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OSCILLATION OF NONLINEAR NEUTRAL DELAY DIFFERENCE EQUATIONS

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Abstract: We present new sufficient conditions for oscillation of all solutions of the nonlinear neutral delay difference equation

$$\Delta(y_n - p_n y_{n-\tau}) + q_n \prod_{i=1}^{m} |y_{n-\delta_i}|^{\alpha_i} \operatorname{sgn} y_{n-\delta_i} = 0, \quad n = 0, 1, 2, \dots .$$

where $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ are sequences of positive real numbers, τ and δ_i for $i = 1, 2, \dots, m$ are nonnegative integers. Our results improve some of the well known oscillation results in the literature.

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1. Introduction

This work is motivated by several investigations [2]-[10], in which comparison and oscillation theorems are given for the linear and nonlinear neutral delay difference equations. Here we will consider the nonlinear neutral delay equation

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$$\Delta(y_n - p_n y_{n-\tau}) + q_n \prod_{i=1}^{m} |y_{n-\delta_i}|^{\alpha_i} \operatorname{sgn} y_{n-\delta_i} = 0, \quad n = 0, 1, 2, \ldots. \tag{1.1}$$

where τ is a positive integer and $\delta_1, \delta_2, \ldots, \delta_m$ are nonnegative integers, $p_n \geq 0$ and $q_n \geq 0$ $(n = 0, 1, \ldots)$ and q_n has a positive subsequence, each α_i is a positive number for $i = 1, \ldots, m$ such that $\alpha_1 + \alpha_2 + \ldots \alpha_m = 1$, and Δ denotes the forward difference operator $\Delta y_n = y_{n+1} - y_n$.

Let μ=max$\{\tau,\ \delta_1, \ldots, \delta_m\}$. By a solution of (1.1) we mean a sequence $\{y_n\}$ which is defined for $n \geq -\mu$ and which satisfies equation (1.1) for $n = 0, 1, 2 \ldots.$ Clearly, if

$$y_n = A_n \quad \text{for} \quad n = -\mu, \ldots, -1, 0 \tag{1.2}$$

are given, then equation (1.1) has a unique solution satisfying the initial conditions (1.2). A nontrivial solution $\{y_n\}$ of (1.1) is said to be oscillatory if for every $N > 0$ there exists an $n \geq N$ such that $y_n y_{n+1} \leq 0$. Otherwise it is nonoscillatory. It is clear that the behavior of the solution of (1.1) depends on the integers τ, $\delta_1, \ldots, \delta_m$ and the sequences $\{p_n\}$ as well as $\{q_n\}$.

When $m = 1$, equation (1.1) reduces to the linear neutral delay difference equation

$$\Delta(y_n - p_n y_{n-\tau}) + q_n y_{n-\delta} = 0, \qquad n = 0, 1, 2, \ldots. \tag{1.3}$$

that has already been studied in [2], [4]-[7], [9], [10], and some oscillation criteria are given under the assumptions that: assume that there exists a positive integer $N > 0$ such that

$$p_{N+i\tau} \leq 1 \quad \text{for} \quad i = 0, 1, 2, \ldots. \tag{1.4}$$

and either

$$\sum_{n=0}^{\infty} q_n = \infty, \tag{1.5}$$

or

$$\sum_{n=0}^{\infty} n q_n \sum_{j=n}^{\infty} q_j = \infty. \tag{1.6}$$

In these papers the hypothesis (1.4) and (1.5) or (1.4) and (1.6) play an essential roles to establish the oscillation criteria for equation (1.3).

Recently Zhang and Cheng [3] and Li [8] considered the equation (1.1) and gave some oscillation criteria. In fact, [3] observed that for equation (1.1) it is sufficient to have an integer N such that (1.4) and (1.6) hold, and then every solution of (1.1) oscillates. Li [8] relaxed assumptions (1.5) and (1.6) by proving, following Zhang and Cheng [3], the equivalence of oscillation for equation (1.1) and the absence of eventually positive solution of the corresponding neutral delay difference inequality

$$\Delta(y_n - p_n y_{n-\tau}) + q_n \prod_{i=1}^{m} |y_{n-\delta_i}|^{\alpha_i} \operatorname{sgn} y_{n-\delta_i} \leq 0, \ n = 0, 1, 2, \dots , \tag{1.7}$$

and proved a comparison theorem for oscillation of (1.1). He then succeed in getting oscillation criteria for equation (1.1), and proved that: if (1.4) holds,

$$\prod_{i=1}^{m} p_{n-\delta_i}^{\alpha_i} q_n \geq q_{n-\tau} \tag{1.8}$$

and

$$\sum_{n=0}^{\infty} q_n \prod_{j=0}^{n-1} (1 + \frac{j q_j}{\tau}) = \infty \,, \tag{1.9}$$

then every solution of (1.1) oscillates. It is interesting to ask, if the conditions (1.4), (1.6), (1.8) and (1.9) can be replaced by others. Also we note that, the oscillation criteria placed separately on the sequence $\{q_n\}$ mimicking the conditions on q_n and placing conditions on p_n which allow extension of arguments used in the case when $p_n = 1$ by used the condition (1.8) (see [3, Theorem 2], [8, Theorem 1]).

Our aim, in this paper, is to give some new oscillation criteria for equation (1.1). Our results relax the conditions (1.4) and (1.8) even further to allow a class of functions satisfying $p_n > 1$ for $n = 0, 1 \dots$, and show that the combined growth of p_n and q_n can give oscillation even when (1.5) and (1.6) fail. In addition our results improve the results of [3] and [8].

2. Main Results

We begin this section by improving the basic lemma (Lemma 1) in [3] and [8].

Lemma 2.1. *Suppose $q_n \geq 0$ and is not identically zero for all large n. Suppose, further that there exist an integer $N > 0$ and a positive constant M_1 such that*

$$\lim_{j\to\infty} [1 + p_{N+j\tau} + p_{N+j\tau} p_{N+(j-1)\tau} + \dots + p_{N+j\tau} p_{N+(j-1)\tau} p_{N+(j-2)\tau} \cdots p_{N+\tau}] = \infty, \tag{2.1}$$

$$\prod_{i=0}^{j} p_{N+i\tau} \le M_1, \qquad j = 0, 1, 2, \dots \,. \tag{2.2}$$

Let

$$x_n = y_n - p_n y_{n-\tau}\,, \tag{2.3}$$

where $\{y_n\}$ *is an eventually, positive solution of (1.1). Then we have, eventually,* $\Delta x_n \le 0$ *and*

$$x_n > 0\,. \tag{2.4}$$

Proof. Let n_1 be a positive integer such that $y_{n-\mu} > 0$ for all $n \ge n_1$. Then by (1.1) and (2.3), we have, eventually, $\Delta x_n = -q_n \prod_{i=1}^{m} y_{n-\delta_i}^{\alpha_i} \le 0$ for all $n \ge n_1$, since $\{q_n\}$ has a positive subsequence, the nondecreasing sequence $\{x_n\}$ is either positive or negative for all $n \ge n_1$. Hence, if (2.4) does not hold, then we would have, eventually $x_n < 0$. Therefore, there exist an integer $n_2 > n_1$ and a constant $\alpha < 0$ such that $x_n < \alpha < 0$ for all $n \ge n_2$. Then, from (2.3) we have

$$y_n \le \alpha + p_n y_{n-\tau}, \qquad n \ge n_2.$$

By choosing k^* so large that $N + k^*\tau \ge n_2$, we see that

$$y_{N+k^*\tau+j\tau} \le \alpha + p_{N+k^*\tau+j\tau} y_{N+k^*\tau+(j-1)\tau}.$$

Applying inductively this gives

$$\begin{aligned} & y_{N+k^*\tau+j\tau} \\ &\quad \le \alpha(1 + p_{N+k^*\tau+j\tau} + p_{N+k^*\tau+j\tau} p_{N+k^*\tau+(j-1)\tau} p_{N+k^*\tau+(j-2)\tau} + \dots \\ &\qquad + p_{N+k^*\tau+j\tau} p_{N+k^*\tau+(j-1)\tau} \cdots p_{N+k^*\tau}) \\ &\qquad + (p_{N+k^*\tau+j\tau} p_{N+k^*\tau+(j-1)\tau} \cdots p_{N+k^*\tau}) y_{N+(k^*-1)\tau}. \end{aligned} \tag{2.5}$$

By conditions (2.1), (2.2) and using (2.5) one can see that

$$y_{N+k^*\tau+j\tau} \to -\infty \quad \text{as} \quad j \to \infty$$

for some constant k^*, and this contradicts the assumption that $y_n > 0$. Then $x_n > 0$, and this completes the present proof. □

Now, we will establish some new oscillation criteria for equation (1.1)

Theorem 2.1. *Assume that (2.1) and (2.2) hold and let* $\delta = \min\{\delta_1, \dots, \delta_m\}$. *If* $\sum_{i=n+1}^{n+\delta} q_i > 0$ *for* $n \geq n_0$ *for some* $n_0 > 0$ *and*

$$\sum_{n=n_0}^{\infty} q_n \left[\left(\sum_{i=n+1}^{n+\delta} q_i \right)^{\frac{1}{1+\delta}} (\delta + 1) - \delta \right] = \infty. \tag{2.6}$$

Then every solution of equation (1.1) oscillates.

Proof. Suppose to the contrary that equation (1.1) has a nonoscillatory solution, without loss of generality we assume that there exists positive integer $n_1 \geq 0$ such that $y_{n-\mu} > 0$ for all $n \geq n_1$. Then by (1.1) and (2.3), we have, eventually, $\Delta x_n = -q_n \prod_{i=1}^{m} y_{n-\delta_i}^{\alpha_i}$ for all $n \geq n_1$. Then, by means of Lemma 2.1, the sequence $\{x_n\}$ defined by (2.3) will satisfy $\Delta x_n \leq 0$ and $x_n > 0$ for all large n, and then we have eventually that $y_n > x_n$. From (1.1) and (2.3) by using the Holder's inequality [1, p. 20], we have

$$\begin{aligned} \Delta x_n &= -q_n \prod_{i=1}^{m} |y_{n-\delta_i}|^{\alpha_i} = -q_n \prod_{i=1}^{m} [x_{n-\delta_i} + p_{n-\delta_i} y_{n-\tau-\delta_i}]^{\alpha_i} \\ &\leq -q_n \prod_{i=1}^{m} x_{n-\delta_i}^{\alpha_i} - q_n \prod_{i=1}^{m} p_{n-\delta_i}^{\alpha_i} \prod_{i=1}^{m} y_{n-\tau-\delta_i}^{\alpha_i} \leq -q_n \prod_{i=1}^{m} x_{n-\delta_i}^{\alpha_i}. \end{aligned}$$

Since $\delta = \min\{\delta_1, \dots \delta_m\}$, $\alpha_1 + \alpha_2 + \dots \alpha_m = 1$, and x_n is nonincreasing we have

$$\Delta x_n + q_n \, x_{n-\delta} \leq 0. \tag{2.7}$$

Let

$$\lambda_n = -\frac{\Delta x_n}{x_n}. \tag{2.8}$$

Since $\{x_n\}$ is nonincreasing sequence, then we have $0 \leq \lambda_n < 1$ for large n. From (2.8) we have $\frac{x_{n+1}}{x_n} = 1 - \lambda_n$ and $\frac{x_n}{x_{n-\delta}} = \prod_{i=n-\delta}^{n-1} (1 - \lambda_i)^{-1}$. Then by (2.7) we have

$$\lambda_n \geq q_n \prod_{i=n-\delta}^{n-1} (1 - \lambda_i)^{-1} \geq q_n \left(1 - \frac{1}{\delta} \sum_{i=n-\delta}^{n-1} \lambda_i \right)^{-\delta}. \tag{2.9}$$

Let $b_n = \sum_{i=n+1}^{n+\delta} q_i$. Then (2.9) can be rewritten as

$$\lambda_n \geq q_n \left(1 - \frac{1}{\delta b_n} b_n \sum_{i=n-\delta}^{n-1} \lambda_i\right)^{-\delta}. \tag{2.10}$$

From (2.10), by using the inequality

$$\left[1 - \frac{1}{\delta} r x\right]^{-\delta} \geq x + \frac{\left[r^{\frac{1}{\delta+1}}(\delta+1) - \delta\right]}{r}, \quad \text{for} \quad r > 0 \quad \text{and} \quad x < \frac{\delta}{r}, \tag{2.11}$$

we have

$$\lambda_n \geq q_n \left[\frac{1}{b_n} \sum_{i=n-\delta}^{n-1} \lambda_i + \frac{1}{b_n}\left((b_n)^{\frac{1}{\delta+1}}(\delta+1) - \delta\right)\right].$$

It follows that

$$\lambda_n b_n - q_n \sum_{i=n-\delta}^{n-1} \lambda_i \geq q_n \left(\left(\sum_{i=n+1}^{n+\delta} q_i\right)^{\frac{1}{\delta+1}} (\delta+1) - \delta\right).$$

Then, for $N > n_1$

$$\sum_{n=n_1}^{N} \lambda_n b_n - \sum_{n=n_1}^{N} q_n \sum_{i=n-\delta}^{n-1} \lambda_i \geq \sum_{n=n_1}^{N} q_n \left(\left(\sum_{i=n+1}^{n+\delta} q_i\right)^{\frac{1}{\delta+1}} (\delta+1) - \delta\right). \tag{2.12}$$

Interchanging the bound of summation, we find

$$\sum_{n=n_1}^{N} q_n \sum_{i=n-\delta}^{n-1} \lambda_i \geq \sum_{n=n_1}^{N-\delta-1} \sum_{n=i+1}^{i+\delta} \lambda_i q_n = \sum_{n=n_1}^{N-\delta-1} \lambda_i \sum_{n=i+1}^{i+\delta} q_n = \sum_{n=n_1}^{N-\delta-1} \lambda_n \sum_{i=n+1}^{n+\delta} q_i. \tag{2.13}$$

Combining (2.12) and (2.13), it follows that

$$\sum_{n=N-\delta}^{N} \lambda_n \sum_{i=n+1}^{n+\delta} q_i \geq \sum_{n=n_1}^{N} q_n \left(\left(\sum_{i=n+1}^{n+\delta} q_i \right)^{\frac{1}{\delta+1}} (\delta+1) - \delta \right) . \tag{2.14}$$

But from (2.7), since $\{x_n\}$ is positive and decreasing, one can prove easily that

$$\sum_{i=n+1}^{n+\delta} q_i \leq 1, \tag{2.15}$$

eventually. Then, form (2.14) and (2.15) we have

$$\sum_{n=N-\delta}^{N} \lambda_n \geq \sum_{n=n_1}^{N} q_n \left(\left(\sum_{i=n+1}^{n+\delta} q_i \right)^{\frac{1}{\delta+1}} (\delta+1) - \delta \right) \to \infty , \quad \text{as N} \to \infty \tag{2.16}$$

by (2.6). But from the definition of λ_n we have

$$\lambda_n = \left(1 - \frac{x_{n+1}}{x_n} \right) .$$

Hence,

$$\sum_{n=N-\delta}^{N} \lambda_n = \sum_{n=N-\delta}^{N} \left(1 - \frac{x_{n+1}}{x_n} \right) < \delta + 1 ,$$

and this contradicts (2.16). Then every solution of (1.1) oscillates. The proof is complete. □

Remark 2.2. Theorem 2.1 improves Theorem 2 in [3] and Theorem 1 in [8].

Theorem 2.2. *Assume that (2.1) and (2.2) hold, and let* $\tau \geq \delta = \min\{\delta_1, \dots \delta_m\}$. *If* $\sum_{i=n+1}^{n+\delta} Q_i > 0$ for $n \geq n_0$, *for some* $n_0 > 0$ *and*

$$\sum_{n=n_0}^{\infty} Q_n \left[\left(\sum_{i=n+1}^{n+\delta} Q_i \right)^{\frac{1}{1+\delta}} (\delta+1) - \delta \right] = \infty , \tag{2.18}$$

where $Q_n = q_n(1+q_n \prod_{i=1}^{m} p_{n-\delta_i}^{\alpha_i})$. *Then every solution of equation (1.1) oscillates.*

Proof. Proceeding as in Theorem 2.1 we assume that equation (1.1) has a nonoscillatory solution $n \geq n_1$. Let x_n defined by (2.3), then we have eventually that $y_n > x_n$. From (1.1), (2.3) and using the Holder's inequality we have

$$\Delta x_n = -q_n \prod_{i=1}^{m} |y_{n-\delta_i}|^{\alpha_i} = -q_n \prod_{i=1}^{m} [x_{n-\delta_i} + p_{n-\delta_i} y_{n-\tau-\delta_i}]^{\alpha_i}$$
$$\leq -q_n \prod_{i=1}^{m} x_{n-\delta_i}^{\alpha_i} + q_n \prod_{i=1}^{m} p_{n-\delta_i}^{\alpha_i} \frac{\Delta x_{n-\tau}}{q_{n-\tau}}. \quad (2.19)$$

Since $\delta = \min\{\delta_1, \ldots \delta_m\}$, $\alpha_1 + \alpha_2 + \ldots \alpha_m = 1$ and x_n is nonincreasing we have

$$\Delta x_n \leq -q_n\, x_{n-\delta} + q_n \prod_{i=1}^{m} p_{n-\delta_i}^{\alpha_i} \frac{\Delta x_{n-\tau}}{q_{n-\tau}}\,. \quad (2.20)$$

We define again

$$\lambda_n = -\frac{\Delta x_n}{x_n}. \quad (2.8)$$

Then $\lambda_n > 0$, and from (2.20) we have

$$\lambda_n \geq q_n \frac{x_{n-\delta}}{x_n} + q_n \prod_{i=1}^{m} p_{n-\delta_i}^{\alpha_i} \frac{1}{q_{n-\tau}} \frac{x_{n-\tau}}{x_n} \lambda_{n-\tau}. \quad (2.21)$$

Since $\{x_n\}$ is nonincreasing sequence, then we have $\frac{x_{n-\delta}}{x_n} \geq 1$ and from (2.21) we find that $\lambda_n \geq q_n$, and for large $n\lambda_{n-\tau} \geq q_{n-\tau}$. Using this and (2.21) we obtain

$$\lambda_n \geq q_n \frac{x_{n-\delta}}{x_n} + q_n \prod_{i=1}^{m} p_{n-\delta_i}^{\alpha_i} \frac{x_{n-\tau}}{x_n}. \quad (2.22)$$

Since $\Delta x_n \leq 0$ and $\tau \geq \delta$, then we have

$$\lambda_n \geq Q_n \frac{x_{n-\delta}}{x_n}. \quad (2.23)$$

Then as in Theorem 2.1 we have

$$\lambda_n \geq Q_n \prod_{i=n-\delta}^{n-1} (1-\lambda_i)^{-1} \geq Q_n \left(1 - \frac{1}{\delta} \sum_{i=n-\delta}^{n-1} \lambda_i\right)^{-\delta}. \quad (2.24)$$

The remainder of the proof is similar to that of Theorem 2.1, and hence is omitted. □

Theorem 2.3. *Assume that (2.1) and (2.2) hold, and let* $\tau \geq \delta = \min\{\delta_1, \dots \delta_m\}$. *If* $\sum\limits_{i=n+1}^{n+\delta} A_i > 0$ for $n \geq n_0$, *for some* $n_0 > 0$, *and*

$$\sum_{n=n_0}^{\infty} A_n \left[\left(\sum_{i=n+1}^{n+\delta} A_i \right)^{\frac{1}{1+\delta}} (\delta+1) - \delta \right] = \infty\,, \tag{2.25}$$

where $A_n = q_n(1 + \prod\limits_{i=1}^{m} p_{n-\delta_i}^{\alpha_i} q_n \prod\limits_{i=1}^{m} p_{n-\tau-\delta_i}^{\alpha_i})$. *Then every solution of equation (1.1) oscillates.*

Proof. Proceeding as in the proof of Theorem 2.2 we obtain (2.22). From (2.22) we have $\lambda_n \geq q_n \prod\limits_{i=1}^{m} p_{n-\delta_i}^{\alpha_i}$, then for large n we have

$$\lambda_{n-\tau} \geq q_{n-\tau} \prod_{i=1}^{m} p_{n-\tau-\delta_i}^{\alpha_i}. \tag{2.26}$$

Substituting from (2.26) in (2.21) we have

$$\lambda_n \geq q_n \frac{x_{n-\delta}}{x_n} + q_n \prod_{i=1}^{m} p_{n-\delta_i}^{\alpha_i} \prod_{i=1}^{m} p_{n-\tau-\delta_i}^{\alpha_i} \frac{x_{n-\tau}}{x_n}. \tag{2.27}$$

Since $\Delta x_n \leq 0$ and $\tau \geq \delta$, then we have

$$\lambda_n \geq A_n \frac{x_{n-\delta}}{x_n}. \tag{2.28}$$

Then as in Theorem 2.1 we have

$$\lambda_n \geq A_n \prod_{i=n-\delta}^{n-1} (1-\lambda_i)^{-1} \geq A_n \left(1 - \frac{1}{\delta} \sum_{i=n-\delta}^{n-1} \lambda_i \right)^{-\delta}.$$

The remainder of the proof is similar to that of Theorem 2.1, and hence is omitted. □

Theorem 2.4. *Assume that (2.1) and (2.2) hold, and let* $\tau \geq \delta = \min\{\delta_1, \dots \delta_m\}$. *If* $\sum_{i=n+1}^{n+\delta} B_i > 0$ for $n \geq n_0$ *for some* $n_0 > 0$ *and*

$$\sum_{n=n_0}^{\infty} B_n \left[\left(\sum_{i=n+1}^{n+\delta} B_i \right)^{\frac{1}{1+\delta}} (\delta+1) - \delta \right] = \infty \,, \tag{2.29}$$

where $B_n = q_n(1 + \prod_{i=1}^{m} p^{\alpha_i}_{n-\tau-\delta_i} q_n \prod_{i=1}^{m} p^{\alpha_i}_{n-2\tau-\delta_i})$. *Then every solution of equation (1.1) oscillates.*

Proof. Proceeding as in the proof of Theorem 2.3 we obtain (2.27). From (2.27) we have

$$\lambda_n \geq q_n \prod_{i=1}^{m} p^{\alpha_i}_{n-\delta_i} \prod_{i=1}^{m} p^{\alpha_i}_{n-\tau-\delta_i}, \tag{2.30}$$

hence for n large enough, we have

$$\lambda_{n-\tau} \geq q_{n-\tau} \prod_{i=1}^{m} p^{\alpha_i}_{n-\tau-\delta_i} \prod_{i=1}^{m} p^{\alpha_i}_{n-2\tau-\delta_i}. \tag{2.31}$$

Substituting from (2.31) in (2.21) and using the fact that $\Delta x_n \leq 0$ and $\tau \geq \delta$, we have

$$\lambda_n \geq B_n \frac{x_{n-\delta}}{x_n}. \tag{2.32}$$

Then, as in Theorem 2.1, we have

$$\lambda_n \geq B_n \prod_{i=n-\delta}^{n-1} (1-\lambda_i)^{-1} \geq B_n \left(1 - \frac{1}{\delta} \sum_{i=n-\delta}^{n-1} \lambda_i \right)^{-\delta}.$$

The remainder of the proof is similar to that of Theorem 2.1, and hence is omitted. □

Note that one can proceed the above approach to give new sufficient conditions for oscillations depending on $\{p_{n-\delta-i\tau}\}$. The details are left to the reader.

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