International Journal of Pure and Applied Mathematics
Volume 1 No. 4 2002, 415-427

A NEW METHOD FOR FINDING THE EIGENPAIRS OF SYMMETRIC INTERVAL MATRICES

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Abstract: A new method for finding a rigorous bound of all eigenpairs of symmetric interval matrices is proposed. The method is based on solving interval quadratic systems using the interval Newton method. The main advantage of this method is that it works on parallel and is simple to use. The algorithm of the method is described, and numerical examples are given.

AMS Subject Classification: 65F15
Key Words: eigenpairs, interval matrices, interval arithmetic, symmetric

1. Introduction

Most practical problems requiring extensive numerical computation involve quantities determined experimentally, by approximate measurements, very often with some estimate of the accuracy of the measured values. Results computed from such inexact initial data will be also of limited precision. Interval

Received: March 10, 2002

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arithmetic is a convenient and effective means of dealing with such problems. At most fundamental level, interval arithmetic operations works with sets: the result of a single arithmetic operation is the set of all possible results as the operands range over the domain.

The starting point for the application of interval analysis was, in retrospect, the desire in numerical mathematics to be able to execute algorithms on digital computers capturing all the round-off errors automatically and, therefore, to calculate strict error bounds automatically. Interval arithmetic, when practical, allows rigor in scientific computations and can provide tests of correctness of hardware and function libraries for floating-point computations.

In this paper we present a new method and novel method for computing a rigorous bound of all eigenpairs of interval symmetric matrices. The method is based on using, so called, the "Quadratic method" developed by Kharab and Elgindi (see [3]). The general idea to transform the calculation of eigenpairs into the problem of solving a system of nonlinear equations by interval Newton's method, goes back to R. Krawczyk in [9] and [10]. The main advantage of this method is that it works in parallel and is simple to use. As an application of this method, model parameter variations in linear time-invariant system and stability analysis of interval matrices will be presented without the use of 2^n stability tests. Both of these subjects have received considerable interest in the last few years; see, for example, [1] and [11].

2. The Algorithm for the Quadratic Method

Let A be an $n \times n$ matrix. Assume, first, that A is symmetric so that its eigenpairs are real. Let X_i be the eigenvector of A which has 1 in the i-th position and λ_i as its corresponding eigenvalue. Then the algebraic eigenvalue problem

$$AX_i = \lambda_i X_i \tag{1}$$

is a nonlinear system of n equations in n unknowns: λ_i, x_i, $i = 1, 2, ..., i-1, i+1, ..., n$.

The i-th equation of (1) is given by

$$a_{ii} + \sum_{p=1}^{i-1} a_{ip}x_p + \sum_{p=i+1}^{n} a_{ip}x_p = \lambda_i. \tag{2}$$

When $j \neq i$ the j-th equation of (1) is

$$a_{ji} + \sum_{p=1}^{i-1} a_{jp}\, x_p + \sum_{p=i+1}^{n} a_{jp}\, x_p = \lambda_i\, x_j. \tag{3}$$

Using (2) and (3), we obtain for $j = 1, ..., i-1, i+1, ..., n$

$$a_{ji} + \sum_{p=1}^{i-1} a_{jp} x_p + \sum_{p=i+1}^{n} a_{jp} x_p = \left[a_{ii} + \sum_{p=1}^{i-1} a_{ip} x_p + \sum_{p=i+1}^{n} a_{ip} x_p \right] x_i. \tag{4}$$

For $j = 1, ..., i-1$, (4) takes the form of

$$f_j = a_{ij} x_j^2 + \left[a_{ii} - a_{jj} + \sum_{p=1}^{j-1} a_{ip} x_p + \sum_{p=j+1}^{i-1} a_{ip} x_p + \sum_{p=i+1}^{n} a_{ip} x_p \right] x_j$$

$$- \left[a_{ji} + \sum_{p=1}^{j-1} a_{jp} x_p + \sum_{p=j+1}^{i-1} a_{jp} x_p + \sum_{p=i+1}^{n} a_{jp} x_p \right] = 0\,, \tag{5}$$

and for $j = i+1, ..., n$

$$f_j = a_{ij} x_j^2 + \left[a_{ii} - a_{jj} + \sum_{p=1}^{i-1} a_{ip} x_p + \sum_{p=i+1}^{j-1} a_{ip} x_p + \sum_{p=j+1}^{n} a_{ip} x_p \right] x_j$$

$$- \left[a_{ji} + \sum_{p=1}^{i-1} a_{jp} x_p + \sum_{p=i+1}^{j-1} a_{jp} x_p + \sum_{p=j+1}^{n} a_{jp} x_p \right] = 0\,. \tag{6}$$

For a fixed i and $j = 1, 2, ..., i-1, i+1, ..., n$, we define $A_j = a_{ij}$, B_j the coefficient of x_j, and C_j the constant term.

Hence, the finding of the eigenvectors of A is reduced to solving of the quadratic system

$$A_j\, x_j^2 + B_j\, x_j + C_j = 0, \quad j = 1, ..., i-1, i+1, ..., n$$

Once the eigenvectors are obtained, the corresponding eigenvalues are readily obtained from Eq. (2).

The $(n-1) \times (n-1)$ Jacobian matrix $J = (q_{lm})$ of the system is defined by

$$(q_{lm}) = \begin{cases} 2A_l x_l + B_l, & \text{if } l = m\,, \\ a_{im} x_l - a_{lm}, & \text{if } l \neq m\,. \end{cases} \tag{7}$$

3. The Algorithm for Interval Matrices

We first give some notation. An interval matrix is a matrix in which all the elements are defined by a closed interval. That is, an $n \times n$ interval matrix $\mathbf{A}_I = [B, C]$ is the set of all matrices defined by

$$\mathbf{A}_I = \{ A = [a_{ij}] \mid b_{ij} \le a_{ij} \le c_{ij}, \quad i, j = 1, 2, ..., n\} .$$

The algorithm for finding a rigorous bound of the eigenpairs of a real symmetric interval matrix $\mathbf{A}_I$ is based on the quadratic method described above with $\mathbf{a}_{ij} = [b_{ij}, c_{ij}]$. That is, we need to solve the quadratic system (5), (6)

$$\mathbf{F}(\mathbf{X}) = \begin{pmatrix} f_1(\mathbf{x}_1, ..., \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, ..., \mathbf{x}_n) \\ \vdots \\ f_{i-1}(\mathbf{x}_1, ..., \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, ..., \mathbf{x}_n) \\ f_{i+1}(\mathbf{x}_1, ..., \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, ..., \mathbf{x}_n) \\ \vdots \\ f_n(\mathbf{x}_1, ..., \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, ..., \mathbf{x}_n) \end{pmatrix} = 0,$$

$$j = 1, ..., i-1, i+1, ..., n , \quad (8)$$

where $f_j(\mathbf{X}) = \mathbf{A}_j\, \mathbf{x}_j^2 + \mathbf{B}_j\, \mathbf{x}_j + \mathbf{C}_j$, $\mathbf{X} = (\mathbf{x}_1, ..., \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, ..., \mathbf{x}_n)$ is known and given by the box $\mathbf{E} = \{\mathbf{x}_j = [\alpha_j, \beta_j]$ for $1 \le i \le n\}$ and $x_j \in [\alpha_j, \beta_j]$.

$$\mathbf{A}_j = \mathbf{a}_{ij} , \quad (9)$$

$$\mathbf{B}_j = \mathbf{a}_{ii} - \mathbf{a}_{jj} + \sum_{p=1}^{j-1} \mathbf{a}_{ip}\mathbf{x}_p + \sum_{p=j+1}^{i-1} \mathbf{a}_{ip}\mathbf{x}_p + \sum_{p=i+1}^{n} \mathbf{a}_{ip}\mathbf{x}_p , \quad (10)$$

$$\mathbf{C}_j = -\left[\mathbf{a}_{ji} + \sum_{p=1}^{j-1} \mathbf{a}_{jp}\mathbf{x}_p + \sum_{p=j+1}^{i-1} \mathbf{a}_{jp}\mathbf{x}_p + \sum_{p=i+1}^{n} \mathbf{a}_{jp}\mathbf{x}_p \right] ,$$

$$j = 1, .., i-1. \quad (11)$$

A similar form is given for $\mathbf{A}_j, \mathbf{B}_j$ and $\mathbf{C}_j$ when $j = i+1, .., n$, using (6).[1]

A successful method for solving the system (8) is the interval Newton generalized bisection method (see [5] for more details). The iteration method consists on first transforming the nonlinear interval system to the linear interval system

$$\mathbf{F}'(\mathbf{X}^{(k)})(\tilde{\mathbf{X}}^{(k)} - X^{(k)}) \ni -F(X^{(k)}) , \quad (12)$$

[1] Interval quantities will be denoted by boldface letters.

where $\tilde{\mathbf{X}}^{(k)}$ is the unknown solution, $X^{(k)} \in \mathbf{X}^{(k)}$ is defined by the geometric center

$$X^{(k)} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) \text{ with } x_j = \frac{\alpha_j + \beta_j}{2}$$

and $\mathbf{F}'(\mathbf{X}_k)$ is a suitable interval extension of the Jacobian matrix over the box $\mathbf{X}^{(k)}$ (with $\mathbf{X}^{(0)} = \mathbf{E}$) given by

$$(\mathbf{f}'_{lm}) = \begin{cases} [2,2]\mathbf{A}_l \mathbf{x}_l + \mathbf{B}_l\,, & \text{if } l = m\,, \\ \mathbf{a}_{im}\mathbf{x}_l - \mathbf{a}_{lm}\,, & \text{if } l \neq m\,. \end{cases} \tag{13}$$

If we solve (12), using interval arithmetic, the resulting box $\tilde{\mathbf{X}}^{(k)}$, which actually just satisfies

$$\mathbf{F}'(\mathbf{X}^{(k)})(\tilde{\mathbf{X}}^{(k)} - X^{(k)}) \supset -F(X^{(k)}),$$

will contain all solutions to every possible system $A(X - X^{(k)}) = -F(X^{(k)})$ with $A \in \mathbf{F}'(\mathbf{X}^{(k)})$.

In order to get improvement, when solving the interval linear system (12), we multiply both sides of the system by a matrix $Y = \{y_{ml}\}$ known as a preconditioner (see [2]). The result is

$$Y\ \mathbf{F}'(\mathbf{X}^{(k)})(\tilde{\mathbf{X}}^{(k)} - X^{(k)}) \ni -YF(X^{(k)})\,. \tag{14}$$

Following [6], we take our preconditioner to be the inverse of the midpoint matrix of $\mathbf{F}'(\mathbf{X})$, known as the inverse midpoint preconditioner.

We now use the interval Gauss-Seidel method to solve the system

$$Y\ \mathbf{F}'(\mathbf{X}^{(k)})(\tilde{\mathbf{X}}^{(k)} - X^{(k)}) = -YF(X^{(k)}).$$

If we write $Y = [\,y_{ml}\,]$ and $\mathbf{F}' = [\,\mathbf{f}'_{jl}\,]$, the iterates of the interval Gauss-Seidel method to replace $\mathbf{X}$ by a smaller $\tilde{\mathbf{X}}$ are given by

$$\tilde{\mathbf{x}}_m = x_m - \left[\sum_{\substack{l=1\\ l\neq i}}^{n} y_{ml} f'_l + \sum_{\substack{j=1\\ j\neq m}}^{n} \left(\sum_{\substack{l=1\\ l\neq i}}^{n} y_{ml}\mathbf{f}'_{lj}\right)(\mathbf{x}_j - x_j)\right] \Bigg/ \sum_{\substack{l=1\\ l\neq i}}^{n} y_{ml}\mathbf{f}'_{lm}\,, \tag{15}$$

provided that $0 \notin \sum_{\substack{l=1\\ l\neq i}}^{n} y_{ml}\mathbf{f}'_{lm}$. Here $\mathbf{f}'_{lm}$ represents the interval in the l-th row and j-th column of $\mathbf{F}'(\mathbf{X}^{(k)})$ given by (13) and

$$f_l = \mathbf{A}_l\,\mathbf{x}_l^2 + \mathbf{B}_l\,\mathbf{x}_l + \mathbf{C}_l\,.$$

We then iterate $\mathbf{X}^{(k+1)}$ by the formula

$$\mathbf{X}^{(k+1)} = \mathbf{X}^{(k)} \cap \tilde{\mathbf{X}}^{(k)}$$

to obtain tighter bounds on all possible roots. It has been shown (see [5] and [2]) that if $\tilde{\mathbf{X}}^{(k)} \subset \mathbf{X}^{(k)}$ then the system (8) has a unique solution in $\mathbf{X}^{(k)}$. Conversely, if $\tilde{\mathbf{X}}^{(k)} \cap \mathbf{X}^{(k)} = \emptyset$ then there are no solutions of the system in $\mathbf{X}^{(k)}$. For more details on interval arithmetic see [12], and [4] and the references therein. A thorough treatment of the use of interval arithmetic in solving nonlinear systems and the existence and uniqueness of a solutions can be found in [6] and [13].

For $\mathbf{X}^{(k)}$, containing a solution of $f(x) = 0$ and the widths of the components of $\mathbf{X}^{(k)}$ are sufficiently small, the width of $\tilde{\mathbf{X}}^{(k)}$ is roughly proportional to the squares of the widths of the components of $\mathbf{X}^{(k)}$. To insure an accurate starting box $\mathbf{X}^{(0)} = \mathbf{E}$, we introduce a continuation method (see [8]) as follows:

We consider the sequence of interval matrices defined by

$$\mathbf{A}_k = \mathbf{D} + t_k \mathbf{P}, \quad 0 \leq t_k \leq 1\,,$$

where $t_k = kh,\ k = 0, 1, .., M,\ M = 1/h$ and

$$D = \begin{bmatrix} \mathbf{a}_{11} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathbf{a}_{nn} \end{bmatrix}, \quad t_k P = \begin{bmatrix} 0 & t_k\mathbf{a}_{12} & \cdots & t_k\mathbf{a}_{1n} \\ t_k\mathbf{a}_{21} & 0 & \cdots & t_k\mathbf{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_k\mathbf{a}_{n1} & \cdots & t_k\mathbf{a}_{n,n-1} & 0 \end{bmatrix}.$$

Given the matrix $\mathbf{A}$, the partition t_k on the interval $[0, 1]$ is defined by selecting h such that the Gerschgorin circles of the matrix $\mathbf{A}_1 = \mathbf{D} + h\mathbf{P}$, centered at the midpoints of the diagonal elements, are disjoint. That is $h = 1/2^s$ where s is given by

$$s = \left[\left[\frac{\ln(r/d)}{\ln 2} + 2\right]\right] \tag{16}$$

with

$$r = \max_i \{\sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|\}, \quad d = \min_{\substack{i\leq j\leq n \\ j\neq i}} \{|\check{a}_{ii} - \check{a}_{jj}|\}\,, \quad \check{a}_{ii} = (b_{ii} + c_{ii})/2.$$

Thus, given h, the method consists in finding the sequence of eigenpairs of $\mathbf{A}_0, \mathbf{A}_1, ..., \mathbf{A}_M$ obtained by solving for each matrix $\mathbf{A}_k$ the nonlinear interval system (8), using the interval Newton method in conjunction with the interval Gauss-Seidel method. Note, that the eigenvalues of $\mathbf{A}_0 = \mathbf{D}$ are given by the diagonal elements of $\mathbf{A}$, and the eigenpairs of $\mathbf{A}$ are given by $\mathbf{A}_M$.

The steps for finding the starting vectors for the application of interval Newton method are:

At t_0 the eigenvalues are known and given by the diagonal elements of $\mathbf{A}$, and the corresponding eigenvectors are

$$\mathbf{x}^{(0)} = \left(\begin{bmatrix} [1,1] \\ [0,0] \\ \vdots \\ [0,0] \end{bmatrix}, \begin{bmatrix} [0,0] \\ [1,1] \\ \vdots \\ [0,0] \end{bmatrix}, \ldots, \begin{bmatrix} [0,0] \\ [0,0] \\ \vdots \\ [1,1] \end{bmatrix} \right).$$

At t_1 Newton's method is applied to the matrix $\mathbf{A}_1$, using as a starting vectors the above eigenvectors to get $\mathbf{x}^{(1)}$. At t_2, the two point interpolation formula

$$\mathbf{y}^{(2)} = 2\mathbf{x}^{(1)} - \mathbf{x}^{(0)} \tag{17}$$

is used to find the starting vector $\mathbf{y}^{(2)}$. Here $\mathbf{x}^{(0)}$ and $\mathbf{x}^{(1)}$ are the eigenvectors of $\mathbf{A}_0$ and $\mathbf{A}_1$, respectively.

For $t_k > t_2$, the three previous eigenvectors $\mathbf{x}^{(k-1)}$, $\mathbf{x}^{(k-2)}$, $\mathbf{x}^{(k-3)}$ of $\mathbf{A}_{k-1}$, $\mathbf{A}_{k-2}$, $\mathbf{A}_{k-3}$, respectively, are used together with the three point interpolation formula

$$\mathbf{y}^{(k)} = 3\mathbf{x}^{(k-1)} - 3\mathbf{x}^{(k-2)} + \mathbf{x}^{(k-3)}, \tag{18}$$

to find the starting vector $\mathbf{y}^{(k)}$ of Newton's method to compute the eigenpairs of $\mathbf{A}_k$. So, the method consists of finding the eigenpairs of the sequence of interval matrices $\mathbf{A}_0$, $\mathbf{A}_1, \ldots$, $\mathbf{A}_M$ by solving at each step a quadratic system using formulas (17), (18) to get the starting vectors.

4. Numerical Examples

The Fortran 90 Module for interval arithmetic for an interval data type, developed by Kearfott [7], has been used in the following numerical examples. The tolerance for the convergence of Newton's method was set to 10^{-8}.

Example 1. Let us find the eigenpairs of the interval matrix

$$\mathbf{A} = \begin{bmatrix} [-10,-9] & [0,1] & [2,3] \\ [0,1] & [-6.5,-5.5] & [3.5,4] \\ [2,3] & [3.5,4] & [-20,-18] \end{bmatrix}.$$

A_i	Starting vectors	eigenvalues	eigenvectors
A_0		[−10, −9] [−6.5, −5.5] [−20, −18]	[1, 1] [0, 0] [0, 0] [0, 0] [1, 1] [0, 0] [0, 0] [0, 0] [1, 1]
A_1	[1, 1] [0, 0] [0, 0] [0, 0] [1, 1] [0, 0] [0, 0] [0, 0] [1, 1]	[−9.983, −8.955] [−6.435, −5.414] [−20.119, −18.094]	[1, 1] [0.048, 0.048] [−0.064, −0.064] [−0.052, −0.052] [1, 1] [−0.071, −.071] [.060, .060] [.074, .074] [1, 1]
A_2	[1, 1] [0.096, 0.096] [−0.128, −0.128] [−0.105, −0.105] [1, 1] [−0.142, −0.142] [0.121, 0.121] [0.148, 0.148] [1, 1]	[−9.961, −8.843] [−6.235, −5.139] [−20.459, −18.363]	[1, 1] [0.115, 0.115] [−0.123, −0.123] [−0.131, −0.131] [1, 1] [−0.138, −0.138] [0.105, 0.105] [0.152, 0.152] [1, 1]
A_3	[1, 1] [0.346, 0.346] [−0.368, −0.368] [−0.393, −0.393] [1, 1] [−0.413, −0.413] [0.314, 0.314] [0.455, 0.455] [1, 1]	[−9.972, −8.708] [−5.893, −4.663] [−20.984, −18.780]	[1, 1] [0.191, 0.191] [−0.173, −0.173] [−0.221, −0.221] [1, 1] [−0.198, −0.198] [0.130, 0.130] [0.231, 0.231] [1, 1]
A_4	[1, 1] [0.229,] [−0.152, −0.152] [−0.271, −0.271] [1, 1] [−0.181, −0.181] [0.075, 0.075] [0.238, 0.238] [1, 1]	[−10.031, −8.584] [−5.419, −3.999] [−21.654, −19.312]	[1, 1] [0.266, 0.266] [−0.216, −0.216] [−0.308, −0.308] [1, 1] [−0.251, −0.251] [0.139, 0.139] [0.309, 0.309] [1, 1]

Table 1: Starting eigenvectors of A_0, A_1, A_2, A_3 and $A_4 = A$ in Newton's solution for Example 1

The stability of interval matrices has received considerable interest in many areas of engineering. There are various versions of sufficient conditions upon stability properties and one of them is that $\forall A \in \mathbf{A}$, all the eigenvalues of A are in the open left-plane (see [1]). Here $h = 1/2^2$ is chosen such that the Gershgorin circles are disjoint. Note that we choose the center $v_i^{(1)}$ of the circles to be at the midpoint of the diagonal elements of $\mathbf{A}_1 = \{\mathbf{a}_{ij}^{(1)}\}$ and $R_i^{(1)} = \sum_{j=1,j\neq i}^{n} \max\{|b_{ij}^{(1)}|, |c_{ij}^{(1)}|\}$, but to be in the safe side one should choose the radius of the circles $R_i^{(1)} = |b_{ii}^{(1)} - c_{ii}^{(1)}| + \sum_{j=1,j\neq i}^{n} \max\{|b_{ij}^{(1)}|, |c_{ij}^{(1)}|\}$. With this choice the circles are disjoint $\forall\, v_i^{(1)} \in [b_{ii}^{(1)}, c_{ii}^{(1)}]$.

A summary of the calculations of a rigorous bound of all eigenpairs of $\mathbf{A}$ is shown in Table 1.

Example 2. The state of equation of a linear time-invariant system is represented by

$$\dot{x}(t) = Ax(t) + Bu(t)\,,$$

where A is given by

$$\mathbf{A} = \begin{bmatrix} [-0.01, 0.01] & [1.90, 2.2] & [-0.01, 0.01] \\ [1.90, 2.2] & [1.98, 2.15] & [-0.01, 0.015] \\ [-0.01, 0.01] & [-0.01, 0.015] & [0.95, 1.05] \end{bmatrix}.$$

In practical situations, the elements of the matrix $\mathbf{A}$ of the dynamic system are known to be within certain closed interval. Among the many form of performance specifications, used in design, the most important requirement is that the system have to be stable. One method for determining the stability of the system is by computing the eigenvalues of $\mathbf{A}$. That is if the eigenvalues of a matrix A are all negative, then the system is said to be stable. The following bounds of all eigenvalues of $\mathbf{A}$ were obtained for the example under consideration:

$$\begin{aligned} \lambda_1 &= [-1.36524366, -1.16042502]\,, \\ \lambda_2 &= [3.15041970, 3.50525136]\,, \\ \lambda_3 &= [0.94998315, 1.05001448] \end{aligned}$$

Note that the eigenvalues of the midpoint matrix of $\mathbf{A}$ are $\lambda_i = -1.26283434$, 3.32783553, 0.99999881 which correspond exactly to the midpoints of the interval eigenvalues of $\mathbf{A}$.

Example 3. Consider the real symmetric interval matrix

$$\mathbf{A} = \begin{bmatrix} [75,79] & [7,7] & [12,15] & [0,2] & [3,5] \\ [7,7] & [69,72] & [3,3] & [15,17] & [14,15] \\ [12,15] & [3,3] & [57,60] & [12,12] & [7,8] \\ [0,2] & [15,17] & [12,12] & [6,8] & [4,5] \\ [3,5] & [14,15] & [7,8] & [4,5] & [44,48] \end{bmatrix}.$$

Note that a symmetric interval matrix may contain nonsymmetric matrices. The matrix $\mathbf{A}$ was randomly generated in the interval $[0, 99]$. It belongs to a class of real symmetric interval matrices whose right endpoint is not smaller than the absolute value of the left endpoint. For a such class of matrices, it has been shown in [1] that the maximal eigenvalue of the single vertex matrix C whose entries are the right endpoint of its intervals, coincides with the maximal eigenvalue of the interval matrix. To check how rigorous the bounds are of our algorithm, we computed the maximal eigenvalue of C using MATLAB and $\lambda_l(\mathbf{A})$ = the interval that contains the largest eigenvalues of $\mathbf{A}$ using our algorithm. The following results were obtained:

$$\begin{aligned} \lambda_{\max}(C) &= 97.7163 \quad \text{and} \quad \lambda_l(\mathbf{A}) = [90.2778, 97.7423]\,, \\ |\lambda_{l\max}(\mathbf{A}) - \lambda_{\max}(C)| &= |97.7423 - 97.7163| = 0.26\,, \end{aligned}$$

which show that the computed interval $\lambda_l(\mathbf{A}) = [90.2778, 97.7423]$ is a rigorous and tight bound for the largest eigenvalues of $\mathbf{A}$.

5. Conclusion

In this paper we have presented a novel method for computing a rigorous bound of the eigenpairs of interval matrices. The method is based on the quadratic method, used in conjunction with the interval Newton method. By introducing a continuation method, we eliminated the main obstacle of finding the initial guess of the interval Newton's method. One of the main advantage of this method is that it works in parallel. That is, the bound of each eigenpair of the interval matrix can be found independently of the others. Therefore, the algorithm will enjoy a great speed advantage.

6. Acknowledgement

The first author wishes to acknowledge the support of King Fahd University of Petroleum & Minerals, Dhahran, Saudi Arabia.

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