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ON GENERALIZED SUBSET-SUM-DISTINCT SEQUENCES

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Abstract: In this paper, we generalize the concept of subset-sum-distinctness to k-SSD, the k-fold version. The classical subset-sum-distinct sets would be 1-SSD in our definition. We establish some properties on the generalized subset-sum-distinct sequences.

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1. Introduction

We begin the paper with an interesting puzzle: suppose there are six piles of money, each consisting of 100 silver coins. All the coins in one pile are counterfeit, but we do not know which pile it is. We know the correct weight of a legal coin, and we know that the counterfeit coin weighs one gram less than a genuine one. Using a spring scale, identify the pile of counterfeit coins by only one weighing.

The answer for this puzzle is easy: After numbering each pile from 1 to 6, choose i coins from each $i-$th pile, $1 \leq i \leq 6$. Weighing them all together, one can tell the pile of counterfeit coins. For example, if it lacks 5 grams to the

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expected weight of $21\ (= 1 + 2 + 3 + 4 + 5 + 6)$ legal coins, then the fifth pile is the counterfeit. Let us change the puzzle a little: all conditions are the same as above but we do not know how many piles are counterfeit. At this time, one may choose 2^{i-1} coins from $i-$th pile for $1 \leq i \leq 6$. Weighing them together, if it lacks $5\ (= 1 + 2^2)$ grams to the expected weight, the first and the third piles are the counterfeit.

When we consider the changed puzzle with the constraint that every pile has only 24 coins, we arrive at the concept of "distinct subset sums". A set of real numbers is said to have distinct subset sums if no two finite subsets have the same sum. To be precise, we define the following.

Definition 1.1. (i) Let A be a set of real numbers. We say that A has the subset-sum-distinct property (briefly SSD-property) if for any two finite subsets X, Y of A,

$$\sum_{x\in X} x = \sum_{y\in Y} y \qquad \Longrightarrow \qquad X = Y$$

Also, we say that A is SSD or A is an SSD-set if it has the SSD-property.

(ii) A sequence of positive integers $\{a_n\}_{n=1}^{\infty}$ is called a subset-sum-distinct sequence (or briefly, an SSD-sequence) if it has the SSD-property.

With this terminology, to answer for the final puzzle, we just need an SSD-set of six positive integrs whose greatest element is less than or equal to 24. In other words, we are very interested in a "dense" SSD-set. In fact, problems related to dense SSD-sets have been considered by many mathematicians in various contexts (see [1, pp. 47-48], [2], [3], [4], [5], [6], [7], [8], [9], [10, pp. 59-60], [11, p. 114, problem C8], [12], [13], [14]). The greedy algorithm generates one of the most natural SSD-sequence $\{1, 2, 2^2, 2^3, \cdots\}$ which is quite sparce. In 1967, on the request of "dense" SSD-sets, John Conway and Richard Guy constructed so called "Conway-Guy sequence" as following ([12]). First, define an auxiliary sequence u_n by

$$u_0 = 0, u_1 = 1 \ \text{ and } \ u_{n+1} = 2u_n - u_{n-r}, \quad n \geq 1$$

where $r =< \sqrt{2n} >$, the nearest integer to $\sqrt{2n}$. Now, for a given positive integer n, we define

$$a_i = u_n - u_{n-i},\ 1 \leq i \leq n.$$

The well known Conway-Guy conjecture is that $\{a_i :\ 1 \leq i \leq n\}$ is SSD for any positive integer n. F. Lunnon showed numerically that they are SSD-sets for $n < 80$ (see [14, p. 307, Theorem 4.6]) and the conjecture was completely resolved affirmatively by T. Bohman in 1996 (see [6]).

Note that, in the last changed puzzle, the Conway-Guy sequence gives the unique answer $\{11, 17, 20, 22, 23, 24\}$ when $n = 6$.

2. A Generalization

Consider the puzzle of counterfeit coin in the introduction again. This time, we assume that each pile has 169 coins and we know that the counterfeit coin weighs one or two grams less than a genuine one. Also we know that any two coins in the same pile have the same weight. But we do not know how many piles are counterfeit. In this case, we need a set $S = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ of positive integers such that $1 \le a_1 \le a_2 \le a_3 \le a_4 \le a_5 \le a_6 \le 169$ and all sums $\sum_{i=1}^{6} \epsilon_i a_i$ are different, when each integer ϵ_i varies from 0 to 2. This means that S is not only subset-sum-distinct but also 2-fold subset-sum-distinct. We define this concept in full generalization.

Definition 2.1. (i) For a set A of real numbers, we say that A has the k-fold subset-sum-distinct property (briefly k-SSD-property) if for any two finite subsets X, Y of A,

$$\sum_{x \in X} \epsilon_x \cdot x = \sum_{y \in Y} \epsilon_y \cdot y \text{ for some } \epsilon_x,\ \epsilon_y \in \{1, 2, \cdots, k\} \quad \text{implies} \quad X = Y.$$

Also, we say that A is k-SSD or A is a k-SSD-set if it has the k-SSD-property.

(ii) A sequence of positive integers $\{a_n\}_{n=1}^{\infty}$ is called a k-fold subset-sum-distinct sequence (briefly, k-SSD-sequence) if it has the k-SSD-property.

Note that a classical SSD-set is just a 1-SSD-set. Note also that the greedy algorithm produces the k-SSD-sequence $1, k+1, (k+1)^2, (k+1)^3, \cdots$.

For the answer of above puzzle, we need a 2-SSD-set of six elements with minimal height. Lots of calculations shows that

$$\{109, 147, 161, 166, 168, 169\}$$

is the unique answer. That is, after choosing 109, 147, 161, 166, 168, 169 coins from the piles, respectively, one weighs them together. If the scale indicates, for example, $722\ (= 2\cdot 109 + 166 + 2\cdot 169)$ grams less than the expected weight of $920\ (= 109 + 147 + 161 + 166 + 168 + 169)$ genuine coins, one can conclude that every coin in the fourth pile lacks one gram and every coin in the first and the sixth piles lacks two grams.

Now we give a lemma which is needed in the next section.

Lemma 2.2. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a* k*-SSD-sequence. Then*

$$a_1 + a_2 + \cdots + a_n \ \geq \ \frac{(k+1)^n - 1}{k}$$

for every $n \geq 1$.

Proof. Let

$$J = \left\{ \sum_{i=1}^{n} \epsilon_i \cdot a_i \ : \ \epsilon_i\text{'s are integers with} \quad 0 \leq \epsilon_i \leq k \right\}.$$

Note that all the elements of J are nonnegaitive integers. By the k-SSD-property of the sequence

$$\sum_{i=1}^{n} \epsilon_i \cdot a_i \ = \ \sum_{i=1}^{n} {\epsilon_i}' \cdot a_i \qquad \Longleftrightarrow \qquad (\epsilon_1, \epsilon_2, \cdots, \epsilon_n) \ = \ ({\epsilon_1}', {\epsilon_2}', \cdots, {\epsilon_n}').$$

Hence $|J| = (k+1)^n$. Because $0 \in J$ and $k \cdot (a_1 + a_2 + \cdots + a_n)$ is the greatest element in J, we have

$$k \cdot (a_1 + a_2 + \cdots + a_n) \ \geq \ (k+1)^n - 1$$

which proves the lemma. □

Ryavec's theorem says that the reciprocal sum of a $1-$SSD sequence has the optimal upper bound 2 (see [5]). In other words, if $\{a_n\}_{n=1}^{\infty}$ is $1-$SSD, then

$$\sum_{n=1}^{\infty} \frac{1}{a_n} \leq 2\,.$$

In the next section we will show that this result can be generalized to k-SSD-sequences by using Tomić 's inequality. One can observe that Ryavec's original generating function method is not useful anymore for the general cases.

3. Dirichlet Series of k-SSD-sequences

In this section, we estimate Dirichlet series $\sum_{n=1}^{\infty} {a_n}^s$, for a given k-SSD-sequence $\{a_n\}_{n=1}^{\infty}$. It will turn out that the following Tomić's inequality is quite useful for these estimation.

Theorem 3.1. (M. Tomić, 1949) *Let*

$$u_1 \geq u_2 \geq \cdots \geq u_m , \qquad v_1 \geq v_2 \geq \cdots \geq v_m ,$$

where u_i'-s and v_i'-s are real numbers. Then

$$\sum_{i=1}^{j} u_i \leq \sum_{i=1}^{j} v_i \qquad for \ \ j = 1, 2, \cdots , m$$

if and only if

$$\sum_{i=1}^{m} f(u_i) \leq \sum_{i=1}^{m} f(v_i)$$

for every convex increasing function f.

Proof. See [15]. □

Remark. In [16], Tomić gave a geometric proof based on a Gauss' theorem on the centroid. But it can be proved easily by using convexity and summation by parts (See [15], [17]).

Corollary 3.2. *Let* $x_1 \leq x_2 \leq \cdots \leq x_m$ and $y_1 \leq y_2 \leq \cdots \leq y_m$, *where x_i'-s and y_i'-s are real numbers. Then the followings are equivalent:*

(i) $\displaystyle\sum_{i=1}^{j} x_i \leq \sum_{i=1}^{j} y_i \qquad for \ \ j = 1, 2, \cdots , m.$

(ii) $\displaystyle\sum_{i=1}^{m} g(x_i) \geq \sum_{i=1}^{m} g(y_i)$ *for every convex decreasing function g.*

(iii) $\sum_{i=1}^{m} h(x_i) \;\leq\; \sum_{i=1}^{m} h(y_i)$ *for every concave increasing function* h.

The following theorem, which was first proved by F. Hanson, J. M. Steele and F. Stenger (see [9]) for $k = 1$, is immediate from the above corollary.

Theorem 3.3. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a* k*-SSD-sequence. Then*

$$\sum_{i=1}^{m} {a_i}^s \leq \sum_{i=1}^{m} (k+1)^{(i-1)s} = \frac{1-(k+1)^{ms}}{1-(k+1)^s} \tag{1}$$

for all positive integers m *and all real numbers* $s < 0$.

Proof. By Lemma 2.2, for all positive integers $j \leq m$ we have

$$a_1 + a_2 + \cdots + a_j \;\geq\; 1 + (k+1) + (k+1)^2 + \cdots + (k+1)^{j-1}. \tag{2}$$

Now, the conclusion follows from Corollary 3.2 when we set $x_i = (k{+}1)^{i-1}$, $y_i = a_i$ and $g(x) = x^s$. □

Taking $s = -1$ and $m \longrightarrow \infty$ we obtain the following generalization of Ryavec's theorem.

Theorem 3.4. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a* k*-SSD-sequence. Then*

$$\sum_{n=1}^{\infty} \frac{1}{a_n} \leq 1 + \frac{1}{k}.$$

One may ask whether one can have the reverse inequality of (1) for non-negative s. In other words, is

$$\sum_{i=1}^{m} {a_i}^s \geq \sum_{i=1}^{m} (k+1)^{(i-1)s} = \frac{1-(k+1)^{ms}}{1-(k+1)^s} \tag{3}$$

for all positive integers m and all $s \geq 0$?

In view of (2), inequality (3) is true for $s = 1$. What about other positive values of s? In connection with this question we have the following theorem.

Theorem 3.5. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a* k*-SSD-sequence and* β *a fixed positive real number. If*

$$\sum_{i=1}^{m} {a_i}^{\beta} \geq \sum_{i=1}^{m} (k+1)^{(i-1)\beta} = \frac{1-(k+1)^{m\beta}}{1-(k+1)^{\beta}} \tag{4}$$

for all positive integers m, *then (3) is true for all positive integers* m, *and all* $0 \leq s \leq \beta$.

Proof. Starting from inequality (4), put $x_i = (k+1)^{i-1}$, $y_i = a_i$ and $h(x) = x^{s/\beta}$ in Corollary 3.2. Note that $h(x)$ is concave and increasing on $x \geq 0$, whence the result follows. □

By the previous theorem, we may define λ_k to be the supremum of all s that satisfy (3) for all k-SSD-sequences $\{a_n\}_{n=1}^{\infty}$ and for all positive integers m. In [2], the author showed that $2 \leq \lambda_1 \leq 3.6906742\cdots$. Here we give lower and upper bounds for λ_2.

Theorem 3.6. $1 \leq \lambda_2 \leq 3.9852202\cdots$.

Proof. By (2), we know that $\lambda_k \geq 1$ for any positive integer k. Hence $\lambda_2 \geq 1$ also. Let

$$\{a_i\}_{i=1}^{7} = \{308, 417, 455, 469, 474, 476, 477\}. \tag{5}$$

By routine calculations, or by using similar construction of Conway-Guy sequence, one can show the set of (5) is 2-SSD. Then

$$\sum_{i=1}^{7} {a_i}^{s} - \sum_{i=1}^{7} 3^{(i-1)s}$$

changes sign from $+$ to $-$ at $s = 3.9852202\cdots$. Note that it is obvious, from Theorem 3.5, that it cannot change sign in the opposite direction. Hence we have the theorem. □

Lastly, we present a multiplicative analogue of the inequality (2).

Theorem 3.7. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a* k*-SSD-sequence. Then*

$$a_1 a_2 \cdots a_n \geq 1 \cdot (k+1) \cdot (k+1)^2 \cdots (k+1)^{n-1} = (k+1)^{n(n-1)/2}$$

for all positive integers n.

Proof. In Corollary 3.2 (ii), put $x_i = (k+1)^{i-1}$, $y_i = a_i$ and $h(x) = \log x$.

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