International Journal of Pure and Applied Mathematics
Volume 1 No. 4 2002, 373-377

A SIMPLE PROOF THAT HOMOCLINICITY IMPLIES HORSESHOE FOR CONTINUOUS INTERVAL MAPS

Ming-Chia Li
Dept. of Mathematics
National Changhua University of Education
Changhua, Taiwan 500, REPUBLIC OF CHINA
e-mail: mcli@math.ncue.edu.tw

Abstract: In [5] Smale, showed that for diffeomorphisms on high dimensions the existence of a transverse homoclinic point implies the existence of a horseshoe and so guarantees chaotic dynamics. In Theorem 16.5 of [2], Devaney showed the same result holds for C^1 maps on the real line with a non-degenerate homoclinic point. In Proposition III.16 of [1], Block and Coppel showed that for a continuous map f on the real line a (possibly degenerate) homoclinic point leads a horseshoe for f^2. In this paper, based on an elementary proof, we extend the result of Block and Coppel.

AMS Subject Classification: 37E05, 37C25
Key Words: homoclinicity, 2-horseshoe, repelling fixed points

*

Throughout this paper we assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous map and p is a fixed point of f, i.e. $f(p) = p$. For the case when p is a periodic point of period n, the argument can be easily extended.

Received: February 26, 2002

Definition 1. The point p is said to be *one-sided repelling*, if there is an interval $I = [p, p + \epsilon)$ or $(p - \epsilon, p]$ such that the ratio $\frac{f(x) - p}{x - p} > 1$ for all $x \neq p$ in I. The point p is said to be *two-sided repelling*, if there is an interval $I = (p - \epsilon, p + \epsilon)$ such that the ratio $\frac{f(x) - p}{x - p} < -1$ for all $x \neq p$ in I. For both cases, we call the point p a *repelling fixed point* and such an interval I a *local unstable set* of p.

It is easy to see that, if f is differentiable at p, then the inequality $|f'(p)| > 1$ implies repulsion at p. Such a criterion is often used as a definition for repelling fixed points, e.g. [2]. Here we have chosen definitions that seem reasonable for maps which is possibly not differentiable.

Definition 2. Let q be a point in a local unstable set of a repelling fixed point p. We call q *a homoclinic point* of p if, $q \neq p$, $f^l(q) = p$ for some integer $l \geq 2$, and there is a choice of backward iterates q_i for $i \leq 0$ such that $q_0 = q$, $f(q_{i-1}) = q_i$ for $i \leq 0$, and q_i converges to p as i goes to $-\infty$.

A homoclinic point together with its backward iterates defined above and its (finite) forward iterates is called a *homoclinic orbit*. In the case, when f is differentiable, a homoclinic point is said to be *nondegenerate*, if $f'(x) \neq 0$ for all points x on the homoclinic orbit. In higher dimensional space, what we have defined as a nondegenerate homoclinic point is called a *transverse homoclinic point*.

Definition 3. We say that f has a *2-horseshoe*, if there are two proper closed intervals I_1 and I_2 in $\mathbb{R}$ with disjoint interiors, such that $f(I_1) \supset I_1 \cup I_2$ and $f(I_2) \supset I_1 \cup I_2$.

There are various names in the literature associated to a 2-horseshoe. For instance, sometimes it is called an *L-scheme* in [4] and an *turbulent* in [1].

In Proposition III.16 of [1], Block and Coppel showed that for a continuous map f on the real line a homoclinic point leads to a horseshoe for f^2. The following theorem is a generalization.

Theorem 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map with a homoclinic point q of a repelling fixed point p. Then there exists a* 2*-horseshoe for f or f^2, provided that p is one-sided or two-sided repelling, respectively.*

The proof is very elementary and uses only mathematical induction.

Proof. Without loss of generality, we assume that $q > p$. Suppose p is one-sided repelling. Then $p < q < f(q)$. Since $f^l(q) = p$ for some integer $l \geq 2$, we can define $n = \min\{i \geq 2 : f^i(q) \leq p\}$. Thus $f^i(q) \neq f^j(q)$ for all $0 \leq i, j \leq n$ with $i \neq j$. Now we proceed the proof by induction on n. If $n = 2$, let $I_1 = [p, q]$ and $I_2 = [q, f(q)]$, then $f(I_1) \supset I_1 \cup I_2$ and $f(I_2) \supset I_1 \cup I_2$. Thus f has a 2-horseshoe. Suppose that the result is true for $n = 2, 3, \ldots, k-1$. If $n = k$, then $f^k(q) \leq p$. Let $m = \max\{0 \leq i \leq k-1 : f^i(q) < f^{i+1}(q)\}$, then $0 \leq m \leq k-2$ because of $q < f(q)$ and $f^k(q) < f^{k-1}(q)$. If $m \neq 0$, then $2 \leq k-m \leq k-1$ and $f(p) \leq p < f^m(q) < f(f^m(q))$ and $f^{k-m}(f^m(q)) \leq p$. By the assumption on induction, f has a 2-horseshoe. If $m = 0$, then $f^k(q) \leq p < f^{k-1}(q) < f^{k-2}(q) < \cdots < f^2(q) < f(q)$. Thus either $q < f^i(q)$ for all $1 \leq i \leq k-1$ or $f^{j+1}(q) < q < f^j(q)$ for some $1 \leq j < k-1$. For the former case, let $I_1 = [p, q]$ and $I_2 = [q, f^{k-1}(q)]$, then $f(I_1) \supset I_1 \cup I_2$ and $f(I_2) \supset I_1 \cup I_2$. Thus f a 2-horseshoe. For the latter case, let $I_1 = [f^{j+1}(q), q]$ and $I_2 = [q, f^j(q)]$, then $f(I_1) \supset I_1 \cup I_2$ and $f(I_2) \supset I_1 \cup I_2$. Thus f a 2-horseshoe. This completes the proof of the first statement of the theorem.

Suppose p is two-sided repelling, and let $I = (p-\epsilon, p+\epsilon)$ be a local unstable set of p which contains q. Then $f(I) \supset I$ and so there is an subinterval $J \subset I$ such that $p \in J$ and $f(J) = I$. By taking a preimage of q if necessary, we can assume that $q \in J$ and $q > p$. Let $g = f^2$. Then the point p is a one-sided repelling fixed point for g, with respect to the local unstable set $I' = [p, p+\epsilon') \subset J$, since for all $x \neq p$ in J

$$\frac{g(x)-p}{x-p} = \frac{f(f(x))-p}{f(x)-p} \cdot \frac{f(x)-p}{x-p} > 1.$$

Clearly, the point q is a homoclinic point of p for g. By the same argument as above, $g = f^2$ has a 2-horseshoe. □

Note that if p is one-sided repelling and the intervals I_1 and I_2 are from the proof of the theorem, then $I_1 \cup I_2$ is a connected compact interval containing q. We define $\Lambda = \bigcap_{i=0}^{\infty} f^i(I_1 \cup I_2)$, then Λ is an invariant set for f. If p is two-sided repelling, then we define the invariant set Λ in the same way by replacing f by f^2.

The result on the existence of a 2-horseshoe gives us the following chaotic characteristics, refer to [3].

Corollary 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map with a homoclinic point q of a repelling fixed point p. Depending on p to be one-sided (or two-sided repelling, respectively), the followings hold:*

1. *The map f (or f^2, respectively) on the invariant set Λ defined above is topologically conjugate to the one-sided shift map on two symbols.*

2. *f has periodic points of all periods (or of all even periods, respectively).*

3. *The topological entropy of f is at least $\log 2$ (or at least $\frac{1}{2}\log 2$, respectively).*

The following example indicates that the existence of a homoclinic point does not imply the existence of period-odd cycles.

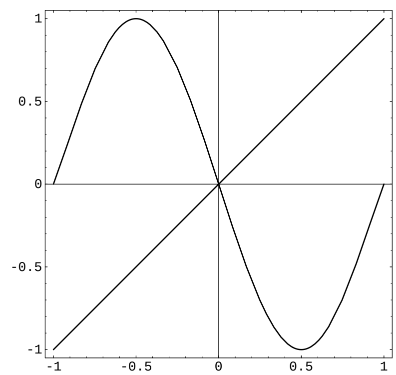

Figure 1

Example 1. Let

$$f(x) = -\frac{16}{9}x^5 + \frac{44}{9}x^3 - \frac{28}{9}x \quad \text{on} \quad [-1, 1],$$

see its figure below. Then 0 is a repelling fixed point of f with $f'(0) < -1$ and $\frac{1}{2}$ is a non-degenerate homoclinic point of 0. It is easy to see that f has a fixed point and periodic points of all even periods, but has no periodic points of odd periods bigger than one.

Acknowledgements

This work was partially supported by NSC grant.

References

[1] L. Block and W. Coppel, *Dynamics in One Dimension*, Lecture Notes in Math., **1513**, Springer, Berlin (1992).

[2] R. Devaney, *An Introduction to Chaotic Dynaimcal Systems*, 2nd ed., Addison-Wesley (1989).

[3] C. Robinson, *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos*, 2nd ed., CRC Press, Boca Raton, FL (1999).

[4] A. N. Sharkovsky, Coexistence of cycles of a continuous map of a line into itself, *Ukrainian Math. J.*, **16** (1964), 61–71.

[5] S. Smale, Differentiable dynamical systems, *Bull. Amer. Math. Soc.*, **73** (1967), 747–817.