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EXISTENCE RESULTS FOR IMPULSIVE LOWER SEMICONTINUOUS DIFFERENTIAL INCLUSIONS

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Abstract: In this paper, a nonlinear alternative of Leray-Schauder type for single valued maps and Schaefer's fixed point theorem are used to investigate the existence of solutions for first order impulsive lower semicontinuous, nonconvex differential inclusions.

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1. Introduction

In this paper, we shall be concerned with the existence of solutions to the initial value problem for first order impulsive differential inclusions of the type

$$y'(t) \in F(t, y(t)), \quad \text{a.e. } t \in J := [0, T], \; t \neq t_k, \quad k = 1, \dots, m, \tag{1}$$

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$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{2}$$

$$y(0) = y_0, \tag{3}$$

where $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a multivalued map with nonempty compact values, $y_0 \in \mathbb{R}^n$, $\mathcal{P}(\mathbb{R}^n)$ is the family of all subsets of $\mathbb{R}^n$, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$, $I_k \in C(\mathbb{R}^n, \mathbb{R}^n)$ $(k = 1, 2, \ldots, m)$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^-)$ and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively. The theory of impulsive differential equations appears as a natural description of several real processes subject to certain perturbations whose duration is negligible in comparison with the duration of the process. Differential equations involving impulse effects occurs in many applications: physics, population dynamics, ecology, biological systems, biotechnology, industrial robotic, pharmacokinetics, optimal control, etc. The reader can see, for instance, the monographs of Lakshmikantham et al [16], and Samoilenko and Perestyuk [17], where numerous properties of their solutions are studied, and detailed bibliographies are given. Recently, by means of the Leray-Schauder alternative for convex valued multivalued maps and a fixed point theorem due to Martelli for condensing multivalued maps, existence results of solutions for first and second order impulsive differential inclusions were given by Benchohra [1], Benchohra and Boucherif [2], [3]. Notice that different tools, such as the topological transversality theorem of Granas [9], the Leray-Schauder alternative [13] and the lower and upper solutions method [4], [5], have been recently used for various initial and boundary value problems for impulsive differential inclusions. However, in all the above works, the right hand side, $F(t, y)$, was assumed to be convex valued. Here we drop this restriction and consider problems with a nonconvex valued right-hand side. Our approach here is based on a fixed point argument combined with a selection theorem, due to Bressan and Colombo [6], for lower semicontinuous multivalued operators. This paper is organized as follows. In Section 2, we present preliminary facts from multivalued analysis and auxiliary results that will be used in the following section. In Section 3 two existence theorems are proposed, and a result for second order impulsive differential inclusions is stated without proof. The results of this paper extend to the multivalued case some results in [12] and to the nonconvex-valued case some results in [1]-[5] and [9].

2. Preliminaries

In this section we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

$C(J, \mathbb{R}^n)$ is the Banach space of all continuous functions from J into $\mathbb{R}^n$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : 0 \le t \le T\}.$$

$L^1(J, \mathbb{R}^n)$ denotes the Banach space of measurable functions $y : J \to \mathbb{R}^n$ which are Lebesgue integrable normed by

$$\|y\|_{L^1} = \int_0^T |y(t)|dt \quad \text{for all} \quad y \in L^1(J, \mathbb{R}^n).$$

$AC^i(J, \mathbb{R}^n)$ is the space of i-times differentiable functions $y : J \to \mathbb{R}^n$, whose i^{th} derivative, $y^{(i)}$, is absolutely continuous.

Let A be a subset of $J \times \mathbb{R}^n$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $N \times D$, where N is Lebesgue measurable in J and D is Borel measurable in $\mathbb{R}^n$. A subset B of $L^1(J, \mathbb{R}^n)$ is decomposable if, for all $u, v \in B$ and $N \subset J$ measurable, the function $u\chi_N + v\chi_{J-N} \in B$, where χ denotes the characteristic function.

Let E be a Banach space, X a nonempty closed subset of E and $G : X \to \mathcal{P}(E)$ a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{x \in X : G(x) \cap C \neq \emptyset\}$ is open for any open set C in E. G has a fixed point, if there is $x \in X$ such that $x \in G(x)$. For more details on multivalued maps we refer to the books of Deimling [7], Gorniewicz [14], Hu and Papageorgiou [15] and Tolstonogov [19].

Definition 2.1. Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1(J, \mathbb{R}^n))$ be a multivalued operator. We say N has property (BC), if

1) N is lower semi-continuous (l.s.c.);

2) N has nonempty closed and decomposable values.

Let $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be a multivalued map with nonempty compact values.
Assign to F the multivalued operator

$$\mathcal{F} : C(J, \mathbb{R}^n) \to \mathcal{P}(L^1(J, \mathbb{R}^n))$$

by letting

$$\mathcal{F} = \{w \in L^1(J, \mathbb{R}^n) : w(t) \in F(t, y(t)) \text{ for a.e. } t \in J\}.$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated with F.

Definition 2.2. Let $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type), if its associated Niemytzki operator $\mathcal{F}$ is lower semi-continous and has nonempty closed and decomposable values.

Next we state a selection theorem due to Bressan and Colombo.

Theorem 2.1. *[6] Let Y be separable metric space and let $N : Y \to \mathcal{P}(L^1(J, \mathbb{R}^n))$ be a multivalued operator which has property (BC). Then N has a continuous selection, i.e. there exists a continuous function (single-valued) $g : Y \to L^1(J, \mathbb{R}^n)$ such that $g(y) \in F(y)$ for every $y \in Y$.*

3. Main Results

In order to define a solution of (1)–(3), we shall consider the space

$$\Omega = \{y : J \to \mathbb{R}^n : y_k \in C(J_k, \mathbb{R}^n),\ k = 0, \ldots, m,\\ \text{and there exist } y(t_k^-) \text{ and } y(t_k^+),\ k = 1, \ldots, m, \text{ with } y(t_k^-) = y(t_k)\},$$

which is a Banach space with the norm

$$\|y\|_\Omega = \max\{\|y_k\|_{J_k}, k = 0, \ldots, m\},$$

where y_k is the restriction of y to $J_k = [t_k, t_{k+1}],\ k = 0, \ldots, m$.

Definition 3.1. A function $y \in \Omega \cap \cup_{k=0}^m AC((t_k, t_{k+1}), \mathbb{R}^n)$ is said to be a solution of (1)–(3), if y satisfies the differential inclusion $y'(t) \in F(t, y(t))$ a.e. on $J - \{t_1, \ldots, t_m\}$ and the conditions $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, and $y(0) = y_0$.

Let us introduce the following hypotheses which are assumed hereafter:

(H1) $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a nonempty compact valued multivalued map such that
  a) $(t, y) \mapsto F(t, y)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable;
  b) $y \mapsto F(t, y)$ is lower semi-continuous for a.e. $t \in J$.

(H2) For each $r > 0$, there exists a function $h_r \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, y)\| := \sup\{|v| : v \in F(t, y)\} \leq h_r(t) \text{ for a.e. } t \in J \text{ and}\\ y \in \mathbb{R}^n \text{ with } |y| \leq r\}.$$

The following lemma is crucial in the proof of our main theorem:

Lemma 3.1. ([10]) *Let $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be a multivalued map with nonempty, compact values. Assume $(H1)$ and $(H2)$ hold. Then F is of l.s.c. type.*

The first result of this section concerns *a priori* estimates on possible solutions of the problem (1)-(3).

Theorem 3.1. *Suppose, in addition to (H1), (H2) the following hypothesis also holds:*

(H3) There exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and $p \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t,y)\| \le p(t)\psi(|y|) \text{ for a.e. } t \in J \text{ and each } y \in \mathbb{R}^n$$

with

$$\int_{t_{k-1}}^{t_k} p(s)ds < \int_{N_{k-1}}^{\infty} \frac{du}{\psi(u)}, \; k = 1, \dots, m+1;$$

where

$$N_0 = |y_0|, \; N_{k-1} = \sup_{y \in [-M_{k-2}, M_{k-2}]} |I_{k-1}(y)| + M_{k-2},$$

$$M_{k-2} = \Gamma_{k-1}^{-1}\Big(\int_{t_{k-2}}^{t_{k-1}} p(s)ds\Big)$$

for $k = 1, \dots, m+1$ and

$$\Gamma_l(z) = \int_{N_{l-1}}^{z} \frac{du}{\psi(u)}, \quad z \ge N_{l-1}, \; l \in \{1, ..., m+1\}.$$

Then for each $k = 1, \dots, m+1$ there exists a constant M_{k-1} such that

$$\sup\{|y(t)| : t \in [t_k, t_{k-1}]\} \le M_{k-1},$$

for each solution y of the problem (1)-(3).

Proof. Let y be a possible solution to (1)-(3). Then $y|_{[0,t_1]}$ is a solution to

$$y'(t) \in F(t, y(t)) \text{ for a.e. } t \in [0, t_1], \quad y(0) = y_0.$$

Since $|y|' \leq |y'|$, then

$$|y(t)|' \leq p(t)\psi(|y(t)|) \quad \text{for a.e. } t \in [0, t_1].$$

Let $t^* \in [0, t_1]$ such that

$$\sup\{|y(t)| : t \in [0, t_1]\} = |y(t^*)|.$$

From

$$\frac{|y(t)|'}{\psi(|y(t)|)} \leq p(t) \text{ for a.e. } t \in [0, t_1]$$

it follows that

$$\int_0^{t^*} \frac{|y(s)|'}{\psi(|y(s)|)} ds \leq \int_0^{t^*} p(s)ds.$$

Using the change of variable formula (see [11]), we have

$$\Gamma_1(|y(t^*)|) = \int_{|y(0)|}^{|y(t^*)|} \frac{du}{\psi(u)} \leq \int_0^{t^*} p(s)ds \leq \int_0^{t_1} p(s)ds.$$

In view of $(H3)$ we obtain

$$|y(t^*)| \leq \Gamma_1^{-1}\left(\int_0^{t_1} p(s)ds\right).$$

Hence

$$|y(t^*)| = \sup\{|y(t)| : t \in [0, t_1]\} \leq \Gamma_1^{-1}\left(\int_0^{t_1} p(s)ds\right) := M_0.$$

Now $y|_{[t_1,t_2]}$ is a solution to

$$y'(t) \in F(t, y(t)) \qquad \text{for a.e.} \qquad t \in [t_1, t_2],$$

$$\Delta y|_{t=t_1} = I_1(y(t_1)).$$

Note that

$$|y(t_1^+)| \leq \sup_{y \in [-M_0, M_0]} |I_{k-1}(y)| + M_0 := N_1.$$

Next

$$|y(t)|' \leq p(t)\psi(|y(t)|) \quad \text{for a.e. } t \in [t_1, t_2],$$

and let $t^* \in [t_1, t_2]$ such that

$$\sup\{|y(t)| : t \in [t_1, t_2]\} = |y(t^*)|.$$

Then

$$\frac{|y(t)|'}{\psi(|y(t)|)} \le p(t).$$

From this inequality it follows that

$$\int_{t_1}^{t^*} \frac{|y(s)|'}{\psi(|y(s)|)} ds \le \int_{t_1}^{t^*} p(s)ds.$$

Proceeding as above we obtain

$$\Gamma_2(|y(t^*)|) = \int_{N_1}^{|y(t^*)|} \frac{du}{\psi(u)} \le \int_{t_1}^{t^*} p(s)ds \le \int_{t_1}^{t_2} p(s)ds.$$

This yields

$$|y(t^*)| = \sup\{|y(t)| : t \in [t_1, t_2]\} \le \Gamma_2^{-1}\left(\int_{t_1}^{t_2} p(s)ds\right) := M_1.$$

We continue this process and also take into account that $y|_{[t_m,T]}$ is a solution to the problem

$$y'(t) \in F(t, y(t)) \qquad \text{for a.e.} \qquad t \in [t_m, T],$$

$$\Delta y|_{t=t_m} = I_m(y(t_m)).$$

We obtain that there exists a constant M_m such that

$$\sup\{|y(t)| : t \in [t_m, T]\} \le \Gamma_{m+1}^{-1}\left(\int_{t_m}^{T} p(s)ds\right) := M_m.$$

Consequently, for each possible solution y to (1)-(3) we have

$$\|y\|_\Omega \le \max\{|y_0|, M_{k-1} : k = 1, \dots, m+1\} := b. \qquad \square$$

Theorem 3.2. *Suppose that hypotheses (H1)–(H3) are satisfied. Then the impulsive initial value problem (1)–(3) has at least one solution.*

Proof. (H1) and (H2) imply by Lemma 3.1 that F is of lower semi-continuous type. Then from Theorem 2.1 there exists a continuous function $f : \Omega \to L^1(J, \mathbb{R}^n)$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in \Omega$.

We consider the problem

$$y'(t)) = f(y(t)), \quad t \in J, \ t \ne t_k, \quad k = 1, \dots, m, \tag{4}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{5}$$

$$y(0) = y_0. \tag{6}$$

Remark 3.1. If $y \in \Omega$ is a solution of the problem (4)–(6), then y is a solution to the problem (1)-(3).

Transform problem (4) - (6) into a fixed point problem by considering the operator $N : \Omega \to \Omega$ defined by:

$$N(y)(t) := y_0 + \int_0^t f(y(s))ds + \sum_{0<t_k<t} I_k(y(t_k)).$$

We shall show that N is a compact operator.

Step 1: N *is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in Ω. Then

$$\begin{aligned} |N(y_n(t)) - N(y(t))| &\le \int_0^t |f(y_n(s)) - f(y(s))|ds \\ &\quad + \sum_{0<t_k<t} |I_k(y_n(t_k)) - I_k(y(t_k))| \\ &\le \int_0^T |f(y_n(s)) - f(y(s))|ds \\ &\quad + \sum_{0<t_k<t} |I_k(y_n(t_k)) - I_k(y(t_k))|. \end{aligned}$$

Since the functions f and $I_k,\ k = 1, \dots, m$ are continuous, then

$$\|N(y_n) - N(y)\|_\Omega \le \|f(y_n) - f(y)\|_{L^1} + \sum_{k=1}^m |I_k(y_n(t_k)) - I_k(y(t_k))| \to 0 \quad \text{as} \quad n \to \infty.$$

Step 2: N *maps bounded sets into bounded sets in* Ω.

Indeed, it is enough to show that there exists a positive constant ℓ such that for each $y \in B_q = \{y \in \Omega : \|y\|_\Omega \le q\}$ we have $\|N(y)\|_\Omega \le \ell$.

Since I_k $(k = 1, \dots, m)$ are continuous, and from (H2) we have

$$\begin{aligned} |N(y)(t)| &\le |y_0| + |\int_0^t |f(y(s))|ds + \sum_{0<t_k<t} |I_k(y(t_k))| \\ &\le |y_0| + \|h_q\|_{L^1} + \sum_{k=1}^m |I_k(y(t_k))| = \ell. \end{aligned}$$

Step 3: *N maps bounded sets into equicontinuous sets of Ω.*

Let $r_1, r_2 \in J$ and B_q be a bounded set of Ω. Then

$$|N(y)(r_2) - N(y)(r_1)| \leq \int_{r_1}^{r_2} h_q(s)ds + \sum_{0<t_k<r_2-r_1} |I_k(y(t_k))|.$$

As $r_2 \to r_1$ the right-hand side of the above inequality tends to zero. Then $N(B_q)$ is equicontinuous.

Set

$$U = \{y \in \Omega : ||y||_\Omega < b+1\},$$

where b is the constant of Theorem 3.1. As consequence of Steps 1 to 3 together with the Arzela-Ascoli theorem, we can conclude that $N : \overline{U} \to \Omega$ is compact.

From the choice of U there is no $y \in \partial U$ such that $y = \lambda N y$ for in $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray Schauder type [8], we deduce that N has a fixed point $y \in U$ which is a solution of problem (4)-(6), and hence a solution to problem (1)-(3). □

We present now a result for the problem (1)-(3) in the spirit of Schaefer's theorem.

Theorem 3.3. *Suppose, in addition to hypotheses (H1), (H2), the following conditions also hold:*

(H4) There exist constants c_k, such that $|I_k(y)| \leq c_k,\ k = 1, \ldots, m$ for each $y \in \mathbb{R}^n$.

(H5) There exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and $p \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t,y)\| \leq p(t)\psi(|y|)$$

for a.e. $t \in J$ and each $y \in \mathbb{R}^n$ with

$$\int_0^T p(s)ds < \int_c^\infty \frac{du}{\psi(u)}, \quad c = |y_0| + \sum_{k=1}^m c_k.$$

Then the impulsive initial value problem (1)–(3) has at least one solution.

Proof. In order to apply Schaefer's theorem, it remains to show that the set

$$\mathcal{E}(N) := \{y \in \Omega : \lambda y = N(y) \text{ for some } \lambda > 1\}$$

is bounded. Let $y \in \mathcal{E}(N)$. Then $\lambda y = N(y)$ for some $\lambda > 1$. Thus

$$y(t) = \lambda^{-1} y_0 + +\lambda^{-1} \int_0^t f(y(s))ds + \lambda^{-1} \sum_{0<t_k<t} I_k(y(t_k)).$$

This implies by (H5) and (H6), that for each $t \in J$, we have

$$\begin{aligned} |y(t)| &\leq |y_0| + \int_0^t p(s)\psi(|y(s)|)ds + \sum_{k=1}^m |I_k(y(t_k))| \\ &\leq |y_0| + \int_0^t p(s)\psi(|y(s)|)ds + \sum_{k=1}^m c_k. \end{aligned}$$

Let $v(t)$ denote the right hand side of the above inequality. Then

$$v(0) = |y_0| + \sum_{k=1}^m c_k \text{ and } v'(t) = p(t)\psi(|y(t)|) \text{ for a.e. } t \in J.$$

Since ψ is a nondecreasing, we have

$$v'(t) \leq p(t)\psi(v(t)) \text{ for a.e. } t \in J.$$

It follows that

$$\int_0^t \frac{v'(s)}{\psi(v(s))} ds \leq \int_0^t p(s)ds.$$

Then we have

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq \int_0^t p(s)ds \leq \int_0^T p(s)ds < \int_{v(0)}^{\infty} \frac{du}{\psi(u)}.$$

This inequality implies that there exists a constant d depended only on the functions p and ψ such that

$$|y(t)| \leq d \text{ for each } t \in J.$$

Hence

$$\|y\|_\Omega := \sup\{|y(t)| : 0 \leq t \leq T\} \leq d.$$

This shows that $\mathcal{E}(N)$ is bounded. As a consequence of Schaefer's theorem ([18]) we deduce that N has a fixed point y, which is a solution to problem (4)–(6). Then from the Remark 3.1, y is a solution to the problem (1)–(3). □

Remark 3.2. We can easily show that the above reasoning with appropriate hypotheses can be applied to obtain existence results for the second order impulsive differential inclusion,

$$y''(t) \in F(t, y(t)), \quad \text{a.e. } t \in J,\ t \neq t_k, \quad k = 1, \dots, m, \tag{7}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{8}$$

$$\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{9}$$

$$y(0) = y_0, \quad y'(0) = y_1, \tag{10}$$

where F, I_k $(k = 1, \dots, m)$, y_0 are as in the problem (1)–(3) and $\overline{I}_k \in C(\mathbb{R}^n, \mathbb{R}^n)$ $(k = 1, \dots, m)$, $y_1 \in \mathbb{R}^n$. The details are left to the reader.

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