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DIMENSION OF THE GLOBAL ATTRACTOR FOR WAVE EQUATIONS WITH NEUMANN BOUNDARY CONDITIONS

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Abstract: We get an upper estimate of Hausdorff dimension for the global attractor for damped hyperbolic equations with Neumann boundary conditions. In particular, this upper estimate remains small for large damping. The results obtained in this work extends results obtained by Shengfan [4] to Neumann problems.

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1. Introduction

In this paper we find an upper estimate of Hausdorff dimension for the global attractor for the damped hyperbolic equation

$$u_{tt} + \beta u_t - \Delta u + f(u) = g, \ x \in \Omega, \ t > 0, \tag{1}$$

with Neumann boundary condition

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$$\frac{\partial u}{\partial \nu} = 0 \text{ in } \partial\Omega, \tag{2}$$

and the initial data

$$u(x,0) = u_0(x),\ \frac{\partial u\,(x,t)}{\partial t} = u_1\,(x)\,,\ x \in \Omega, \tag{3}$$

where β is a positive real number, $u : \Omega \times [0,\infty) \to \Re$, $\Re$ is a set of real numbers, and Ω is an open bounded set in $\Re^n$ with smooth boundary $\partial\Omega$, $g \in L^2(\Omega)$, and the initial data is in $H^1(\Omega) \times L^2(\Omega)$.

The function $f \in C^2\,(\Re, \Re),\ f = I + h,\ I$ is an identity map and h satisfies the growth condition

$$\begin{aligned} &\limsup_{|u| \to +\infty} \tfrac{-h(u)}{u} \leq 0, \\ &|h'\,(u)| \leq k\,, \quad |\ h'\,(u_1) - h'\,(u_2)\ |\leq k_1\ |\ u_1 - u_2\ |^{\delta}, \end{aligned} \tag{4}$$

for any $u, u_1, u_2 \in \Re,\ k, k_1, \delta > 0$. Equations (1) and (2) are equivalent to

$$u_{tt} + \beta u_t - \Delta u + u + h\,(u) = g,\ x \in \Omega,\ t > 0, \tag{5}$$

with Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \text{ in } \partial\Omega. \tag{6}$$

We also need to write equations (5) and (6) as an abstract evolution equation. For notation, we let $\|\,.\,\|_{L^2(\Omega)}, \|\,.\,\|_{H^1(\Omega)}$ and $\|\,.\,\|_{H^2(\Omega)}$ denote the classical norms in $L^2(\Omega), H^1(\Omega)$ and $H^2(\Omega)$, respectively.

Let $A : D\,(A) \subset L^2\,(\Omega) \to L^2\,(\Omega)$ be $-\Delta + I$ with homogeneous Neumann boundary condition and $D\left(A^{\frac{1}{2}}\right) = H^1(\Omega)$ and $D(A) = \{u \in H^2(\Omega) : \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\}$. The initial conditions for (5), (6) will be taken in the space $E \equiv H^1\,(\Omega) \times L^2\,(\Omega)\,,$ where the norm in $H^1\,(\Omega)$ is defined by $\|\varphi\|_{H^1(\Omega)} = \|\nabla\varphi\|_{L^2(\Omega)} + \|\varphi\|_{L^2(\Omega)},\ \varphi \in H^1\,(\Omega)\,.$

Equations (5) and (6) with initial data in E can be written as an abstract evolution equation:

$$u_{tt} + \beta u_t + Au = h\,(u) + g,\ x \in \Omega,\ t > 0. \tag{7}$$

We need more notation, to explain the results more precisely. Suppose Z is a Banach space, C, D are any subsets of Z, let

$$\text{dist}\,(C, D) = \sup_{c \in C} \inf_{d \in D} \|\ c - d\ \|_Z\ .$$

We say that a semigroup $T(t)$ on Z has a global attractor $\mathcal{A}$ if $\mathcal{A}$ is a compact, invariant set ($T(t)\mathcal{A} = \mathcal{A}\ \forall t \geq 0$) and $\operatorname{dist}(T(t)B, \mathcal{A}) \to 0$ as $t \to \infty$ for each bounded set B in Z.

Using results, for example from Hale [1] and [2], Temam [5] and Hale and Raugel [3] that the evolutionary equation (7) defines a C^0-semigroup

$\{T(t), t \geq 0\}$ on E, where $T(t)(u_0, u_1) = (u, u_t)$ with (u, u_t) is the solution with initial data $(u_0, u_1) \in E$. Moreover, Hale [2] showed that under the conditions (4) and $\beta > 0$ there is a global attractor B in E.

In the paper, by Shengfan [4], was shown that if we replace Neumann boundary condition with Dirichlet boundary conditions and replace assumptions (4) by the following assumptions:

$$\begin{aligned}
&f(0) = 0, \\
&\lim_{|s| \to +\infty} \sup \tfrac{f(s)}{s} \leq 0, \\
&|f'(s)| \leq k, \ \ |f'(s_1) - f'(s_2)| \leq k_2 |s_1 - s_2|^{\delta},
\end{aligned} \tag{8}$$

for any s, s_1, $s_2 \in R$, where k, k_1, $\delta > 0$, then there is an upper bound of the Hausdorff dimension for the global attractor B and the Hausdorff dimension of the attractor, remains small for large damping. The method of proof used was pioneered by G. Wang and S. Zhu [6].

In this paper, we use an approach similar to the method used by G. Wang and S. Zhu [6] and Shengfan [4]. We find an upper bound of the Hausdorff dimension for the global attractor of equations (1)-(3) with the assumptions (4). Our results will also show that the Hausdorff dimension of B should remain small for large damping. The main result is:

Theorem 1.1. *For system (1)-(3) with $\beta > 0$. The Hausdorff dimension $d_H(B)$ of the global attractor B satisfies:*

$$d_H(B) \leq \min \left\{ l : \frac{1}{l} \sum_{j=1}^{l} \lambda_j^{-1} \leq \frac{2\lambda_1 \beta^2}{k^2 \left(\beta + \sqrt{\beta^2 + 4\lambda_1}\right) \sqrt{\beta^2 + 4\lambda_1}} \right\},$$

where λ_i's are defined in Section 2.

Note that, $d_H(B)$ remains small for sufficiently large damping β, because

$$\frac{2\lambda_1 \beta^2}{k^2 \left(\beta + \sqrt{\beta^2 + 4\lambda_1}\right) \sqrt{\beta^2 + 4\lambda_1}} \to \frac{\lambda_1}{k^2}$$

as $\beta \to +\infty$.

2. Basic Results

In this section, we present the preliminary results: existence and uniqueness of solutions, and the existence of the global attractor for equations (1)-(3).

The unbounded operator $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ is self-adjoint, positive definite, densely defined and linear operator and its eigenvalues $\{\lambda_i\}_{i\in N}$ satisfy

$$0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_m \leq ..., \qquad \lambda_m \to +\infty \ (m \to +\infty).$$

Let

$$\begin{aligned} &(u,v)_{L^2(\Omega)} = \textstyle\int_\Omega uvdx, \\ &\|u\|_{L^2(\Omega)} = (u,u)^{\frac{1}{2}}_{L^2(\Omega)}, \ \forall u,v \in L^2(\Omega), \end{aligned} \tag{9}$$

$$\begin{aligned} &(u,v)_{H^1(\Omega)} = \textstyle\int_\Omega \nabla u \nabla v dx + (u,v)_{L^2(\Omega)}, \\ &\|u\|_{H^1(\Omega)} = (u,u)^{\frac{1}{2}}_{H^1(\Omega)}, \ \forall u,v \in H^1(\Omega), \end{aligned} \tag{10}$$

and

$$(y_1,y_2)_E = (u_1,u_2)_{H^1(\Omega)} + (v_1,v_2)_{L^2(\Omega)}, \ \forall y_i = \begin{pmatrix} u_i \\ v_i \end{pmatrix} \in E, \ i = 1,2.$$

$$\|y\|_E = (y,y)^{\frac{1}{2}}_E, \quad \forall y = \begin{pmatrix} u \\ v \end{pmatrix} \in E$$

denote the usual inner products in $L^2(\Omega)$, $H^1(\Omega)$ and $E = H^1(\Omega) \times L^2(\Omega)$, respectively.

Consider equations (1)-(3) as an abstract evolutionary equation in E:

$$\begin{aligned} &Y_t = CY + G(Y), \ Y \in E, \\ &Y(0) = Y_0 \in E, \end{aligned} \tag{11}$$

where

$$\begin{aligned} &Y = \begin{pmatrix} u \\ v \end{pmatrix}, \ C = \begin{pmatrix} 0 & I \\ -A & -\beta I \end{pmatrix}, \\ &D(C) = D(A) \times H^1(\Omega), \ G\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ h(u) + g \end{pmatrix}. \end{aligned} \tag{12}$$

Note that C is an unbounded closed operator and generates a C_0-semigroup on E for details, see Hale [2], and Hale and Raugel [3].

Lemma 2.1. *The semigroup $\{T(t), t \geq 0\}$ determined by system (11) possesses a global attractor B in E .*

Proof. For the proof see Hale [2] or Hale and Raugel [3]. □

The first step will be to study the differentiability of $T(t)$.

Lemma 2.2. *Consider the linearized of equations (1), (2)*

$$\begin{aligned} &U_{tt} + \beta U_t + U - \Delta U = h'(u) U, \ x \in \Omega, \ t \geq 0, \\ &\tfrac{\partial U(x,t)}{\partial \nu} = 0 \ \forall t > 0, \ x \in \partial\Omega, \\ &U(x,0) = \xi, \ U_t(x,0) = \eta, \ x \in \Omega, \end{aligned} \tag{13}$$

where $\{u, u_t\}$ is a solution of (1)-(3). Then (13) is well-posed problem in E, the mapping $T(t)$ is Fréchet differentiable on E for any $t > 0$, and its differential at $\varphi = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$ is the linear operator on E, $\begin{pmatrix} \xi \\ \eta \end{pmatrix} \to \begin{pmatrix} U(t) \\ V(t) \end{pmatrix}$, where $\begin{pmatrix} U \\ V \end{pmatrix}$ is the solution of (13) .

Proof. From assumptions on h it follows that (13) is a well-posed problem in E, for details see Hale [2].

Next, we show a Lipschitz property of the mapping

$$T(t) : \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \to \begin{pmatrix} u(t) \\ u_t(t) \end{pmatrix} \text{ on bounded sets of } E.$$

Suppose $\varphi_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in E$ and $\tilde{\varphi}_0 = \varphi_0 + \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} u_0 + \xi \\ u_1 + \eta \end{pmatrix} \in E$.

Then it follows from [2] that the solution

$$T(t)\varphi_0 = \varphi(t) = \begin{pmatrix} u(t) \\ u_t(t) \end{pmatrix} \in E \quad \text{and}$$

$$T(t)\tilde{\varphi}_0 = \tilde{\varphi}(t) = \begin{pmatrix} \tilde{u}(t) \\ \tilde{u}_t(t) \end{pmatrix} \in E.$$

Since there is a global attractor B for $T(t)$, $t \geq 0$ in E, then $\varphi(t)$, $\tilde{\varphi}(t)$ are uniformly bounded in E for $t \geq 0$.

Let $\psi = \tilde{u} - u$. Then the difference satisfies

$$\psi_{tt} + \beta\psi_t + \psi - \Delta\psi = h(\tilde{u}) - h(u). \tag{14}$$

By assumptions (4), it is known that $f : H^1(\Omega) \to L^2(\Omega)$ is compact, see Hale [2]. Taking the scalar product of (14) with $\psi_t = \tilde{u}_t - u_t$ in $L^2(\Omega)$, and by Lemma (2.1), we have

$$\begin{aligned} &\tfrac{1}{2}\frac{d}{dt}\left(\|\psi_t\|^2_{L^2(\Omega)} + \|\psi\|^2_{H^1(\Omega)}\right) \\ &= -\beta\|\psi_t\|^2_{L^2(\Omega)} + (h(\tilde{u}) - h(u), \psi_t)_{L^2(\Omega)} \\ &\leq -\beta\|\psi_t\|^2_{L^2(\Omega)} + (h'(u + \vartheta(\tilde{u} - u))\psi, \psi_t)_{L^2(\Omega)} \\ &\leq -\beta\|\psi_t\|^2_{L^2(\Omega)} + k(\sqrt{\lambda_1})^{-1}\|\psi\|_{H^1(\Omega)}\|\psi_t\|_{L^2(\Omega)} \\ &\leq c\left(\|\psi_t\|^2_{L^2(\Omega)} + \|\psi\|^2_{H^1(\Omega)}\right), \end{aligned} \tag{15}$$

where $\vartheta \in (0,1)$, $c > 0$.

Thus

$$\frac{d}{dt}\left(\|\psi_t\|^2_{L^2(\Omega)} + \|\psi\|^2_{H^1(\Omega)}\right) \leq 2c\left(\|\psi_t\|^2_{L^2(\Omega)} + \|\psi\|^2_{H^1(\Omega)}\right).$$

So, we have the Lipschitz property

$$\begin{aligned} \|\tilde{\varphi}(t) - \varphi(t)\|^2_E &= \|\tilde{u}_t(t) - u_t(t)\|^2_{L^2(\Omega)} + \|\tilde{u}(t) - u(t)\|^2_{H^1(\Omega)} \\ &\leq \exp(2c_3 t)\left(\|\eta\|^2_{L^2(\Omega)} + \|\xi\|^2_{H^1(\Omega)}\right), \text{ for all } t \geq 0. \end{aligned}$$

Consider the difference $\theta = \tilde{u} - u - U$ with U the solution of the linearized equation (13) of (1)-(3). Using the definitions of $\tilde{u}, u$ and U, we obtain,

$$\theta(0) = \theta_t(0) = 0, \tag{16}$$

and θ satisfies

$$\theta_{tt} + \theta - \Delta\theta = h(\tilde{u}) - h(u) - h'(u)(\tilde{u} - u) + h'(u)\theta - \beta\theta_t. \tag{17}$$

By mean value theorem, we have

$$\theta_{tt} + \theta - \Delta\theta = (h'(u + \vartheta_1(\tilde{u} - u)) - h'(u)(\tilde{u} - u)) - \beta\theta_t\,, \tag{18}$$

where $\vartheta_1 \in (0, 1)$.

Taking the scalar product of each side of (17) with θ_t in $L^2(\Omega)$, we find

$$\begin{aligned} &\tfrac{1}{2}\frac{d}{dt}\left(\|\theta_t\|^2_{L^2(\Omega)} + \|\theta\|^2_{H^1(\Omega)}\right) \\ &= ((h'(u + \vartheta_1(\tilde{u} - u)) - h'(u))(\tilde{u} - u), \theta_t)_{L^2(\Omega)} \\ &+ (h'(u)\,\theta - \beta\theta_t, \theta_t)_{L^2(\Omega)}\,. \end{aligned} \tag{19}$$

So, we get

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\left(\|\theta_t\|^2_{L^2(\Omega)} + \|\theta\|^2_{H^1(\Omega)}\right) \\ &\leq \|\theta_t\|_{L^2(\Omega)}(k_1\|\tilde{u} - u\|^{1+\delta}_{L^2(\Omega)} + k\|\theta\|_{L^2(\Omega)} - \beta\|\theta_t\|_{L^2(\Omega)}) \\ &+\|\theta_t\|_{L^2(\Omega)}(k_1\|\tilde{u} - u\|^{1+\delta}_{L^2(\Omega)} + k\|\theta\|_{L^2(\Omega)})\,. \end{aligned} \tag{20}$$

Then we get,

$$\begin{aligned} &\frac{d}{dt}\left(\|\theta_t\|^2_{L^2(\Omega)} + \|\theta\|^2_{H^1(\Omega)}\right) \\ &\leq c_2\left(\|\theta_t\|^2_{L^2(\Omega)} + \|\theta\|^2_{H^1(\Omega)}\right) + c_3\|\tilde{u} - u\|^{2+2\delta}_{H^1(\Omega)}\,, \end{aligned} \tag{21}$$

where $c_2, c_3 > 0$. From the Gronwall inequality we obtain

$$\|\theta_t\|^2_{L^2(\Omega)} + \|\theta\|^2_{H^1(\Omega)} \leq \tfrac{c_2}{c_3}\exp(c_2 t)\int_0^t \left(\|\tilde{u}(\tau) - u(\tau)\|^{2+2\delta}_{H^1(\Omega)}\right) d\tau \tag{22}$$

$$\leq c_4 \exp(c_5 t)\,[(\|\xi\|^2_{H^1(\Omega)} + \|\eta\|^2_{L^2(\Omega)})^{1+\delta}], \text{for all } t \geq 0,$$

where $c_4, c_5 > 0$,

$$\|\tilde{\varphi}(t) - \varphi(t) - U(t)\|^2_E \leq c_4 \exp(c_5 t)\left[\left\|\begin{bmatrix}\xi \\ \eta\end{bmatrix}\right\|^{2+2\delta}_E\right].$$

Therefore,

$$\frac{\|\tilde{\varphi}(t) - \varphi(t) - U(t)\|^2_E}{\left\|\begin{bmatrix}\xi \\ \eta\end{bmatrix}\right\|^2_E} \leq c_4 \exp(c_5 t)\left[\left\|\begin{bmatrix}\xi \\ \eta\end{bmatrix}\right\|^{2\delta}_E\right] \to 0,$$

as $\begin{bmatrix} \xi \\ \eta \end{bmatrix} \to 0$ in E. □

3. Proof of the Main Theorem

Let us consider the system (1)-(3). Let $\varphi = \begin{bmatrix} u \\ v \end{bmatrix}$, $v = \dot{u} + \epsilon u$, where ϵ is chosen as

$$\epsilon = \frac{\beta\lambda_1}{\beta^2 + 4\lambda_1}; \tag{23}$$

then the system can be written as

$$\begin{aligned} &\dot{\varphi} + \Lambda\varphi = G(\varphi), \\ &\varphi(0) = \begin{bmatrix} u_0 \\ u_1 + \epsilon u_0 \end{bmatrix}, \end{aligned} \tag{24}$$

$$G(\varphi) = \begin{bmatrix} 0 \\ -h(u) + g \end{bmatrix}, \ \Lambda = \begin{bmatrix} \epsilon I & -I \\ A - \epsilon(\beta - \epsilon) I & (\beta - \epsilon) I \end{bmatrix}. \tag{25}$$

Lemma 3.1. *Let* $\left\{ \begin{bmatrix} \xi_i \\ \eta_i \end{bmatrix} \right\}_{j=1}^{\ell}$ *be an orthonormal family of elements of* E*. Then*

$$\sum_{j=1}^{\ell} |\xi_i|^2 \le \sum_{j=1}^{\ell} \lambda_j^{-1}. \tag{26}$$

Proof. See Lemma 6.3 in Chapter 6 in [5]. □

To estimate the Hausdorff dimension of the global attractor B for (24) in E. We consider the first variation equation of (24)

$$\Psi' = -\Lambda\Psi + G'(\varphi)\Psi, \ \Psi(0) = \begin{bmatrix} \xi \\ \eta \end{bmatrix} \in E, \tag{27}$$

where $\Psi = \begin{bmatrix} U \\ V \end{bmatrix} \in E$ and $\varphi = \begin{bmatrix} u \\ v \end{bmatrix}$ is a solution of equation (24) with Dirichlet boundary condition (2) and initial condition (3) and

$G'(\varphi) = \begin{bmatrix} 0 & 0 \\ h'(u) & 0 \end{bmatrix}$, $D(\Lambda) = D(A) \times H^1(\Omega)$.

Since the relation of $T(t)$ with $T_\epsilon(t)$ defined by (24) is made by reversible transformation $u = u, v = u_t + \epsilon u$, it easy to show (Lemma 2.2) that (27) is a well-posed problem in E, the mapping $T_\epsilon(t)$ is Fréchet differentiable on E for any $t > 0$, and its differential at $\varphi = \begin{bmatrix} u_0 \\ u_1 + \epsilon u_0 \end{bmatrix}$ is the linear operator on E, $\begin{bmatrix} \xi \\ \eta \end{bmatrix} \to \begin{bmatrix} U(t) \\ V(t) \end{bmatrix}$, where $\begin{bmatrix} U \\ V \end{bmatrix}$ is the solution of equation (27).

Lemma 3.2. *If $\varphi = \begin{bmatrix} u \\ v \end{bmatrix} \in E$, then*

$$(\Lambda\varphi, \varphi) \geq \sigma \|\varphi\|_E^2 + \frac{\beta}{2} \|v\|_{L^2(\Omega)}^2, \tag{28}$$

where

$$\sigma = \frac{\lambda_1 \beta}{\left(\beta + \sqrt{\beta^2 + 4\lambda_1}\right)\sqrt{\beta^2 + 4\lambda_1}}. \tag{29}$$

Proof. Since

$$(\Lambda\varphi, \varphi) - \sigma \|\varphi\|_E^2 + \frac{\beta}{2} \|v\|_{L^2(\Omega)}^2$$

$$= (\epsilon - \sigma) \|u\|_{H^1(\Omega)}^2 + \left(\frac{\beta}{2} + \epsilon - \sigma\right) \|v\|_{L^2(\Omega)}^2 - \epsilon(\beta - \epsilon)(u, v)_{L^2(\Omega)}$$

$$\geq (\epsilon - \sigma) \|u\|_{H^1(\Omega)}^2 + \left(\frac{\beta}{2} + \epsilon - \sigma\right) \|v\|_{L^2(\Omega)}^2 - \frac{\beta\epsilon}{\sqrt{\lambda_1}} \|u\|_{H^1(\Omega)} \|v\|_{L^2(\Omega)},$$

and simple computations by (23) and (29) give

$$4(\epsilon - \sigma)\left(\frac{\beta}{2} + \epsilon - \sigma\right) = \frac{\epsilon^2\beta^2}{\lambda_1}.$$

Thus, this is end of the proof. □

Lemma 3.3. *Consider the system (24). Let Φ denote a set of ℓ vectors $\{\Phi_i\}_{i=1}^{\ell}$ which are orthonormal in E. If*

$$\sup_{\Phi\subset E}\sup_{\varphi\in B}\sum_{i=1}^{\ell}((-\Lambda+G'(\varphi))\Phi_i,\Phi_i)_E\leq 0, \tag{30}$$

then the Hausdorff dimension of the global attractor B is less than or equal to ℓ, that is

$$\dim_H(B)\leq \ell.$$

Proof. By direct application of Theorem V. 3.3, equation (V. 3.47)-(V. 3.49) and identity (VI. 6.24) of [5], the above lemma follows. □

Lemma 3.4. *The Hausdorff dimension $d_H(B)$ of B for system (24), (2), and (3) in E satisfies*

$$d_H(B)\leq\left\{\ell:\ell\in N,\frac{1}{\ell}\sum_{j=1}^{\ell}\lambda_i^{-1}\leq\frac{2\beta\sigma}{k^2}\right\}. \tag{31}$$

Proof. Let $\ell\in N$ be fixed. Consider ℓ solutions $\Psi_1,\Psi_2,..,\Psi_\ell$ of equation (27). At a given time τ, let $Q_\ell(\tau)$ denote the orthogonal projector in E onto the space spanned by Ψ_1, Ψ_2,..., Ψ_ℓ. Let $\Phi_j(\tau)=\begin{bmatrix}\xi_j\\ \eta_j\end{bmatrix}\in E, j=1,2,...,\ell$ be an orthonormal basis of $Q_l(\tau)E$. From (28) and $\|\Phi_j\|_E=1$ we have

$$-(\Lambda\Phi_j,\Phi_j)_E\leq-\sigma-\frac{\beta}{2}\|\eta_j\|^2_{L^2(\Omega)} \tag{32}$$

and

$$\begin{aligned}&(G'(\varphi)\Phi_j,\Phi_j)_E\\&=(h'(\varphi)\xi_j,\eta_j)_{L^2(\Omega)}\\&\leq k\|\xi_j\|_{L^2(\Omega)}\|\eta_j\|_{L^2(\Omega)}\\&\leq\frac{k^2}{2\beta}\|\xi_j\|^2_{L^2(\Omega)}+\frac{\beta}{2}\|\eta_j\|^2_{L^2(\Omega)}.\end{aligned} \tag{33}$$

Hence, by (26), (32) and (33)

$$\sum_{i=1}^{\ell}((-\Lambda+G'(\varphi))\Phi_i,\Phi_i)_E\leq-\frac{\ell k^2}{2\beta}(\frac{2\beta\sigma}{k^2}-\frac{1}{\ell}\sum_{j=1}^{\ell}\lambda_j^{-1}). \tag{34}$$

If

$$\frac{1}{\ell}\sum_{i=1}^{\ell}\lambda_j^{-1} \leq \frac{2\beta\sigma}{k^2},$$

then

$$\sum_{i=1}^{\ell}((-\Lambda + G'(\varphi))\Phi_i, \Phi_i)_E \leq 0.$$

By Lemma 3.3, the proof is completed. □

Using Lemma 3.4, (29) and (31) the proof of the main theorem is completed.

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