and (33) into (25) we find:

$$\left.\begin{aligned} M[P_3] = &-\textstyle\sum_{n=1}^{\infty}\lambda_n[2h^2(h+K)A_{1n}\frac{\partial}{\partial x}(\frac{1}{g_x})]\psi_n\theta e^{-\theta\frac{\sqrt{\lambda_n}}{g_x}} \\ &-\textstyle\sum_{n=1}^{\infty}\lambda_n[2h^2(h+K)A_{2n}\frac{\partial}{\partial x}(\frac{1}{g_x})]\psi_n\theta e^{\theta_2\frac{\sqrt{\lambda_n}}{g_x}} \\ &+\textstyle\sum_{n=1}^{\infty}\sqrt{\lambda_n}[-\Lambda hA_{1n}\psi_n + \frac{\partial}{\partial x}(h^2(h+K)A_{1n}\psi_n) \\ &+h^2(h+K)g_x\frac{\partial}{\partial x}(\frac{A_{1n}\psi_n}{g_x})]e^{-\theta\frac{\sqrt{\lambda_n}}{g_x}} + \sum_{n=1}^{\infty}\sqrt{\lambda_n}[\Lambda hA_{2n}\psi_n \\ &-\frac{\partial}{\partial x}(h^2(h+K)A_{2n}\psi_n) - h^2(h+K)g_x\frac{\partial}{\partial x}(\frac{A_{2n}\psi_n}{g_x})]e^{\theta\frac{\sqrt{\lambda_n}}{g_x}} \end{aligned}\right\}. \tag{36}$$

If we propose to construct a particular solution of (36) in the form of an eigenfunction expansion

$$P_3 = \sum_{n=1}^{\infty} \left[B_{1n}(x) e^{-\theta \frac{\sqrt{\lambda_n}}{g_x}} + B_{2n}(x) e^{\theta \frac{\sqrt{\lambda_n}}{g_x}} \right] \psi_n(x,z), \tag{37}$$

the requirement that $\mid P_3/P_2 \mid$ be bounded for *all* $-\infty < \theta < \infty$ leads to the conditions that

$$\int_{-1/2}^{1/2} \lambda_n A_{in} h^2 (h+K) \frac{\partial}{\partial x} \left(\frac{1}{g_x} \right) \psi_n^2 dz = 0 \tag{38}$$

and

$$\int_{-1/2}^{1/2} \sqrt{\lambda_n} h^2 (h+K) \{ -\Lambda h A_{in} \psi_n^2 + \frac{\partial}{\partial x} (h^2 (h+K) A_{in} \psi_n) \psi_n$$

$$+ h^2 (h+K) g_x \frac{\partial}{\partial x} \left(A_{in} \psi_n / g_x \right) \psi_n \} dz = 0, \tag{39}$$

for $i = 1, 2, \quad n \geq 0$ and all $0 \leq x \leq 1$.

The relationship (38) gives

$$g_x = \text{constant};$$

and since we require that

$$g(x) \sim x \quad \text{as} \quad x \to 0,$$

this leads us to choose

$$g(x) = r(x) = x. \tag{40}$$

The relationship (39) then gives the following differential equation for the A_{in}

$$\frac{dA_{in}}{dx} + C_n(x) A_{in} = 0, \tag{41}$$

where the coefficient function $C_n(x)$ is given by

$$C_n(x) = \frac{< (3hh_x + 2Kh_x - \Lambda) h \psi_n + 2h^2 (h+K) \psi_{nx}, \psi_n >}{< 2h^2 (h+K) \psi_n, \psi_n >}, \tag{42}$$

and $<.,.>$ is the inner product associated with the system (34), (35), namely

$$<\phi,\psi>=\int_{-1/2}^{1/2}h^2(x,z)\left[h(x,z)+K\right]\phi(z)\psi(z)dz.$$

The differential equation (41) may be solved to give

$$A_{in}(x)=c_{in}M_{in}(x),$$

where the c_{in} are constants, and the functions $M_{in}(x)$ are defined by

$$M_{1n}(x)=e^{-\int_0^x C_n(s)ds},$$

$$M_{2n}(x)=e^{\int_x^1 C_n(s)ds} \tag{43}$$

for $n=1,2,\ldots$. Thus, $M_{1n}(0)=1$, $M_{2n}(1)=1$.

Combining the above with (40), we may write the expansion (33) for $v_2=v_2(x,z,\epsilon)$ as

$$v_2=\sum_{n=1}^{\infty}\left[c_{1n}M_{1n}(x)e^{\sqrt{\lambda_n(x)}\left(\frac{1-2x}{2\epsilon}\right)}+c_{2n}M_{2n}(x)e^{-\sqrt{\lambda_n(x)}\left(\frac{1-2x}{2\epsilon}\right)}\right]\times\psi_n(x,z). \tag{44}$$

At $x=0$, the c_{1n} terms in (44) are exponentially large as $\epsilon\to 0$. At $x=1$, the c_{2n} terms are similarly large as $\epsilon\to 0$. To accommodate $O(1)$ boundary values at $x=0$ and $x=1$ respectively, the c_{in} in (44) may be rescaled to give

$$v_2=\sum_{n=1}^{\infty}\left[\alpha_{1n}M_{1n}(x)e^{\frac{1}{2\epsilon}(\chi_n(x)-\chi_n(0))}+\alpha_{2n}M_{2n}(x)e^{\frac{1}{2\epsilon}(\chi_n(1)-\chi_n(x))}\right]\times\psi_n(x,z), \tag{45}$$

where

$$\chi_n(x)=(1-2x)\sqrt{\lambda_n(x)} \tag{46}$$

and the α_{in} are $O(1)$ constants.

Combining these results, we have the expansion for the pressure field $p(x,z,\epsilon)$ as

$$p(x,z,\epsilon) = 1 + \epsilon^2 \left[\hat{p}_2(x,z) + v_2(x,z,\epsilon)\right] + O(\epsilon^3). \tag{47}$$

Applying the boundary conditions at $x = 0$ and $x = 1$ (to the order considered) gives

$$\left.\begin{aligned} \alpha_{1n} &=< -\hat{p}_2(0,z), \psi_n(0,z) > \\ \alpha_{2n} &=< -\hat{p}_2(1,z), \psi_n(1,z) > \end{aligned}\right\}, \tag{48}$$

determining the $O(1)$ constants α_{in}.

5. Particular Cases

We now consider the pressure field in bearings for which the profile $h(x,z)$ has a particularly simple form. Two cases are of particular interest in application - the wedge bearing and the shallow crowned bearing.

5.1. The Wedge Slider Bearing

Here, $h(x,z)$ is independent of z; i.e.

$$h(x,z) = A(x), \quad A(0) = 1, \quad A(1) = \gamma. \tag{49}$$

We find, from (30) that

$$\hat{p}_2(x,z) = \frac{\Lambda A'(x)}{2A^2(x)\left[A(x) + K\right]}(z^2 - 1/4). \tag{50}$$

For such a class of bearings, the system (34), (35) becomes

$$\frac{\partial^2 \psi_n}{\partial z^2} + \lambda_n \psi_n = 0; \;\; \psi_n(x, \pm 1/2) = 0, \tag{51}$$

so that the eigenfunctions are given by

$$\psi_n(x,z) = \begin{cases} \frac{\sqrt{2}}{A(x)(A(x)+K)^{1/2}} \cos(n\pi z) & \text{for } n = 1,3,5,\ldots \\ \dfrac{\sqrt{2}}{A(x)\left(A(x)+K\right)^{1/2}} \sin(n\pi z) & \text{for } n = 2,4,6,\ldots \end{cases} \tag{52}$$

with corresponding eigenvalues

$$\lambda_n = n^2\pi^2, \;\; n = 1, 2, \ldots\;. \tag{53}$$

Noting from (52), that

$$\psi_{nx} = -\frac{1}{2}A'\frac{3A+2K}{A(A+K)}\psi_n,$$

we obtain from (42)

$$C_n(x) = -\frac{\Lambda}{A(A+K)}. \tag{54}$$

Thus

$$\left.\begin{aligned} M_{1n}(x) &= e^{\frac{1}{2}\Lambda\int_0^x A(s)^{-1}(A(s)+K)^{-1}ds}\,, \\ M_{2n}(x) &= e^{-\Lambda\frac{1}{2}\int_x^1 A(s)^{-1}(A(s)+K)^{-1}ds}\,. \end{aligned}\right. \tag{55}$$

The expansion (47) for the pressure field becomes

$$\left.\begin{aligned} &p(x,z) = 1 + \epsilon^2[\hat{p}_2(x,z) \\ &+\sum_{n=1}^{\infty}\{\alpha_{1n}M_{1n}(x)e^{-\sqrt{\lambda_n}x/\epsilon} + \alpha_{2n}M_{2n}(x)e^{-\sqrt{\lambda_n}(1-x)/\epsilon}\} \\ &\times\psi_n(x,z)] + O(\epsilon^3), \end{aligned}\right\} \tag{56}$$

where α_{1n}, α_{2n} are given by (48), adapted to the present case.
From (48) we obtain

$$\alpha_{1n} = \begin{cases} 0, & n = 2, 4, 6, \ldots, \\ \dfrac{(-1)^{\frac{n-1}{2}}2\sqrt{2}\Lambda A'(0)}{\pi^3 n^3\sqrt{1+K}}, & n = 1, 3, 5, \ldots; \end{cases} \tag{57}$$

$$\alpha_{2n} = \begin{cases} 0, & n = 2, 4, 6, \ldots, \\ \dfrac{(-1)^{\frac{n-1}{2}}2\sqrt{2}\Lambda A'(1)}{\pi^3 n^3\gamma\sqrt{\gamma+K}}, & n = 1, 3, 5, \ldots. \end{cases} \tag{58}$$

The results (50), (52), (55), (56), (57) and (58) now completely determine the uniformly valid expansion (56) for the pressure. Note that in the expansion (56), the factors M_{in} and the eigenfunctions $\psi_n(x,z)$ vary slowly with x. If we note that for $x = O(\epsilon)$; i.e, in the layer at $x = 0$,

$$A(x) = 1 + O(\epsilon); \quad M_{1n} = 1 + O(\epsilon);$$

and for $1 - x = O(\epsilon)$; i.e. in the layer at $x = 1$,

$$A(x) = \gamma + O(\epsilon); \quad M_{2n} = 1 + O(\epsilon);$$

the layer term in (56) may be shown to coincide with the corresponding term in [2], computed by matching techniques. Thus, in (56) the factor M_{in} and the eigenfunctions ψ_n take into account, variation is beyond the layer regions; although such influence is rapidly diminishing, due to the exponentially decaying factors.

5.2. The Shallow Crowned Slider Bearing

In this case,

$$h(x,z) = A(x)(1 + \delta z^2),\ A(0) = 1,\ A(1) = \gamma; \tag{59}$$

where the function $A(x)$ is positive and δ is small; i.e. $0 \leq |\delta| << 1$.

Note that if $\delta = 0$ this reduces to the profile of Section 5.1.

Since h is even in z, we have $F_1(x, \frac{1}{2}) = 0$, and (30) gives $\hat{p}_2$ as

$$\hat{p}_2(x,z) = \Lambda A'(x) A(x)^{-2} \left[G(x,z) - G(x, -\frac{1}{2}) \right], \tag{60}$$

where

$$G(x,z) =$$

$$\begin{cases} \dfrac{2A(x) - K}{6K^2\delta} ln \left(A(x) + \dfrac{K}{1 + \delta z^2} \right) - \dfrac{1}{3K\delta(1 + \delta z^2)};\ K > 0 \\ -\dfrac{2 + \delta z^2}{6A(x)\delta(1 + \delta z^2)^2};\ K = 0. \end{cases} \tag{61}$$

For small δ, we can solve the Sturm-Liouville problem (34), (35) for ψ_n by perturbation expansions in δ. Thus, for each n we propose

$$\psi_n = \psi_{n0} + \delta\psi_{n1} + \delta^2\psi_{n2} + \dots,$$

$$\lambda_n = \lambda_{n0} + \delta\lambda_{n1} + \delta^2\lambda_{n2} + \dots.$$

Carrying out the relevant calculations, we find

$$\lambda_n = n^2\pi^2 + \frac{(2K+3A(x))}{(K+A(x))}\delta + ... \tag{62}$$

Note that now the eigenvalues λ_n depend on x, though only at the $O(\delta)$ level.

The eigenfunctions ψ_n are found to be given by

$$\psi_n(x,z) =$$

$$\begin{cases} \frac{\sqrt{2}}{A(x)\sqrt{K+A(x)}}\left[1 - \frac{2K+3A(x)}{2(K+A(x))}\delta z^2 + O(\delta^2)\right]\cos(n\pi z) \\ \qquad\qquad\qquad \text{for } n = 1,3,5,\ldots \\ \dfrac{\sqrt{2}}{A(x)\sqrt{K+A(x)}}\left[1 - \dfrac{2K+3A(x)}{2(K+A(x))}\delta z^2 + O(\delta^2)\right]\sin(n\pi z) \\ \qquad\qquad\qquad \text{for } n = 2,4,6,\ldots \end{cases} . \tag{63}$$

The factors M_{in} are given by (43) with

$$C_n(x) =$$

$$\frac{-\Lambda(A+K) + \frac{\delta}{2}(\frac{1}{6} - \frac{1}{n^2\pi^2})\left[\Lambda(5A+3K) - 2AA'(K^2+3A+5K)\right]}{(A+K)\left[2A(A+K) - \frac{\delta}{2}(\frac{1}{6} - \frac{1}{n^2\pi^2})A(3A+2K)\right]} . \tag{64}$$

Applying the above to (48), we obtain expansions in δ for the coefficients α_{in}, that take the form

$$\alpha_{in} = \alpha_{in,0} + \delta\alpha_{in,1} + O(\delta^2)\,,$$

where explicitly

$$\alpha_{1n,0} = \begin{cases} 0, & n = 2,4,6,\ldots, \\ \dfrac{2\sqrt{2}\Lambda A'(0)(-1)^{\frac{(n-1)}{2}}(3+2K)}{3n^3\pi^3\{(1+K)^{2/3}}, & n = 1,3,5,\ldots; \end{cases} \tag{65}$$

$$\alpha_{2n,0} = \begin{cases} 0, & n = 2,4,6,\ldots, \\ \dfrac{2\sqrt{2}\Lambda A'(1)(-1)^{\frac{(n-1)}{2}}(3\gamma+2K)}{3\gamma^{3/2}n^3\pi^3[\gamma(\gamma+K)^2]^{1/3}}, & n = 1,3,5,\ldots; \end{cases} \tag{66}$$

$$\alpha_{1n,1} = \begin{cases} 0, & n = 2,4,6,\ldots, \\ \dfrac{-(-1)^{\frac{(n-1)}{2}} \Lambda A'(0)\sqrt{2}[3+2K]}{(1+K)^{2/3}} & \\ \qquad \times\{\dfrac{4}{n^5\pi^5} + \dfrac{1}{4n^3\pi^3}\} & n = 1,3,5,\ldots, \end{cases} \tag{67}$$

$$\alpha_{2n,1} = \begin{cases} 0, & n = 2,4,6,\ldots, \\ \dfrac{-(-1)^{\frac{(n-1)}{2}} \Lambda A'(1)\sqrt{2}[3\gamma+2K]}{2[\gamma(\gamma+K)^2]^{1/3}\gamma^{3/2}} & \\ \qquad \times\{\dfrac{4}{n^5\pi^5} + \dfrac{1}{4n^3\pi^3}\} & n = 1,3,5,\ldots. \end{cases} \tag{68}$$

Obviously, the leading terms (65), (66) above coincide with those for the previous case, the wedge slider, while (67), (68) provide $O(\delta)$ corections due to the presence of the crowning.

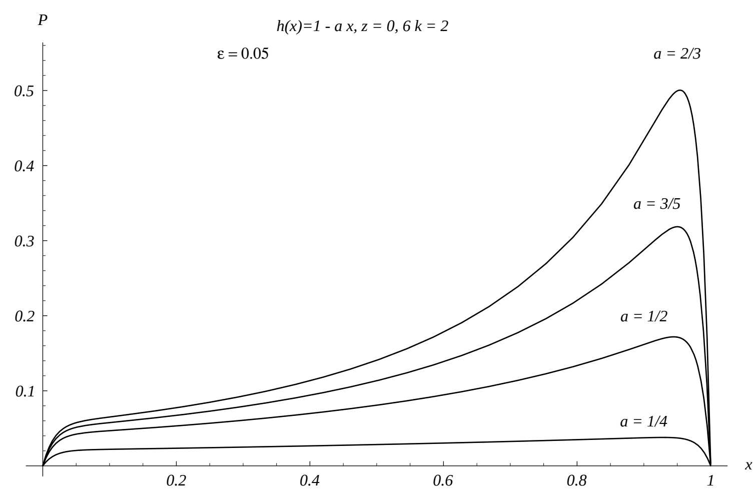

Figure 2: Scaled excess pressure, $P = (p-1)/\epsilon^2$, as given by (56), along the midline $(z = 0)$, for a linear bearing profile $h = 1 - ax$, with $\epsilon = 0.05$, $6k = 2$, $\Lambda = 2$, $a = 1/4, 1/2, 3/5, 2/3$.

6. Discussion

The expansion (47), with $\hat{p_2}$ given by (30) and v_2 by (45) provides a straightforward, explicit and readily computed approximate expression for the pressure field in the bearing, for given profile function $h(x,z)$. For the case of relatively simple geometry considered in Section 5, the Sturm-Liouville system (34), (35) takes the particularly simple form (51) and may be solved easily. The resulting expansion (56) displays the characteristic layer structure already found by matching techniques [2], with some small, but significant modifications.

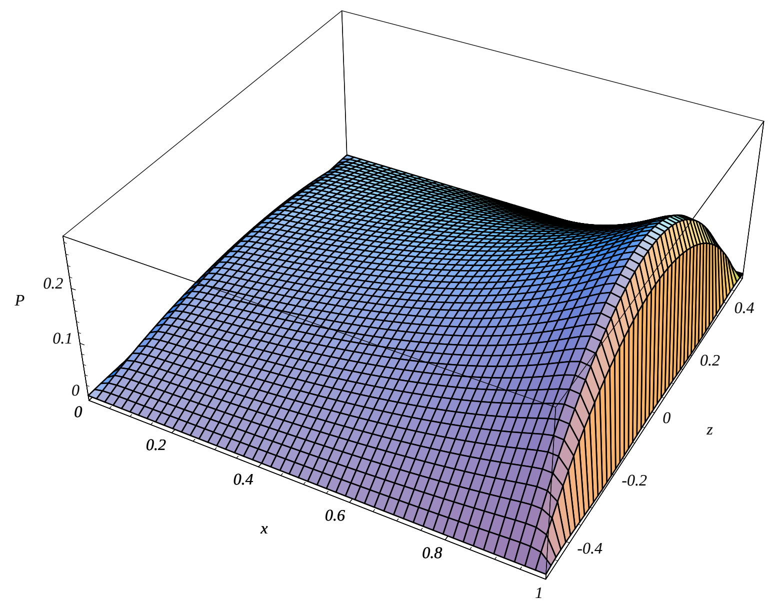

Figure 3: Scaled excess pressure, $P = (p-1)/\epsilon^2$, as given by (56), for $h = 1-ax$, $\epsilon = 0.05$, $6k = 1$, $\Lambda = 2$ and $a = 1/2$.

For bearings of more complicated geometries, where the profile $h(x,z)$ is not separable as in Section 5, the situation is more complicated, and may need to be analyzed by numerical techniques. Morover, it becomes an important question whether the more general expansion derived from (47) using the v_2 of (45) does indeed display the leading and trailing edge boundary layer structure seen in the profiles of Section 5.

This will *not* be so if, for example, for some $n = m$, say, the function $\chi_m(x) - \chi_m(0)$ becomes positive at some x on $0 < x < 1$; for then, the contribution of the "layer" term at $x = 0$ corresponding to α_{1m} will become exponentially large - unless, of course $\alpha_{1m} = 0$. This condition,

$$< -\hat{p_2}(0, z), \psi_m(0, z) >= 0,$$

poses a restriction on the possible profiles allowable, for the expected layer structure to exist. Such a condition is not seen in the analysis by matched expansions, since the boundary layer terms involve only the values of the eigenvalues λ_n at $x = 0$ and $x = 1$; and so do not take account of the intermediate values when the eigenvalues vary with x.

Figure 2 illustrates the "normalized" excess pressure profile , $(p - 1)/\epsilon^2$, along the centreline $z = 0$ of the bearing, obtained from (56), for a simple linear wedge bearing $h(x, z) = 1 - ax$, and various positive values of a. Clearly, the greater the values of a (within reasonable limits), the greater the overall pressure generated in the bearing - and, correspondingly, the greater the load-bearing capacity of the bearing.

Figure 3 shows a three-dimensional plot of the pressure in the above linear bearing, when $a = 0.5$. Again, the leading and trailing layers are clearly in evidence.

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Appendix A

From the $O(\epsilon^{-2})$ terms:

$$h^3 g_z^2 \frac{\partial}{\partial \xi}(P_0 \frac{\partial P_0}{\partial \xi}) + h^3 r_z^2 \frac{\partial}{\partial \eta}(P_0 \frac{\partial P_0}{\partial \eta}) - 2h^3 g_z r_z \frac{\partial}{\partial \xi}(P_0 \frac{\partial P_0}{\partial \eta})$$

$$+ K h^2 g_z^2 \frac{\partial^2 P}{\partial \xi^2} + K h^2 r_z^2 \frac{\partial^2 P}{\partial \eta^2} - 2K h^2 g_z r_z \frac{\partial^2 P}{\partial \xi \partial \eta} = 0, \quad \text{(A1)}$$

$$(r_z \neq 0).$$

Applying the change of variables:

$$\begin{cases} \alpha_1 = \xi + \dfrac{\eta}{2r_z} \\ \alpha_2 = \xi - \dfrac{\eta}{2r_z} \end{cases}, \qquad (A2)$$

we see that equation $(A1)$ is transformed into

$$g_z^2 \frac{\partial}{\partial \alpha_2}\left[(hP_0 + K)\frac{\partial P_0}{\partial \alpha_2}\right] = 0. \qquad (A3)$$

If

$$\frac{\partial}{\partial \alpha_2}\left[(hP_0 + K)\frac{\partial P_0}{\partial \alpha_2}\right] = 0$$

then, simple P_0 is bounded on $-\infty < \alpha_2 < \infty$, we obtain

$$P_0 = N_0(x, z).$$

From the $O(\epsilon^{-1})$ terms, we obtain

$$h^2 g_z^2 \left[hN_0(x, z) + K\right] \frac{\partial^2 P_1}{\partial \alpha_2^2} = 0$$

and if $g_z \neq 0$, this gives $P_1 = N_1(x, z)$.

Continuing in this way, we deduce that if $r_z \neq 0$, $g_z \neq 0$, then none of $P_0, P_1, ...$ can depend on ξ or η - contradictory to our original assumption. If only <u>one</u> of r_z or g_z is zero - say $r_z \neq 0$ and $g_z = 0$, then $(A1)$ yields the result that P_0 cannot depend on η - similarly, $P_1, P_2, ...$ cannot depend on η. Again, this contradicts our original assumption, and we deduce that $g_z = r_z = 0$; i.e. $r = r(x)$, $g = g(x)$.