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NULL CURVES AND 2-SURFACES OF GLOBALLY NULL MANIFOLDS

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Abstract: In this paper we develop a technique which provides a way to reduce problems of null geometry to problems of Riemannian geometry for null 2-surfaces and 3-dimensional globally null manifold (see Definition 1) with an integrable screen distribution.

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1. Introduction

During last quarter of the twentieth century the research on massless objects has produced considerable insight information on our physical universe. Unfortunately, very limited information is available (other than some papers dealing with specific problems) on the general geometric theory of null curves (which, in particular, represent one dimensional massless particles), null 2-surfaces and higher dimensional null (lightlike) manifolds, needed as a mathematical foundation and its use in physics. Also, compared to extensive research on global Riemannian and Lorentzian geometries (see Berger [2] and Beem et al [1]), the

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study on global geometry of null manifolds is quite rare (see, however, Duggal [4], [5]). It is reasonabie to believe thai any global study of null manifolds, must be based on extensive knowledge of null curves and 2-surfaces, which is the objective of this paper. Traditional way of studying of geometric objects using submanifold theory and physical objects in 4-dimensional spacetimes has recently changed. For example, for physical study, the dimension of landing space depends on the type of problem. Now one needs eleven dimensions to unite all the forces of the universe with a single equation, which is still at preliminary stages. Consequently, the study of geometric and or physical objects (independent of their landing manifold) has recently drawn considerable attention. In line with this latest trend, we assume, in this paper, that although the degenerate metric of a null manifold M must come from some non-degenerate metric of a higher dimensional manifold, say $\bar{M}$, but M need not be landed in $\bar{M}$. This seems to be a reasonable assumption for a global study of null curves and 2-surfaces since they do naturally arise as geometric and physical objects. For specific examples and computing metric components, we use a suitable landing manifold with a nondegenerate metric.

An outline of the paper is as follows. In Section 2, we define globally null manifolds (M, g) which admit a global null vector field, a complete Riemannian hypersurface and a metric (Levi-Civita) connection with respect to the degenerate metric g of M. For proof of Theorem 1 and details we refer Duggal [4]. Section 3 is devoted to a study of null curves, denoted by C, in a globally null manifold M. We construct a global Frenet frame (see Proposition 1) and recall that there exists a special parameter p with respect to which C is a null geodesic curve on M. It is known that M is a global product manifold of a 1-dimensional null manifold and a complete Riemannian manifold, if and only if its screen bundle space $S(TM)$ is integrable (see Theorem 2). The material presented in this section can be effectively used in reducing the problems of null geometry to problems of Riemannian geometry, which further opens a way to study on global null geometry by using several key results of Riemannian geometry (see, for example, Berger [2]). Some work in this direction has been done by the present author [5]. In Section 4 we study null 2-surfaces N of a 3-dimensional globally null manifold M. We show that the geometry of N is essentially the same as that of the family of its spacelike curves (see Theorem 3). In particular, we study ruled null surfaces in 4-dimensional Minkowski or curved spacetimes $\bar{M}$. Initially, the notion of ruled null surfaces was introduced by Schild [10] as a null geodesic string on the null cone (with one dimension suppressed) of $\bar{M}$. We show that Schild's null geodesic 2-surface can be obtained by null geodesics of a sequence of a class of 4-dimensional globally null manifolds, tangent to the null cone of $\bar{M}$. Finally, in Section 5 we study spacelike 2-surfaces of a globally null manifold. We show that the geometry of a 3-dimensional globally null manifold M is the same as the Riemannian geometry of its spacelike 2-surface generated by its integrable screen distribution $S(TM)$. This is a step further than a problem discussed by Duggal in [5], on constant curvature of globally null manifolds. The results, in this paper, are not restricted to a constant scalar curvature of M. The paper contains examples of its main results.

2. Globally Null Manifolds

Let (M, g) be a real n-dimensional smooth and paracompact manifold, where g is a symmetric tensor field of type $(0, 2)$. Denote by $Rad\,TM$ a radical (null) distribution of the tangent bundle space TM of M, which is defined by

$$Rad\,TM = \{\xi \in \Gamma(TM)\,;\, g(\xi, X) = 0 \quad \text{for all} \quad X \in \Gamma(TM)\}, \tag{1}$$

where $\Gamma(TM)$ denotes the set of all tangent vector fields on M. The dimension, say r, of $Rad\,TM$ is called nullity degree of g. Clearly, g is degenerate or non-degenerate on M iff $r > 0$ or $r = 0$, respectively. We say that (M, g) is a lightlike manifold if $0 < r \leq n$. For a lightlike M, a complementary distribution $S(TM)$ to $Rad\,TM$ in TM is called (for details, see Duggal [4]) a screen distribution on M, whose existence is secured for paracompact M. It is easy to see that $S(TM)$ is semi-Riemannian, and we have

$$TM = Rad\,TM \oplus S(TM). \tag{2}$$

Example 1. Let S^3_1 be the unit pseudo sphere of Minkowski space $\mathbf{R}^4_1$, given by $-t^2 + x^2 + y^2 + z^2 = 1$. Cut S^3_1 by the hypersurface $t - x = 0$ and obtain a lightlike surface M of S^3_1 with $Rad\,TM$ spanned by a null vector $\xi = \partial_t + \partial_x$. Take a screen distribution $S(TM)$ spanned by a spacelike vector $W = z\,\partial_y - y\,\partial_z$. Thus, M is lightlike with $r = 1$ and $S(TM)$ Riemannian.

Theorem 1. (Duggal, [4]) *Let (M, g) be an n-dimensional lightlike manifold, with $Rad\,TM$ of rank $r = 1$. Then, $Rad\,TM$ is a Killing distribution, and there exists a metric (Levi-Civita) connection ∇ on M with respect to the degenerate tensor field g.*

In this paper we study a special class of lightlike manifolds, introduced by the present author in [4] as follows:

Definition 1. (Duggal [4]) A lightlike (M, g) is called a globally null manifold, if it admits a global null vector field and a complete Riemannian hypersurface.

Since the 1-dimensional $Rad\,TM$ is obviously integrable, using Theorem 1 we construct an $(n+1)$-dimensional manifold, denoted by $(\bar{M}, \bar{g})$, with local coordinates (x, x^a, y), where (x, x^a) are coordinates on a globally null manifold (M, g), induced by the foliation determined by $Rad\,TM$ and (y) is a coordinate on 1-dimensional fiber of its vector bundle structure. In this way g can be taken as a degenerate metric on a family of globally null hypersurfaces M, induced by the metric $\bar{g}$ of $\bar{M}$. The embedding condition (see O'Neill [9]) implies that $\bar{g}$ must be a Lorentz metric. Since null spaces do arise quite naturally, which may not be embedded in any higher dimensional manifold, in this paper we have the following two approaches:

(a) Consider null and spacelike spaces in a globally null manifold which require no mention of an ambient manifold.

(b) For the purpose of computing the coefficients of the degenerate metric g of M, we assume that g is an induced metric of the Lorentz metric $\bar{g}$ of the manifold $\bar{M}$, constructed using 1-dimensional $Rad\,TM$, as explained above.

The approach (b) will also be followed for any example of a null curve, a 2-surface and or a globally null manifold, as explained below:

Example 2. Let $(\bar{M}, \bar{g})$ be an $(n+1)$-dimensional globally hyperbolic spacetime [1], with the line element of the metric $\bar{g}$ given by

$$ds^2 = -\,dt^2 + dx^1 + \bar{g}_{ab}\,d\,x^a\,d\,x^b, \;\; (\,a, b = 2, \ldots, n\,) \tag{3}$$

with respect to a coordinate system $(\,t\,, x^1\,, \ldots, x^n\,)$ on $\bar{M}$. Choose the range $0 < x^1 < \infty$ so that the metric (3) is non-singular. Take two null coordinates u and v such that $u = t + x^1$ and $v = t - x^1$. Thus, (3) transforms into a non-singular metric:

$$ds^2 = -\,du\,dv + \bar{g}_{ab}\,d\,x^a\,x^b.$$

The absence of du^2 and dv^2 in this transformed metric implies that $\{v =$ constant$\}$ and $\{\,u =$ constant$\}$ are lightlike hypersurfaces of N. Let $(M, g, r = 1, v =$ constant$)$ be one of this lightlike pair and let D be the 1-dimensional distribution generated by the null vector $\{\partial_v\}$, in $\bar{M}$. Denote by L the 1-dimensional integral manifold of D. A leaf M' of the $(n-1)$-dimensional screen distribution of M is Riemannian with metric $d\Omega^2 = \bar{g}_{ab}\, x^a\, x^b$ and is the intersection of the two lightlike hypersurfaces. In particular, there will be many global timelike vector fields in globally hyperbolic spacetimes $\bar{M}$. If one is given a fixed global *time function*, then its gradient is a global timelike vector field in a given $\bar{M}$. With this choice of a global timelike vector field in $\bar{M}$, we conclude that both its lightlike hypersurfaces admit a global null vector field. Now, the celebrated Hopf-Rinow theorem allows to assume that M' is a complete Riemannian hypersurface of M. Thus, there exists a pair of globally null hypersurfaces of a globally hyperbolic spacetime. In particular, a Minkowski space and a De-Sitter spacetime $\bar{M}$ have a pair of globally null hypersurfaces. Proceeding similar to above example for 4-dimensional $\bar{M}$, one can show that Robertson-Walker, Reissner-Nordström and Kerr spacetimes, all have pairs of globally null hypersurfaces (see Hawking-Ellis [6]).

3. Null Curves

Let C be a smooth null curve in a globally null n-dimensional manifold $(M,\, g)$ given by

$$x^i = x^i(t),\ t \in I\ \subset \mathbf{R},\ i \in \{1, \ldots, n\},$$

for a coordinate neighborhood $\mathcal{U}$ on C. Then, the tangent vector field

$$\frac{d}{dt} = (\,\frac{dx^1}{dt}, \ldots, \frac{dx^n}{dt}\,)$$

on $\mathcal{U}$ satisfies

$$g\,(\,\frac{d}{dt},\, \frac{d}{dt}\,) = 0, \qquad \text{i.e.,} \qquad g_{ij}\,\frac{dx^i}{dt}\,\frac{dx^j}{dt} = 0,$$

where $g_{ij} = g\,(\partial_i,\, \partial_j)$ and $i,\, j \in \{1, \ldots, n\}$. Denote by $T\,C$ the tangent bundle of C, which is a vector sub bundle of TM, and is of rank 1. Then

$$T\,C^{\perp} = \{V \in TM\,;\, g\,(\,V,\, \xi\,) = 0\} = TM\,,$$

where ξ is null vector field tangent over C. We consider a class of null curves, such that $Rad\,TM = T\,C$, and both are generated by ξ. Then we have

$$TM = Rad\,TM \oplus S(TM) = T\,C \oplus S(TM). \tag{4}$$

Proposition 1. *Let (M, g) be an n-dimensional globally null manifold. Then, there exists a quasi-orthonormal frame*

$$F = \{\xi, W_1, \dots, W_{n-1}\}, \quad g(W_a, W_a) = \delta_{ab}, \quad g(\xi, W_a) = 0, \tag{5}$$

for all $a \in \{1, \dots, n-1\}$, *along a null curve C, generated by a null vector field ξ on M, where $\Gamma(S(TM))$ is spanned by an orthonormal frame $\{W_1, \dots, W_{n-1}\}$.*

Proof. Since g is a metric tensor on M (see Theorem 1), we have

$$(L_X g)(Y, Z) = X(g(Y, Z)) - g([X, Y], Z) - g(Y, [X, Z]) = 0$$

for any $X, Y, Z \in \Gamma(TM)$. Using this, $\frac{d}{dt} \equiv \xi$ null and (4), we obtain the following differential equations:

$$\begin{aligned}
\nabla_\xi \xi &= h\,\xi, \\
\nabla_\xi W_1 &= -k_1 \xi + k_3 W_2 + k_4 W_3, \\
\nabla_\xi W_2 &= -k_2 \xi - k_3 W_1 + k_5 W_3 + k_6 W_4, \\
\nabla \xi W_3 &= -k_4 W_1 - k_5 W_2 + k_7 W_4 + k_8 W_5,
\end{aligned} \tag{6}$$

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$$\begin{aligned}
\nabla_\xi W_{n-2} &= -k_{n-1} W_{n-4} - k_n W_{n-3} + k_{n+2} W_{n-1} + k_{n+3} W_n, \\
\nabla_\xi W_{n-1} &= -k_{2n-4} W_{n-3} - k_{2n-3} W_{n-2},
\end{aligned}$$

provided $n \geq 5$, where h and $\{k_1, \dots, k_{2n-3}\}$ are smooth functions on $\mathcal{U}$ and $\{W_1, \dots, W_{n-1}\}$ is an orthonormal basis of $\Gamma(S(TM)_{\mathcal{U}})$. For $n < 5$ above equations reduce to the following cases:

Case 1 $(n = 2)$.

$$\begin{aligned}
\nabla_\xi \xi &= h\,\xi, \\
\nabla_\xi W_1 &= -k_1 \xi.
\end{aligned}$$

Case 2 $(n = 3)$.

$$\begin{aligned}
\nabla_\xi \xi &= h\,\xi, \\
\nabla_\xi W_1 &= -k_1 \xi + k_3 W_2, \\
\nabla_\xi W_2 &= -k_2 \xi - k_3 W_1.
\end{aligned}$$

Case 3 $(n = 4)$.

$$\begin{aligned} \nabla_\xi \xi &= h\,\xi, \\ \nabla_\xi W_1 &= -k_1\,\xi + k_3\,W_2 + k_4\,W_3, \\ \nabla_\xi W_2 &= -k_2\,\xi - k_3\,W_1 + k_5\,W_3, \\ \nabla_\xi W_3 &= -k_4\,W_1 - k_5\,W_2. \end{aligned}$$

In general, for any $n > 1$ we call F (given by (5)) a Frenet frame on M along C with respect to the screen distribution $S(TM)$. The functions $\{k_1, \ldots, k_{2n-3}\}$ and the differential equations (6) (along with three cases for $n < 5$) are called curvature functions of C and Frenet equations for F, respectively. Thus, F, given by (5), is a quasi-orthonormal Frenet frame which completes the proof. □

Example 3. Let $\mathbf{R}^4_1$ be a 4-dimensional Minkowski spacetime with a Lorentz metric of signature $(-\ +\ ++)$ and local coordinates $(x,\, x^1,\, x^2,\, y)$. Following the approach (b), as stated in Section 2, let $(M,\, g)$ be a globally null hypersurface of $\mathbf{R}^4_1$ such that $(x,\, x^1,\, x^2,\, y =$ constant$)$ are coordinates on M, induced by $Rad\,TM$. Consider a curve C in M defined by

$$x = f(t)\ ,\ \ x^1 = -f(t)\ ,\ \ x^2 = a_{_1}\ ,\ \ y = a_{_2}\ ,\quad p \in I \subset \mathbf{R},$$

where $a_{_1}$ and $a_{_2}$ are suitable constants. Then,

$$\frac{d}{dt} = (\,f'(t)\ ,\, -f'(t)\ ,\, 0\ ,\, 0\,) \quad \text{and} \quad g(\frac{d}{dt},\, \frac{d}{dt}) = 0.$$

Thus, C is a null curve of M, generated by a null vector field, say $\xi \equiv \frac{d}{dt}$. Choose a quasi-orthonormal set $\{\xi,\, W_1,\, W_2\}$ on M along C, where

$$W_1 = (\,b\,f(t)\,,\, -b\,f(t)\,,\, 1\,,\, 0\,)\ ;\ \ W_2 = (\,c\,f(t)\,,\, -c\,f(t)\,,\, 0\,,\, 1\,)$$

are orthonormal spacelike vectors which generate a screen distribution of M, b and c are suitable constants. Following are three Frenet equations:

$$\nabla_{_\xi}\,\xi = h\,\xi\ ,\quad h \equiv \frac{f''(t)}{f(t)}\,,$$
$$\nabla_{_\xi}\,W_1 = b\,\xi + 0\,W_2\ ,\quad \nabla_{_\xi}\,W_2 = c\,\xi + 0\,W_1$$

such that, according to the case 2 $(n = 3)$, $k_1 = -b$, $k_2 = -c$, $k_3 = 0$.

Consider, with respect to a given screen distribution $S(TM)$, two Frenet frames F and $F^\star$ along two neighborhoods $\mathcal{U}$ and $\mathcal{U}^\star$, respectively, with non-null intersection. Then we have

$$\xi^\star = \frac{dt}{dt^\star}\,\xi\,, \tag{7}$$

$$W_a^\star = A_a^b\,W_b, \quad a,b \in \{1,\ldots,n-1\}, \tag{8}$$

where A_a^b are smooth functions on $\mathcal{U}\cap\mathcal{U}^\star$ and the matrix $[\,A_a^b\,(x)\,]$ is an element of the orthogonal group $\mathbf{O}(n)$ for any $x \in \mathcal{U}\cap\mathcal{U}^\star$.

Proposition 2. *Let C be a null curve of an n-dimensional globally null manifold M, and F, F^* be two Frenet frames on $\mathcal{U}$ and $\mathcal{U}^*$ with curvature functions $\{k_1,\ldots,k_{2n-3}\}$ and $\{k_1^*,\ldots,k_{2n-3}^*\}$, respectively, induced by the same screen vector bundle $S(TC^\perp)$. Suppose $\mathcal{U}\cap\mathcal{U}^* \neq \phi$ and $\Pi_{\alpha=1}^{2n-2}\,k_\alpha \neq 0$ on $\mathcal{U}\cap\mathcal{U}^*$. Then at any point of $\mathcal{U}\cap\mathcal{U}^*$ we have*

$$k_1^\star = k_1\,A_1, \quad k_2^\star = k_2\,A_2, \tag{9}$$

$$k_a^* = A_a\,k_a\,\frac{dt}{dt^*}\,, \quad a \in \{3,\ldots,2n-3\}, \quad \textit{where} \quad A_a = \pm 1. \tag{10}$$

Proof. Since $\left[A_a^b\right]$ is an orthogonal matrix we infer $A_a^a = A_a = \pm 1$ and $A_a^b = 0$ for all $a,b \in \{1,\ldots,n-1\}$. Then from the second and the third equations in (6) with respect to both F and F^* and taking into account that $k_a \neq 0$ implies $k_a^* \neq 0$ for $a = 4,5$ we obtain the relations (9) and (10) for $a = 1,\ldots,5$. Similarly, we obtain all the relations of (10). □

Corollary 1. *Let C be a null curve as given in Proposition 2. Then, k_1 and k_2 are invariant functions up to a sign, with respect to any parameter transformations on C.*

Next, let $F = \left\{\frac{d}{dt}, N, W_1,\ldots,W_{n-1}\right\}$ and $\bar F = \left\{\frac{d}{d\bar t}, \bar N, \bar W_1,\ldots, \bar W_{n-1}\right\}$ be two Frenet frames with respect to $\left(t, S(TC^\perp), \mathcal{U}\right)$ and $\left(\bar t, \bar S(TC^\perp), \bar{\mathcal{U}}\right)$, respectively. Then the general transformations that relate elements of F and $\bar F$ on $\mathcal{U}\cap\bar{\mathcal{U}} \neq \phi$, are given by

$$\frac{d}{d\bar t} = \frac{dt}{d\bar t}\,\frac{d}{dt}\,, \tag{11}$$

$$\bar W_a = B_a^b\left(W_b - \frac{dt}{d\bar t}\,c_b\,\frac{d}{dt}\right), \tag{12}$$

where c_b and B_a^b are smooth functions on $\mathcal{U}\cap\bar{\mathcal{U}}$ and the $(n-1)\times(n-1)$ matrix $\left[B_a^b(x)\right]$ is an element of $O(m)$ for each $x\in\mathcal{U}\cap\bar{\mathcal{U}}$. Thus, by using (11) and the first equation in (6) for both F and $\bar{F}$, we obtain

$$\bar{h}=\frac{d^2t}{d\bar{t}^2}\,\frac{d\bar{t}}{dt}+h\,\frac{dt}{d\bar{t}}\,. \tag{13}$$

Proposition 3. (Duggal, [4]) *Let C be a null curve of a globally null manifold M, with Frenet equations (6). Then, it is possible to find a special parameter, say p, on C such that the function h vanishes. Moreover, C is a null geodesic with respect to such a special parameter p.*

Consider two Frenet frames F and $\bar{F}$ for two screen distributions $S(TM)$ and $\bar{S}(TM)$, respectively. Using the transformation equations (11) and (12) we conclude that the Proposition 2 will also hold for any screen distribution. Also, let p and $\bar{p}$ be two special parameters induced by t and $\bar{t}$, with respect to the same screen distribution. Then for both p and $\bar{p}$, one can obtain a special parameter $\bar{p}=a\,p+b$, $a\neq 0$. Thus, we have the following result:

Corollary 2. *Let M be a globally null manifold. The existence of a null geodesic curve C of M, is independent of both the parameter transformations on C and the screen distribution transformations.*

Example 4. Let C be the null curve as given in Example 3. It follows that C is a null geodesic, if $h\equiv\frac{f''(t)}{f(t)}=0$. This implies that $f''(t)=0$. Thus $f(t)=c_1t+c_2$ is a linear function. For this case we may take $t=p$, a special parameter. Other two Frenet equations (from Example 3) will be same.

Theorem 2. (Duggal, [4]) *Let $(M,\,g)$ be a globally null manifold. Then, the following assertions are equivalent:*

(a) *The screen distribution S(TM) is integrable.*

(b) *$M=L\times M'$ is a global product manifold, where M' is a leaf of $S(TM)$ and L is a 1-dimensional integral manifold of a null curve C in M.*

(c) *$S(TM)$ is parallel with respect to the metric connection ∇ on M.*

A Mathematical Model. Theorem 1 tells us that any lightlike manifold with $\dim(Rad\,TM)=1$, has a null Killing vector field, say ξ. In particular, for a globally null manifold M, ξ is globally defined. Using this information, we assume that M has a smooth 1 - parameter group G of isometries, whose orbits are global null curves in M, such that ξ is the infinitesimal generator of G. Let M' be the orbit space of the action $G \approx L$, where we denote L by a 1 - dimensional null line in M. Then M' is a smooth Riemannian hypersurface of M and the projection

$$\pi : M \to M'$$

is a principle L - bundle, with null fiber G. The global existence of null vector field implies that M' is Hausdorff and paracompact. The metric g restricted to the screen bundle space $S(TM)$, induces a Riemannian metric g' on M'. Since null vector field ξ is non - vanishing on M, we can take $\xi = \frac{\partial}{\partial t}$ a global null coordinate vector field for some global function t on M. Thus the null function t induces a diffeomorphism on M such that (M, g) is a global product manifold of the form

$$M = L \times M', \quad g = \pi^\star g'.$$

There exists a connection 1 - form η for the L - bundle π, determined by g and the null coordinate function t, such that $\eta(\xi) = 0$. Thus we have a mathematical model of globally null product manifolds (M, g) with an orbit data (M', g', t, η).

Proposition 4. *Under the hypothesis of Theorem 1, if the screen distribution $S(TM)$ of M, is integrable, then the first two curvature functions k_1 and k_2 vanish in the Frenet equations (6).*

Proof. $S(TM)$ integrable implies from (a) of Theorem 2 that M is a product manifold. Also it follows from (c) of Theorem 2, that there exists a metric connection ∇ on M, which preserves $Rad\,TM$ and $S(TM)$. Thus k_1 and k_2 (in the second and third equations of (6)) must vanish. □

Remark 1. Using Theorem 2, one can take (M', g') a complete spacelike hypersurface of the globally null manifold M with induced Riemannian metric g' expressed by

$$g' = w^1 \otimes w^1 + \ldots + w^{n-1} \otimes w^{n-1},$$

where $\{w^1, \ldots, w^{n-1}\}$ are duals of the orthonormal basis $\{W_1, \ldots, W_{n-1}\}$ of $\Gamma\, S(TM)$. Clearly, g' being Riemannian metric its inverse exists and is also Riemannian. In this way, any tensor (including degenerate metric g) on M can be projected onto its screen distribution, and all the analysis on M can be done on its integral spacelike hypersurface M'. In particular, for above mathematical model one can use the orbit data (M', g', t, η) to study null geometry of M. Consequently, Theorem 2 provides a way to reduce, as far as possible, problems of null geometry to problems of Riemannian geometry. The present author has recently used this idea and the technique of warped product to study some properties of 4-dimensional globally null manifolds. For details see Duggal [5], where we solved the following two problems:

(a) Let $M = (L \times B \times_f F, g)$ be a 4-dimensional globally null warped product manifold, $B = (a, b)$ an open connected subset of real line with positive definite metric dr^2 and $-\infty \leq a < b \leq +\infty$, and the 2-dimensional fiber space F be of constant scalar curvature. Then the metric g admits a warping function $f(r)$ for which M has a constant scalar curvature.

(b) Let $M = (L \times B \times_f F, g)$ be a 4-dimensional globally null warped product manifold, (B, g_B) a Riemannian surface with scalar curvature S^B and $F = (a, b)$ an open connected subset of real line with positive definite metric dx^2 and $-\infty \leq a < b \leq +\infty$. Then the metric g admits infinitely many warped functions for which M has constant scalar curvature.

Remark 2. It follows from the Remark 1 that a specific technique on the global study of null curves, presented in this paper and its use in [5], has been effective in reducing the problems of null geometry to problems of Riemannian geometry. We certainly hope that this opens a way to further study on global null geometry by using several key results of Riemannian geometry (see, for example, Berger [2]). In next section we show how this specific technique can be used to study curvature properties of null 2-surfaces.

4. Null 2-surfaces

Let (N, h) be a 2-surface of an $(n+2)$-dimensional globally null (M, g) manifold, $n > 0$, where h is the induced tensor field on N of g, i.e.

$$h(X, Y) = g(X, Y), \quad \text{for all} \quad X, Y \in \Gamma(TN).$$

It follows from Section 2 that N is a null 2-surface of M, if

$$Rad\, TM = Rad\, TN\,,$$

which we assume in this section. Since $S(TM)$ is Riemannian, it is always possible to decompose it such that

$$S(TM) = TH \oplus TV, \tag{14}$$

where TH and $TV = TH^{\perp}$ are horizontal and vertical distributions of $S(TM)$, respectively. In this section, we assume that $\dim(TH) = 1$ and, therefore, $\dim(TV) = n$. Thus using (2) and (14), we have

$$TM_{|N} = (RadTN \oplus TH) \oplus TV = TN \oplus TV. \tag{15}$$

Since both the distributions $RadTN$ and TH are of rank 1 on N, they are integrable. Therefore, there exists an atlas of local charts

$$\{\mathcal{U}\,;\, u^0\,, u^1,\, u^2, \ldots,\, u^{n+1}\,\}\,,$$

such that $\{\frac{\partial}{\partial u^0}, \frac{\partial}{\partial u^1}\} \in \Gamma(TN_{|\mathcal{U}})$. Thus the matrix of the degenerate metric g on M with respect to the natural frames field $\{\frac{\partial}{\partial u^o}, \ldots, \frac{\partial}{\partial u^{n+1}}\}$, is as follows

$$[g] = \begin{bmatrix} 0 & 0 \\ 0 & g_{ij}(u^o, \ldots, u^{n+1}) \end{bmatrix}, \tag{16}$$

where

$$g_{ij} = g\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right), \quad i, j \in \{1, \ldots, n+1\}, \quad \det[g_{ij}] \neq 0$$
$$\text{and} \quad g_{a1} = g_{1a} = 0$$

for all $a \in \{2, \ldots, n+1\}$. According to the general transformations on a foliated manifold, we have

$$\begin{aligned} \bar{u}^o &= \bar{u}^o(u^o, u^1, \ldots, u^{n+1}), \\ \bar{u}^i &= \bar{u}^i(u^1, \ldots, u^{n+1})\,. \end{aligned}$$

Using above transformations, a well-known procedure is available to obtain a local field of frames on M adapted to the decomposition (15). However, with respect to the tangent bundle space TM, we show that there exists a pseudo-orthonormal Frenet frame $F = \{\xi, W_1, W_2, \ldots, W_{n+1}\}$ on M along N, adapted to the decomposition (15), such that TN is spanned by $\{\xi, W_1\}$ and TV is spanned by $\{W_2, \ldots, W_{n+1}\}$. At this point, for simplicity and for physical reasons, we restrict to $n = 1$ so that M is a 3-dimensional globally null manifold (higher dimensional case is too lengthy but straightforward). Following the approach (b), as stated in Section 2, we assume that the degenerate metric g comes from a Lorentzian metric $\bar{g}$ of a 4-dimensional Minkowski spacetime $\mathbf{R}^4_1$. Suppose N is given by

$$x^A = x^A(u, v), \quad A \in \{0, 1, 2\}\,. \tag{17}$$

Then the tangent bundle of N is spanned by

$$\left\{ \frac{\partial}{\partial u} = \frac{\partial x^A}{\partial u} \frac{\partial}{\partial x^A} ; \quad \frac{\partial}{\partial v} = \frac{\partial x^A}{\partial v} \frac{\partial}{\partial x^A} \right\} .$$

By considering a vector field

$$\xi = \alpha \, \frac{\partial}{\partial u} + \beta \, \frac{\partial}{\partial v} \, , \tag{18}$$

we find that N is null, if and only if the homogeneous linear system with (α, β) as variables

$$\begin{aligned} &\alpha \left(\sum_{a=1}^{2} \left(\frac{\partial x^a}{\partial u} \right)^2 - \left(\frac{\partial x^0}{\partial u} \right)^2 \right) \quad + \quad \beta \left(\sum_{a=1}^{2} \frac{\partial x^a}{\partial u} \frac{\partial x^a}{\partial v} - \frac{\partial x^0}{\partial u} \frac{\partial x^0}{\partial v} \right) = 0, \\ &\alpha \left(\sum_{a=1}^{2} \frac{\partial x^a}{\partial u} \frac{\partial x^a}{\partial v} - \frac{\partial x^0}{\partial u} \frac{\partial x^0}{\partial v} \right) \quad + \quad \beta \left(\sum_{a=1}^{2} \left(\frac{\partial x^a}{\partial v} \right)^2 - \left(\frac{\partial x^0}{\partial v} \right)^2 \right) = 0 \, , \end{aligned}$$

has non-trivial solutions. Denote

$$D^{AB} = \begin{vmatrix} \frac{\partial x^A}{\partial u} & \frac{\partial x^B}{\partial u} \\ \frac{\partial x^A}{\partial v} & \frac{\partial x^B}{\partial v} \end{vmatrix} . \tag{19}$$

Proposition 5. *A 2-surface N of a 3-dimensional globally null manifold M is null, if and only if on each coordinate neighborhood $\mathcal{U} \subset N$ we have*

$$\sum_{a=1}^{2} (D^{0\,a})^2 = \sum_{1 \le a < b \le 2} (D^{a\,b})^2. \tag{20}$$

Next, we choose from the above homogeneous system

$$\alpha = \sum_{a=1}^{2} \frac{\partial x^a}{\partial u} \frac{\partial x^a}{\partial v} - \frac{\partial x^0}{\partial v} \frac{\partial x^0}{\partial v} ; \quad \beta = \left(\frac{\partial x^0}{\partial v} \right)^2 - \sum_{a=1}^{2} \left(\frac{\partial x^a}{\partial v} \right)^2 , \tag{21}$$

so that at least one of the quantities from the right hand side of (21) is non-zero. Then by direct calculations we find that $Rad\,TN$ is spanned by

$$\xi = \xi^A \, \frac{\partial}{\partial x^A} ; \quad \xi^A = \sum_{B=0}^{2} \epsilon_B \, D^{AB} \, \frac{\partial x^B}{\partial v} \, , \tag{22}$$

where $\{\epsilon_B\}$ is the signature of the basis $\left\{\frac{\partial}{\partial x^B}\right\}$ with respect to the Minkowski metric $\bar{g}$. The corresponding 1-dimensional screen distribution $S(TN) = TH$ (see equation (15)) is spanned by a spacelike vector field

$$U = \frac{\partial x^0}{\partial v}\frac{\partial}{\partial u} - \frac{\partial x^0}{\partial u}\frac{\partial}{\partial v} = D^{10}\frac{\partial}{\partial x^1} + D^{20}\frac{\partial}{\partial x^2}\,. \tag{23}$$

Finally, by using the decomposition (15) we obtain the spacelike vector bundle $TH^{\perp} = TV$ spanned by

$$V = D^{20}\frac{\partial}{\partial x^1} - D^{10}\frac{\partial}{\partial x^2}. \tag{24}$$

Summing up, we have a quasi-orthonormal Frenet frame field, adapted to the composition (15), as follows:

$$F = \left\{\xi,\ W_1 = \frac{1}{(\Delta)^{1/2}}U,\ W_2 = \frac{1}{(\Delta)^{1/2}}V\right\}, \quad \Delta = \sum_{a=1}^{2}(D^{0a})^2. \tag{25}$$

The adapted Frenet frame (25) on M will relate the material of Section 3 with the following discussion on the geometry of null 2-surfaces.

Using the terminology of differential geometry, we say that $\mathbf{x} = \mathbf{x}(u,\, v)$ is a class $C^m\,(m \geq 1)$ regular parametric representation of N, defined on a coordinates neighborhood $\mathcal{U}$, if:

(1) $\mathbf{x}$ is of class $C^m \in \mathcal{U}$.

(2) At least one 2×2 determinant D^{AB} of (19) is non-zero.

A curve $\mathbf{x}(u,\, v = v_0)$ is called a u-parameter curve on N. Similarly, a curve $\mathbf{x}(u = u_0,\, v)$ is called a v-parameter curve on N. To cover any possible singular points, we take necessary overlapping allowable coordinate patches and use elementary topology, so that N is at least smooth. If this is not possible, then we restrict the parameters to regular points. Thus subject to above topological constraints, this parametric representation can cover a regular N with a null family and a spacelike family of curves. Then $\mathbf{x}_u(u_0,\, v_0)$ and $\mathbf{x}_v(u_0,\, v_0)$ are vectors tangent to the u-parameter and the v-parameter curves, respectively, at their intersecting point $P(u_0,\, v_0)$ of N. In this paper we assume that u-parameter curve is null.

Null Tangent Plane. Let $P \in M$ and ℓ be a null vector of $T_P\, M$. A plane $T_P(N)$ of $T_P\, M$ is called a null plane directed by ℓ, if it contains ℓ, $g_P(\ell,\, W) = 0$ for any $W \in T_P(N)$, and there exists $W_0 \in T_P(N)$ such that $g(W_0,\, W_0) \neq 0$. In particular, given a regular parametric representation $\mathbf{x}(u, v)$ of a null surface N and any point $P \in N$, a null plane $T_P(N)$ through P parallel to $\mathbf{x}_u$ and $\mathbf{x}_v$ at P is called the null tangent plane to N at P. One can verify, that it is independent on the patch containing P and that a nonzero vector of M is tangent to N at P if and only if it is parallel to $T_P(N)$. Thus $T_{\mathbf{x}}(N)$ at any $\mathbf{x}$ on N is given by

$$\mathbf{y} = \mathbf{x} + A\,\mathbf{x}_u + B\,\mathbf{x}_v, \quad -\infty < A,\, B > \infty \tag{26}$$

Example 5. Construct a 4-dimensional Minkowski spacetime $(\mathbf{R}^4_1, \bar{g})$, with local coordinates $(x^0, x^1, x^2, y = a)$, where (x^0, x^1, x^2) are coordinates on a 3-dimensional globally null hypersurface (M, g) of $\mathbf{R}^4_1$ and a is a constant. Let N be a surface of M given by

$$\mathbf{x} = \mathbf{x}(u, v) = \big(x^0 = f(u, v),\; x^1 = \; -f(u, v),\; x^2 = -\,v^2\,\big), \tag{27}$$

where f is an arbitrary smooth function of two variables u and v and of class C^m $(m \geq 1)$. This parametrisation is admissible since from (21) $\alpha = 0$ but $\beta = -4\,v^2 \neq 0$. Then from (19) we obtain

$$D^{10} = 0\,, \quad D^{20} = v\,f_u \neq 0\,, \quad D^{12} = -v\,f_u \neq 0, \quad f_u \equiv \frac{\partial f}{\partial u}.$$

Using above values in (20) we verify that N is a null surface of M. Moreover, since not all D^{AB}'s vanish, N is a regular null surface of class C^m $(m \geq 1)$. From (19) and (22)-(24) we obtain

$$\xi = -\,v^2\,f_u\,\left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}\right), \quad U = f_u\,\frac{\partial}{\partial x^2}\,, \quad V = f_u\,\frac{\partial}{\partial x^1}.$$

Thus we have the following quasi-orthonormal Frenet frames field on M along the null surface N:

$$F = \Big\{\,\xi,\, W_1 = \frac{\partial}{\partial x^2},\, W_2 = \frac{\partial}{\partial x^1}\,\Big\}.$$

Also

$$\mathbf{x}_u = (\,f_u,\; -f_u,\; 0); \quad \mathbf{x}_v = (\,f_v,\; -f_v,\; -2v)$$

are null and spacelike vectors, respectively. Thus u-parameter and v-parameter curves are null and spacelike, respectively. N is regular, of class C^m $(m \geq 1)$, since not all D^{AB}'s vanish. $\{\,\mathbf{x}_u,\, \mathbf{x}_v\,\}$ is a linearly independent set. In particular, let $f(u, v) = u^2 - v^2$ and $P(1, 1)$ a point on N. Then

$$\mathbf{x}(1, 1) = (0,\; 0,\; -1); \quad \mathbf{x}_u(1, 1) = (2,\; -2,\; 0)\,;$$
$$\mathbf{x}_v(2, 1) = (-2,\; 2,\; -2).$$

Thus

$$\mathbf{y} = \mathbf{x}(1,\,1) + A\,\mathbf{x}_u(1,\,1) + B\,\mathbf{x}_v(1,\,1) \\ = \big(\,2A - 2B\,,\ 2B - 2A\,,\ -(1+B)\big)$$

is the equation of tangent plane at $P(1,\,1) \in N$.

Contrary to the Riemannian or semi-Riemannian case, unfortunately, the normal vector field $\mathbf{x}_u \times \mathbf{x}_v$ falls back in $T_P(N)$. Thus one fails to use, in the usual way, the theory of non-degenerate surfaces to study the geometry of N. To overcome this difficulty, we proceed as follows:

Let $\mathbf{x}(u,\,v)$ be a coordinate patch on N of class ≥ 1. Then, the differential $d\mathbf{x} = \mathbf{x}_u\,du + \mathbf{x}_v\,dv$ is parallel to $T_{\mathbf{x}}(N)$ at $\mathbf{x}(u,\,v)$ and the quantity

$$\begin{aligned} \mathbf{I} = d\mathbf{x} \bullet d\mathbf{x} &= (\mathbf{x}_u\,du + \mathbf{x}_v\,dv) \bullet (\mathbf{x}_u\,du + \mathbf{x}_v\,dv) \\ &= h_{11}\,d\,v^2, \end{aligned}$$

since $\mathbf{x}_u \bullet \mathbf{x}_u = \mathbf{x}_v \bullet \mathbf{x}_v = 0; \quad (\mathbf{x}_v \bullet \mathbf{x}_v) \equiv h_{11} \neq 0.$

As in the Riemannian case we call the function $\mathbf{I} = h_{11}\,d\,v^2$ the first fundamental form of the null surface N, whose only one surviving component is always in the direction of the family, say $\phi(u,\,v) = c$ (constant), of spacelike curves on N. It is known that $\mathbf{I}$ is independent of any coordinates transformation, however the coefficient h_{11} varies from point to point on N. Since any 2-dimensional manifold is Einstein, the Ricci tensor of N will also have one surviving component in the direction of $\phi(u,\,v) = c$. Thus we state

Theorem 3. *Let $(N,\,h)$ be a null 2-surface of a 3-dimensional globally null manifold $(M,\,g)$. Then the geometry of N is essentially the geometry of its 1-dimensional spacelike integral manifold, say N', generated by the family of its spacelike curves $\phi(u,\,v) = c$.*

Using above theorem and a well-known procedure to compute curvature quantities of a family of spacelike curves, one can find curvature properties of a null 2-surface N.

Ruled Null Surfaces. Let $\alpha(u)$ be a null curve in a 4-dimensional Lorentz manifold $(\bar{M},\,\bar{g})$, where $u \in I \subset \mathbf{R}$. Consider a null vector field $\ell(u)$ of $\alpha(u)$. Then, by definition

$$\mathbf{x}(u,\,v) = \alpha(u) + v\,\ell(u)\,; \quad v \in I \subset \mathbf{R},$$

is called a ruled null 2-surface, say N, of $\bar{M}$, which is generated by $\ell(u)$ and α is called the base curve of N. Its u-parameter and v-parameter curves both are families of null curves. Therefore, it is also called a totally null surface with $Rad\,TN = TN$. In particular, N is ruled by null geodesics if α is a geodesic curve. The notion of a null geodesic ruled surface was first introduced by Schild [10] in the form of a geodesic null string of a null hypersurface of a 4-dimensional Minkowski or curved spacetime. By null strings we mean 2-dimensional ruled null surfaces on the null cone (with one dimension suppressed) of $\bar{M}$. Since Schild's paper, there has been considerable work done on geodesic and non-geodesic null strings (see, for example, IIyenko [7] and many others cited therein).

Example 6. Consider a Minkowski spacetime $(\mathbf{R}^4_1, \bar{g})$ with the distance element given by

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

Here we set $x^0 = t$, the time coordinate and x^1, x^2 and x^3 are the three space coordinates. It is well-known, that for this metric, the spacelike hypersurfaces ($t =$ constant) are a family of Cauchy hypersurfaces which cover the whole of $\mathbf{R}^4_1$. Thus, $\mathbf{R}^4_1$ is a product space

$$(\mathbf{R}^4_1 = \mathbf{R} \times B,\ \bar{g} = -dt^2 \oplus G),$$

where (B, G) is a 3-dimensional Euclidean space. It is important to mention that not every spacelike hypersurface of $\mathbf{R}^4_1$ is a Cauchy hypersurface (for details seen Hawking-Ellis [6, page 119]). Choose a spherical coordinate system (t, r, θ, ϕ) with $x^1 = r \sin\theta \sin\phi,\ x^2 = r \sin\theta \cos\phi,\ x^3 = r\cos\theta$. Then, above metric transforms into

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

which is singular at $r = 0$ and $\sin\theta = 0$. We, therefore, choose the ranges $0 < r < \infty$ $0 < \theta < \pi$ and $0 < \phi < 2\pi$ for which it is a regular metric. Actually two such charts are needed to cover the full $\mathbf{R}^4_1$. Now we take two null coordinates u and v, with respect to a pseudo-orthonormal basis, such that $u = t + r$ and $w = t - r$ $(u \geq w)$. Thus above metric transforms as

$$ds^2 = -du dw + \frac{1}{4}(u - w)^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where $-\infty < u, w < \infty$. The absence of du^2 and dw^2 in above transformed metric imply that the hypersurfaces $\{u = \text{constant}\}$ and $\{w = \text{constant }\}$ are null hypersurfaces since $v_{;a}v_{;b}\eta^{ab} = 0 = u_{;a}u_{;b}\eta^{ab}$. Thus, there exists a pair of null hypersurfaces of $\mathbf{R}^4_1$. Relating this example with the discussion on globally null manifolds, we say that a leaf of the 2-dimensional screen distribution S is topologically a 2-sphere S^2, with coordinate system $\{\theta, \phi\}$, and can be seen as the intersection of the two hypersurfaces u = constant and v = constant. In relativity, the null coordinates $u(w)$ are called advanced (retarded) time coordinates and are physically related to incoming (outgoing) spherical waves traveling at the speed of light. Suppose $\alpha(u)$ is the null curve representing incoming spherical waves and $\ell(u)$ any of its null tangent vector field. Then, by definition $\mathbf{x}(u, v) = \alpha(u) + v\,\ell(u)\,; \quad v \in I \subset \mathbf{R}$ is a ruled surface (also called null string where one dimension suppressed) on the null cone of $\mathbf{R}^4_1$. Similarly, one can construct another null string using retarded coordinate w.

Since a ruled surface N has $\dim(Rad\,TN) = 2$ and any globally null manifold M has exactly 1-dimensional null distribution, M can not carry any ruled null surface. However, in the following we show that there is a direct interplay between ruled null surfaces of $\bar{M}$ and 4-dimensional globally null manifolds. Let (M, g) be a class of 4-dimensional globally null manifolds with integrable screen distribution $S(TM)$. Then, it follows from Theorem 2 that $M = L \times M'$ is a global product manifold, where (M', g') is a 3-dimensional integral manifold of $S(TM)$. We first deal with the geometry of 3-dimensional M' which, by Definition 1, is a compact Riemannian manifold. Here we follow Yamabe [11] for the existence of constant curvature metrics on (M', g'). Denote by $\mathcal{M}$ the space of all smooth Riemannian metrics on M' and $\mathcal{M}_1 \subset \mathcal{M}$ of metrics satisfying $vol_{g'} = 1$. Define the total scalar curvature or Einstein-Hilbert action $\mathcal{S} : M' \to \mathbf{R}$ by

$$\mathcal{S}(g') = v^{-1/3} \int_{M'} S^{M'}\, dV_{g'},$$

where $S^{M'}$ is the scalar curvature of M', $dV_{g'}$ is the volume element and v is the volume of M'. The critical points of $\mathcal{S}$ are *Einstein metrics*. Moreover, only in dimension 3 these Einstein metrics are of constant scalar curvature. There is a well-known procedure to obtain Einstein manifolds. Following Yamabe [11], suppose $[g']$ is a conformal class of any metric $g' \in \mathcal{M}_1$. Then there exists a metric $g' \in \mathcal{M}_1$ which achieves its infimum $\mu[g'] \equiv \mathcal{S}|_{[g'] \cap \mathcal{M}_1}$. Such metrics are called *Yamabe metrics*. However, there are restrictions on the existence of Yamabe metrics. Denote by $\sigma(M') = \sup(\mu[g'])_{\mathcal{C}_1}$, where $\mathcal{C}_1$ is the subset of unit volume Yamabe metrics. If $\sigma(M') \leq 0$, it has been proved by Besse [3] that any Yamabe metric $g_0 \in \mathcal{C}_1$ such that $S^{M'}_{g_0} = \sigma(M')$ is Einstein. Otherwise, this problem still remains open. Under these restrictions, it is reasonable to say that there exists a 4-dimensional globally null manifold (M, g) whose 3-dimensional compact Riemannian hypersurface (M', g') is an Einstein manifold with a constant curvature, say k, and g' is a Yamabe metric. Indeed, using the information in Section 2, one can construct null manifold $(M,\, g)$ by gluing the Riemannian metric g' with the degenerate metric g as follows:

$$g = \begin{pmatrix} O_{1\,,\,1} & O_{1\,,\,3} \\ O_{3\,,\,1} & g' \end{pmatrix}$$

Now consider a ruled null surface N of a 4-dimensional spacetime manifolds $\bar{M}$ of constant curvature, such that its Cauchy hypersurface, say $(\Sigma,\, \Sigma_{\bar{g}})$, is conformal to M', that is, its induced metric $\Sigma_{\bar{g}} \in [g']$. With this construction, it follows that $(M = L \times M',\, g)$ is tangent to the null cone $\wedge_{\bar{M}}$ of $\bar{M}$, that is,

$$T_x(M) \cap T_x(\wedge_{\bar{M}}) = L_x - \{0\} \tag{28}$$

for any common point of contact x of the pair $(M,\, \bar{M})$.

Definition 3. Let $(M,\, \bar{M})$ be a pair of 4-dimensional globally null and spacetime manifolds, satisfying (28) and $\alpha : [a,b] \to M$ be a null curve segment. A piecewise smooth variation f, defined by a two parameter function

$$f : [a,\, b] \times (-\epsilon,\, \epsilon) \to \bar{M}$$

is said to be admissible if all the neighboring curves $f_v : [a,\, b] \to \bar{M}$, given by $f_v(u) = f(u,\, v)$ are null for each $v \neq 0$ in $(-\epsilon,\, \epsilon)$ and $f_0(u) = f(u,\, 0) = \alpha(u)$ is the null curve common to M and $\bar{M}$ for all $a \leq u \leq b$.

We call u-parameter null curves $f\,(u,\, v = \text{constant})$, v-parameter null curves $f\,(u = \text{constant},\, v)$ and $\alpha(v)$ the *longitudinal*, the *transversal* and the *base* curves respectively. Thus given a point $x \in \alpha$, we have an admissible net of neighboring longitudinal and transversal curves, all of them belonging to the null cone $\wedge_{\bar{M}}$ with the single base null curve $\alpha(u)$ common to M. Let $\{\mathcal{N}_{\alpha(x)}\}$ denote the set of all nets for all points $x \in \alpha$. Now, consider a maximum sequence of globally null manifolds $\{\, M_i,\, g_i\,\}$ such that each M_i satisfies the equation (28) and $\{g'_i\}$ is a maximum sequence of unit volume Yamabe metrics on each M'_i. In this way we cover the surface of the cone $\wedge_{\bar{M}}$ with a global net of all its longitudinal and transversal null curves. Let $\ell(u)$ be a null tangent vector field of $\alpha(u)$. Then as $\alpha(u)$ moves on the null cone $\wedge_{\bar{M}}$, it will generate a ruled null 2-surface N, which establishes a link between the pair $(M,\, \bar{M})$ with its common u-parameter curve $\alpha(u)$. Observe that, based on Proposition 3, it is possible to consider a maximal sequence of special parameters $\{p^i\}$ such that each longitudinal curve is a null geodesic. With this possibility, all longitudinal curves of the global net are geodesics, which means that f is a 1-parameter family of null geodesics. Consequently, N is a null geodesic string in the sense of Schild [10]. Also, see a recent paper by Low [8] in which he has studied some aspects of the causal structure of 4-dimensional Lorentz manifold with respect to its space of null geodesics.

5. Spacelike 2-surfaces

Let $(\mathbf{R}_1^{n+2}, \bar{g})$ be a Minkowski spacetime with Lorentz metric $\bar{g}$ of signature $(-+\ldots+)$ and local coordinates $(x^0, \ldots, x^n, x^{n+1})$. Consider a hypersurface (M, g) of $\mathbf{R}_1^{n+2}$, defined by

$$x^A = f^A(u^0, \ldots, u^n)\ ; \quad \text{rank}\left[\frac{\partial f^A}{\partial u^\alpha}\right] = n+1\,,$$

where $A \in \{0, \ldots, n+1\}$, $\alpha \in \{0, \ldots, n\}$ and $\{f^A\}$ are smooth functions on a coordinate neighborhood $\mathcal{U} \subset M$. Set

$$D^A = \begin{vmatrix} \frac{\partial f^0}{\partial u^0} & \cdots & \frac{\partial f^{A-1}}{\partial u^0} & \frac{\partial f^{A+1}}{\partial u^0} & \cdots & \frac{\partial f^{n+1}}{\partial u^0} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial f^0}{\partial u^n} & \cdots & \frac{\partial f^{A-1}}{\partial u^n} & \frac{\partial f^{A+1}}{\partial u^n} & \cdots & \frac{\partial f^{n+1}}{\partial u^n} \end{vmatrix}.$$

Proposition 6. *A hypersurface M of $\mathbf{R}_1^{n+2}$ is lightlike, if and only if on each $\mathcal{U}$, functions $\{f^A\}$ satisfy*

$$(D^0)^2 = \sum_{a=1}^{n+1} (D^a)^2\,. \tag{29}$$

In this case, the distribution $Rad\,TM = TM^\perp$ is spanned by

$$\xi = D^0 \frac{\partial}{\partial x^0} + \sum_{a=1}^{n+1} (-1)^{a-1}\, D^a \frac{\partial}{\partial x^a}\,. \tag{30}$$

Proof. The natural frames field on $\mathcal{U}$ is given by

$$\frac{\partial}{\partial u^\alpha} = \frac{\partial f^A}{\partial u^\alpha}\frac{\partial}{\partial x^A}\ ,\ \alpha \in \{0, \ldots, n\}\,.$$

Then it is easy to check that ξ, given by (30), belongs to $\Gamma(TM^{\perp}_{|\mathcal{U}})$. Hence, M is lightlike if and only if $\bar{g}(\xi, \xi) = 0$, which is equivalent with (29).

Note that a vector field V of $T\mathbf{R}^{n+2}_1$ defined by $V = -\,D^0 \dfrac{\partial}{\partial x^0}$, is nowhere tangent to M, since $\bar{g}(V, \xi) = (D^0)^2$. Choose a null vector field ℓ, given by

$$\ell = (D^0)^{-2}\,\{\,V + \frac{1}{2}\,\xi\,\}. \tag{31}$$

It follows that $\bar{g}\,(\,\ell,\,\xi\,) = 1$. Denote by NM the vector bundle spanned by ℓ. Thus we have the following decomposition

$$\begin{aligned} T(R^{n+2}_1) &= TM \oplus NM \\ &= S(TM) \oplus Rad\,TM \oplus NM, \end{aligned} \tag{32}$$

where NM is called a null transversal vector bundle over M. At this point we need the following information on the differential geometry of a hypersurface of any semi-Riemannian manifold $(\bar{M},\, \bar{g})$. Let $\bar{\nabla}$ and ∇ be metric and torsion-free linear connections on $\bar{M}$ and M respectively. Using the first form of (32), we obtain

$$\begin{aligned} \bar{\nabla}_X\, Y &= \nabla_X\, Y + H(X,\, Y)\,, \\ \bar{\nabla}_X\, V &= -A_V\, X + \nabla^t_X\, V\,, \end{aligned} \tag{33}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(tr(TM))$, where $\nabla_X\, Y$ and $A_V\, X$ belong to $\Gamma(TM)$ while $H(X,\, Y)$ and $\nabla^t_X\, V$ belong to $\Gamma(NM)$). Define a local symmetric $\mathcal{F}(\mathcal{U})$-bilinear form B and a local 1-form τ on $\mathcal{U}$ by

$$\begin{aligned} B(X,\, Y) &= \bar{g}(H(X,\, Y),\, \xi), \quad \tau(X) = \bar{g}(\nabla^t_X \ell,\, \xi), \;\; A \\ H(X,\, Y) &= B(X,\, Y)\ell\,, \quad \nabla^t_X\, \ell = \tau(X)\,\ell \end{aligned}$$

for all $X, Y \in \Gamma(TM_{|\mathcal{U}})$. Hence, on $\mathcal{U}$, the equations (33) become

$$\begin{aligned} \bar{\nabla}_X\, Y &= \nabla_X\, Y + B(X,\, Y)\ell\,, \\ \bar{\nabla}_X\, \ell &= -A_\ell\, X + \tau(X)\ell\,, \end{aligned} \tag{34}$$

where we call H and B global and local second fundamental forms of M and (33)-(34) the global and local Gauss and Weingarten equations respectively. Next, if P denotes the projection morphism of TM on $S(TM)$ with respect to the decomposition (2) of the Section 2, then a procedure similar to above provides the following equations

$$\begin{aligned} \nabla_X\, PY &= \overset{*}{\nabla}_X\;\, PY + B'(X,\, PY)\xi\,, \\ \nabla_X\, \xi &= -\,\overset{*}{A}_\xi\;\, X - \tau(X)\,\xi\,, \end{aligned} \tag{35}$$

respectively, which are local Gauss and Weingarten equations for $S(TM)$, where B' denotes its local second fundamental form, $\overset{*}{\nabla}$ is linear connection on $S(TM)$ and $\overset{*}{A}_\xi \; X \in \Gamma(S(TM))$. □

Theorem 4. *The screen distribution $S(TM)$ on any lightlike hypersurface M of a Minkowski spacetime $\mathbf{R}_1^{n+2}$ is integrable.*

Proof. Consider a screen distribution $S(TM)$ on M and take $X, Y \in \Gamma(S(TM))$. Then, taking into account that $\bar{\nabla}$ is the Levi-Civita connection on $\mathbf{R}_1^{n+2}$, using Gauss equation (34) and (31) we obtain

$$\begin{aligned}
\bar{g}([X,Y],\,\ell) &= (D^0)^{-1}\;\bar{g}\left(\bar{\nabla}_X Y - \bar{\nabla}_Y X, \frac{\partial}{\partial x^0}\right) \\
&= -(D^0)^{-1}\left\{\bar{g}\left(X, \bar{\nabla}_Y \frac{\partial}{\partial x^0}\right) - \bar{g}\left(Y, \bar{\nabla}_X \frac{\partial}{\partial x^0}\right)\right\} \\
&= 0.
\end{aligned}$$

Hence, $[X, Y] \in \Gamma(S(TM))$, that is, $S(TM)$ is integrable. □

Note. Since Theorem 4 holds for any lightlike manifold, we conclude that there always exists an integrable screen distribution $S(TM)$ of a globally null manifold $M \subset \mathbf{R}_1^{n+2}$, which we now assume.

Suppose $(M',\, g')$ is a 2-surface of M, where g' is the induced tensor field on M' of g, i.e., $g'(X,\,Y) = g(X,\,Y)$, for all $X, Y \in \Gamma(TM')$. It follows from Section 2 that M' is a spacelike 2-surface of M if

$$TM' \subseteq S(TM) \iff Rad\,TM \cap TM' = \emptyset$$

which we assume. At this point, for physical reasons, we restrict to $n = 2$ so that $\dim(M) = 3$ and $TM' = S(TM)$. This means that M' is a leaf of an integrable screen distribution $S(TM)$ of M.

Theorem 5. *Suppose $(M',\, g')$ is a spacelike 2-surface of a 3-dimensional globally null manifold $M \subset \mathbf{R}_1^4$. Then, the geometry of M essentially reduces to the Riemannian geometry of its 2-surface M'.*

Proof. Since M has an integrable screen distribution $S(TM)$, based on Remark 1 (Section 3) the degenerate metric g can be identified with the Riemannian metric g' of its 2-surface M'. Therefore, M and M' have the same first fundamental forms. For a relation between their second fundamental forms, using the second set of (34) and (31), for $X \in \Gamma(TM)$, we get

$$\tau(X) = \bar{g}(\bar{\nabla}_X \ell, \xi) = \frac{1}{2} (D^0)^{-2} \bar{g}(\bar{\nabla}_X \xi, \xi) = 0 . \tag{36}$$

Hence, the two Weingarten equations of (34) and (35) reduce to

$$\bar{\nabla}_X \ell = -A_\ell X , \quad \nabla_X \xi = - \overset{*}{A}_\xi ,$$

respectively. Also, it follows from (33) and (34) that

$$B(X, \xi) = \bar{g}(\bar{\nabla}_X \xi, \xi) = 0.$$

Using this and (31) we obtain

$$\bar{\nabla}_X \ell = \frac{1}{2} \bar{\nabla}_X \xi = \frac{1}{2} \nabla_X \xi = - \frac{1}{2} \overset{*}{A}_\xi \; X .$$

Now using above reduced Weingarten equations we get

$$A_\ell = \frac{1}{2} \overset{*}{A}_\xi .$$

As a consequence of above relation, we see that the second fundamental forms B and B' of M and $S(TM)$, respectively, are related to

$$B'(X, PY) = \frac{1}{2} B(X, PY) \quad \text{for all} \quad X, Y \in \Gamma(TM) . \tag{37}$$

Denote by H' the global second fundamental form of M'. Then,

$$\bar{\nabla}_X Y = \nabla'_X Y + H'(X, Y) \quad \text{for all} \quad X, Y \in \Gamma(TM') .$$

where ∇' is a torsion-free linear connection on M'. By using Gauss equations (34)-(35) and (37), we obtain

$$\bar{\nabla}_X Y = \nabla'_X Y + B(X, Y) U ,$$

where $U = \{ \frac{1}{2}\xi + \ell \}$ is a spacelike unit vector field in $(TM')^{\perp}$. Hence,

$$H'(X, Y) = B(X, Y) U, \quad X, Y \in \Gamma(TM') \tag{38}$$

is a relation between the second fundamental forms B and H' of M and M', respectively. Since τ vanishes and ξ (and therefore, ℓ) is a global vector field (see Definition 1) on M (which means that U is also globally defined), (38) is a global relation. This result and identical first fundamental forms of M and M' imply that the geometry of M reduces to the geometry of M'. □

Remark 3. The results can be generalized for higher dimensions. If the Minkowski space is replaced by a semi-Euclidean space $\mathbf{R}_q^{n+2}$, the proofs of Theorems 4 and 5 are expected to be quite involved and difficult. On the other hand, if the Minkowski space is replaced by an arbitrary Lorentz manifold, of dimension 4 or higher, then there is no guarantee for the existence of integrable distribution of its lightlike hypersurface. Thus, the results do not hold for an arbitrary ambient Lorentz manifold. However, in particular, we do have several physical spacetimes of general relativity for which Theorems 4 and 5 hold. Here we mention some of them.

Physical Examples. De Sitter spacetime, Schwarzchild and Robertson-Walker spacetimes [6] all have lightlike hypersurface with an integrable 2-dimensional screen distribution whose integral manifold is a 2-sphere S^2. Also, consider the following global wave solution of a spacetime, with global coordinate system $(u,\, v,\, x,\, y\,)$ and metric

$$ds^2 = 2\,du\,dv + [(x^2 - y^2)A(u) - 2xy\,B(u)]\,du^2 + dx^2 + dy^2\,,$$

where $A(u)$ and $B(u)$ are arbitrary functions. This spacetime admits plane wave solutions of the empty Einstein field equations [6, page 178]. It has a family of lightlike hypersurfaces defined by $u =$ constant. Let one such hypersurface be denoted by $(M,\, g)$. It follows from the above metric that M has a 2-dimensional oriented plane, with metric $d\Omega^2 = dx^2 + dy^2$, as a leaf of a integrable screen distribution $S(TM)$ of M.

Finally, we propose the following problem:

Suppose $(M,\, g)$ is an $(n+1)$-dimensional globally null hypersurface of an $(n+2)$-dimensional Lorentz manifold $(\bar{M},\, \bar{g})$. Suppose M has an integrable screen distribution S(TM), whose leaf is an n-dimensional Riemannian hypersurface M' of M. Find a class of Lorentz manifolds $\bar{M}$ such that the null geometry of M, embedded in $\bar{M}$, is same as the Riemannian geometry of M'.

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