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CHAOTIC PROPERTIES OF NEURAL NETWORKS: CHAOTIC RESPONSE OF RMS SERIES AND EFFECTS ON THE ABSOLUTE ERROR DISTRIBUTION

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Abstract: This paper investigates chaotic properties of neural models of the chaotic attractors of the logistic and Henon maps, to determine efficient training strategies. Specifically, series of the RMS error are submitted to Lyapunov exponent investigation in a two-phase process. At first the networks are trained for different initial values of the weight vector, and then, the best and the worst of the resulting networks are submitted to additional training. In both cases, the Dominant Lyapunov Exponent is calculated for the RMS series of the training set. The positive value of the exponent implies the chaotic response of the RMS. To make final conclusions for the chaotic response of the error the stability, the performance of the calculated exponents is investigated. Although the RMS attractor is narrow, it seems to have significant effects in the resulting level of the absolute error, which strongly affects the actual response of the neural model.

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1. Introduction

Chaotic behavior emanates from nonlinearity and complexity in a dynamic system. It is investigated by examining whether the system is sensitive to initial conditions or not. The property of sensitivity is measurable through Lyapunov exponents, which represent the average exponential rate of expansion (or contraction) of the axis of the ellipsoid produced when the measured system evolves acting on a unit sphere of initial conditions. There may exist more than one positive Lyapunov exponents representing chaotic response in an N-dimensional space, but the dominant one governs the performance of the system. The dominant Lyapunov exponent corresponds to the most expansive axis of the ellipsoid. Various methods have been applied for its calculation. Schuster [24], Yorke [30] and Hilborn [9] have described calculating methods for systems with known evolution equations. The estimation of Lyapunov exponents from data series is more complicated. J. Wright [29], Allan Wolf, J.B.Swift, H.L. Swinney and J.A. Vastano [28], J.P Eckman and S. Oliffson Kamphorst D. Ruelle and S. Ciliberto [6] have proposed algorithms for the estimation of the exponents.

2. Previous Work and Overview of Current Work

2.1. Previous Work

There are two basic research branches considered, in the common field of chaos and neural networks. The first is related to neural modeling of chaotic attractors and the second pertains in neural networks chaotic properties.

Lapedes A. and Farber R. [16], S.F. Masri. A.G. Chassiakos and T.K. Caughey [23], George J. Mpitsos and Robert M. Burton. Jr [17], Ramazan Gencay [19], I-Cheng Yeh [10] and Kofidis, Roumeliotis and Adamopoulos [22], [13] have published works related with the first branch.

Steve Renals and Richard Rohwer [21], John F. Kolen and Jordan B. Pollack [15], Han L.J Van der Maas, Paul F.M.J. Verschure and Peter C.M. Molenaar [26], Thomas B. Kepler, Sumeet Datt, Robert B. Meyer and L.F. Abbot [11], Paul F.M.J Verschure [27], Francois Chapeau-Blondeau and Gilbert Chauvet [4], K. Aixara, T. Takabe and M. Toyoda [1], E.K. Blum and Xin Wang [3], G. Randons, H.G. Schuster and D. Werner [20], and N. Kofidis, M. Roumeliotis and M. Adamopoulos [12], [14], have worked on neural networks' dynamics.

Name of Input Module	Number of Cells	Cell Input	Cell Output
Main Input	1	X_n	X_n
$FLN - S_k$	4	X_n	$\sin(k\pi X_n)$, $k = 1, 2, 3, 4$
$FLN - E_k$	4	X_n	$X_n \sin(k\pi X_n)$, $k = 1, 2, 3, 4$

Table 1: Input Modules

2.2. Overview

This paper discusses an investigation of the dynamic properties of the RMS error of neural models of the logistic [5], [7] and Henon [8] chaotic attractors. For this reason, well performing neural structures of the attractors are trained. Two cases are considered: the case of basic and the case of additional training. In the first case the networks are trained starting with different initial values of the weight vector. For each training effort the RMS of the data set is calculated and added to the RMS data series, which is submitted to Lyapunov exponent investigation. In the second case additional training is applied to the networks to see if the performance of the networks increases or decreases. Again the RMS series are created and submitted to the same chaotic analysis. For both cases the effects of the RMS dynamic properties to the actual stepwise response are considered, while the distribution of the absolute error is used in measuring the actual response of the network The absolute error, E_{abs}, is calculated as the absolute value of the difference between the actual output value produced by the network and the desired one, as given by the following equation:

$$E_{abs} = |X_{out} - X_{des}|\,, \tag{1}$$

where X_{out} is the actual and X_{des} is the desired output of the network. From the distribution of the absolute error over pre-chosen error levels, a statistical overview of the networks' failure emanates. More specifically, for a given error level questions like "what is the percentage of output data with absolute error greater than" can be answered. For example, in Figure 4.2, a failure of 20% for the 0.0005 error level (depicted as $\%ABS > 0.0005$), means that the network responds with error greater than 0.0005 for 20% of the testing data.

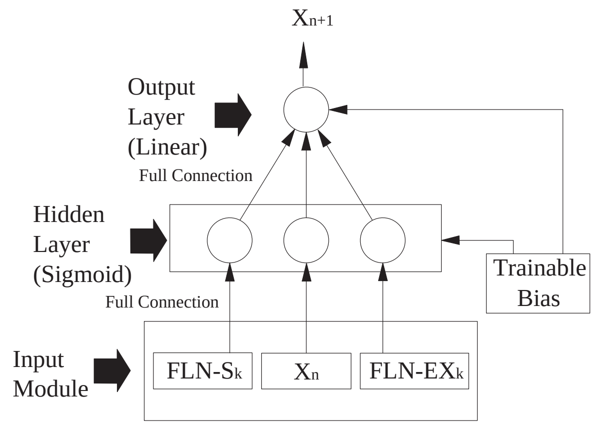

Figure 3.1: FNL-Back propagation network that models the logistic map

3. The Neural Models

3.1. The Logistic Map Neural Model

The neural model of the logistic map is presented by Figure 3.1. A detailed description of the network can be found in authors' previous works [13], [22]. It is a simple network with one tanh-hidden layer and functional link (FLN) inputs. The structure of the input modules is described in Table 2.1.

3.2. The Henon Map Neural Model

The neural model of the Henon attractor has almost the same structure as the neural model of the logistic map, as can be seen in Figure 3.2. It retains almost the same architecture as the logistic map model and is thoroughly described in [22]. The two networks selected to simulate the attractors in discussion were trained using the Back Propagation training algorithm. The training parameters are presented in [13] and [22].

4. The RMS and Absolute Error Fluctuation

A critical aspect of training strategy is the initial conditions of the network and the number of training epochs that retain desirable actual performance. In this

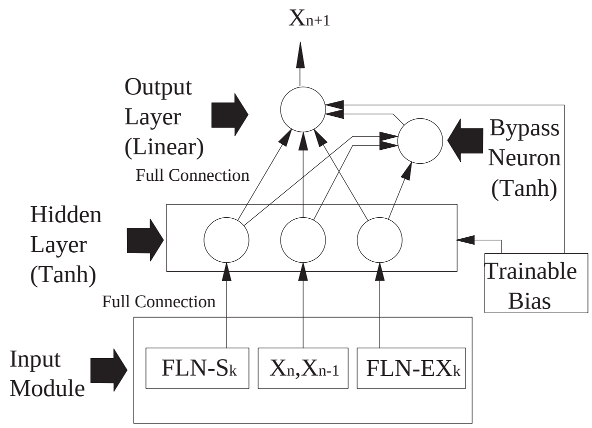

Figure 3.2: FNL-Back propagation network that models the Henon map

section the fluctuation of the resulting RMS and its affection to the absolute error are investigated. Two cases are considered: the case of basic training in which the networks were applied with different, randomly chosen initial weight vectors for each training effort, and the case of additional training in which the networks were submitted to a step by step additional training, to investigate whether their performance is changed. The step of additional training was fixed to 500 epochs. The resulting RMS of the training data set and distribution of the Absolute error over critical levels were calculated for each training effort. The RMS and the absolute error series are depicted in the following for each case.

For both cases, the chaotic manner of the resulting RMS is investigated in the next section.

4.1. The Logistic Map Neural Model

This section presents the results for basic and additional training for the Logistic Map Neural Model. In this case the neural model was trained using 21 randomly chosen initial weight vectors. The resulting RMS and Absolute Error series are presented in Figure 4.1 and Figure 4.2.

The minimum and maximum values of the RMS are 0.000514 and 0.001505 respectively. Although small (0.000895), the fluctuation of the RMS seriously affects the absolute error as shown in Figure 4.2. The failure of the network

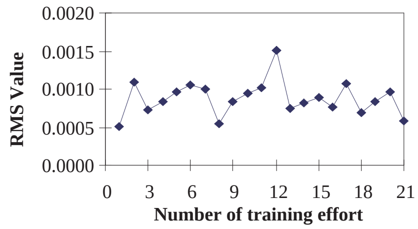

Fig 4.1: Neural Model of the Logistic Map. RMS series created when altering the initial weight Vector

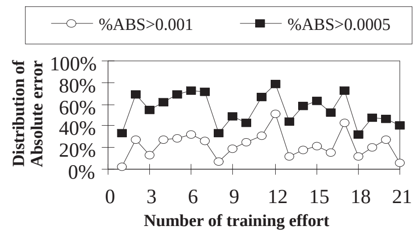

Fig 4.2: Neural Model of the Logistic Map. Distribution of absolute error calculated when altering the initial weight Vector.

	Worst Model			Best Model		
	Worst Value	Best Value	Difference	Worst Value	Best Value	Difference
RMS	0.00161	0.001468	0.00015	0.00053	0.00051	0.00002
$\%ABS > 0.001$	55%	46%	10%	6%	3%	3%
$\%ABS > 0.0005$	80%	74%	6%	37%	28%	9%

Table 2: Neural model of the Logistic Map: Additional Training

raises from $2.80\% - 51.60\%$ for the "0.001" error level and from $33\% - 78\%$ for the "0.0005" error level.

When additional training is applied no more improvement or worsening of the network's performance is observed. The response of the network is captured within limits posed by the local minima which seem to be targeted by the initial value of the weight vector. Figures 4.3, 4.4, 4.5 and 4.6 show the RMS series and the Absolute error distribution over critical error levels for the worst and the best behaving neural model correspondingly. A decay of the fluctuation of the RMS and the corresponding Absolute error can be observed in both cases.

Table 4.1 presents the exact limits of fluctuation for the worst and best neural model, when they are submitted to additional training. As shown in the next Section 5, the chaotic manner of the RMS is present too.

4.2. The Henon Map Neural Model

The behavior of the RMS and its affection to the distribution of the Absolute error is, for the Henon neural model, similar to the behavior stated in previous

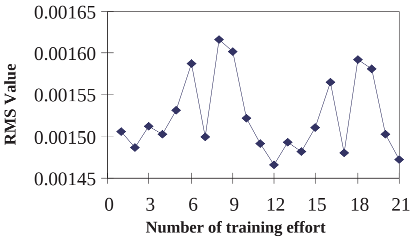

Fig.4.3: RMS series created during addiional training of the worst Neural Model of the Logistic Map

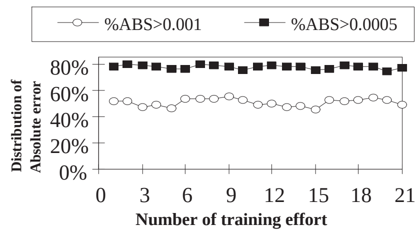

Fig.4.4: Distribution of absolute error created during additional training of the worst Neural Model of the Logistic Map.

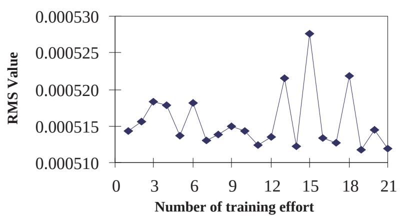

Fig.4.5: RMS series created during additional training of the best Neural Model of the Logistic Map

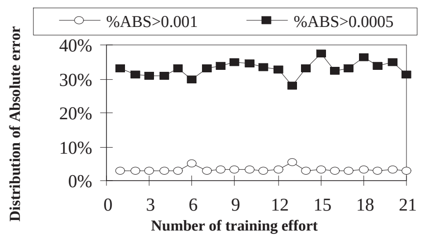

Fig.4.6: Distribution of absolute error created during additional training of the best Neural Model of the Logistic Map.

section. A picture of the error's response can be seen in Figures 4.7, 4.8, 4.9, 4.10, 4.11, and 4.12, while certain data values are presented in Tables 3 and 4. It should be noted that the lower RMS value does not always mean better performance of the model. The actual response of the network is related to the critical error level, which corresponds to pre-determined requirements. For example the neural model with RMS=0.00056 asserts the best performance when the error level of "0.001" is considered (failure 5.60%, next best value 10.20%). But when "0.0005" is selected as critical error level, the model with RMS=0.00061 responds better (failure 23.20% with next best value 32.40%). The previous statements are related to the two best models of the Henon map.

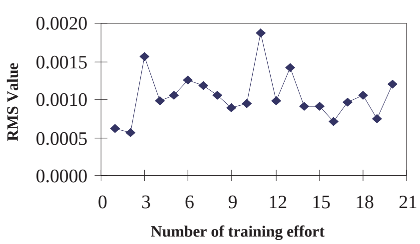

Fig 4.7: Neural Model of the Henon Map. RMS series created when altering the initial

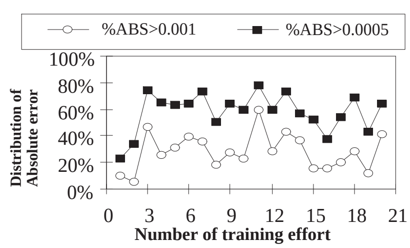

Fig 4.8: Neural Model of the Henon Map. Distribution of absolute error created when altering the initial weight Vector.

	Worst Value	Best Value	Difference
RMS	0.00187	0.00056	0.00131
$\%ABS > 0.001$	59%	6%	54%
$\%ABS > 0.0005$	78%	23%	55%

Table 3: The Henon Neural model: Effects of altering the initial weight vector

	Worst Value	Best Value	Difference
RMS	0.00187	0.00056	0.00131
$\%ABS > 0.001$	59%	6%	54%
$\%ABS > 0.0005$	78%	23%	55%

Table 4: The Henon Neural model: Effects of altering the initial weight vector

In additional training, the influence of RMS fluctuation decreases in both cases (worst and best model). The chaotic manner is present too as shown in the following section.

5. Estimation of the Dominant Lyapunov Exponent and Stability Investigation

The estimation of Lyapunov exponents is based on Alan Wolf' work [28]. The essential idea of the method is the reconstruction of the underlying attractor using the so-called embedding scheme [25]. The application of the method in data series requires the a priori determination of the following parameters.

a) Embedding dimension **D**: The dimension of the embedding space, the choice of which is based on assumptions made for the dimension of the original attractor.

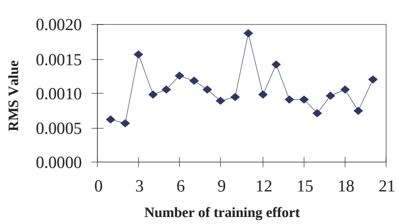

Fig 4.9: RMS series created during additional training of the worst Neural Model of the Henon Map

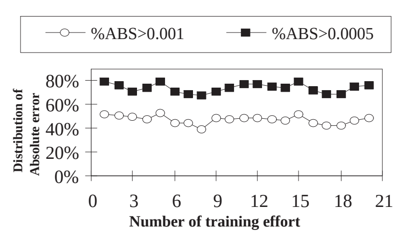

Fig 4.10: Distribution of absolute error created during additional training of the worst Neural Model of the Henon Map.

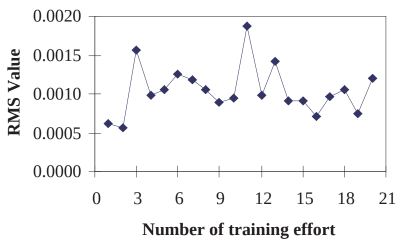

Fig 4.11: RMS series created during additional training of the best Neural Model of the Henon Map

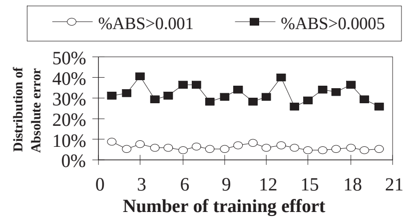

Fig 4.12: Distribution of absolute error created during additional training of the best Neural Model of the Henon Map

	Worst Model			Best Model		
	Worst Value	Best Value	Difference	Worst Value	Best Value	Difference
RMS	0.00183	0.00153	0.00030	0.00060	0.00053	0.00008
$\%ABS > 0.001$	53%	39%	13%	9%	5%	4%
$\%ABS > 0.005$	80%	68%	12%	41%	26%	15%

Table 5: The Henon model Results of additional training

b) Delay time t_d: It is actually the number of intermediate samples skipped during vector construction process.

c) Evolution time t_e: Corresponds to the time (number of samples) that trajectories, starting from nearby initial conditions, are being followed to reveal local stretching of the attractor.

d) Maximum distance d_{max}: Corresponds to the maximum permissible distance, in the embedding space, between trajectories starting from nearby initial conditions. Suggested values [28] are $5\% - 20\%$ of the range of the data series.

Applying fixed values to the above parameters, a value, not always a representative one, is calculated for the exponent. To ensure that chaotic performance is present, stability investigation techniques should be applied to the system. There are two aspects of stability examined in this paper.

1) The stability of the evolution path of the exponent, during its calculation. In this case the trend of the evolving exponent to remain positive or not is examined.

2) The sensitivity of the exponent to the fluctuation of the above presented parameters. Results of a complete exploration of the exponent's sensitivity should be represented in terms of a diagram or a table in a four dimensional

space. Considering this impossible, the investigation took place retaining, in each case, three of the parameters fixed at their reference values, defined in the following, and altering the fourth one.

In the current work as in previous ones [12], [14] a set of parameter values is considered as the representative set. This set is referred as reference set and consists of parameter values, which satisfy theoretical [28] and experimental requirements. In many cases, in this work, reference set applies a representative value and a representative evolution path of the exponent. Here, representative has to do with the proximity of the value (or path) under discussion to the average exponents' value (or path). When another set of parameter values is closer to the average, it is clearly depicted through corresponding diagrams and tables.

It should also be pointed out that since the number of steps of the exponents' calculation is not fixed, there is a requirement of carefully and selectively interpolating values to some of the evolution curves, when average evolution path of the exponent is to be estimated. These intermediate steps should be considered as smoothing noise, not values emerging from the algorithmic process. Since the recognition of these steps is not possible in the diagrams that follow, the evolution paths should be considered as a macroscopic picture of the process. It is meaningless to zoom in or try to mine certain intermediate data values.

5.1. Sensitivity of the RMS to the Initial Weight Vector

To investigate whether the RMS series produced when the logistic and Henon models are sensitive to the initial value of the weight vector, the dominant exponent is calculated and submitted to stability investigation. For both models, the reference set is:

$$(D, t_d, t_e, d_{max}) = (2, 1, 2, 10\%)\,. \tag{2}$$

The evolution of the dominant Lyapunov exponent is depicted in Figures 5.1, 5.2, 5.3, and 5.4 for the RMS series of both neural models. In Figures 5.1 and 5.2 a real evolving procedure is presented for all cases examined. As can be seen, the number of calculation steps is not constant. In all cases the exponent retains positive values. No path leading to negative values seems to emerge.

The Average and reference evolution paths for the Dominant Lyapunov exponent are presented in Figures 5.3 and 5.4.

As shown in these diagrams, the reference path is very close to the average evolution of the Lyapunov exponent and can be considered as the representative

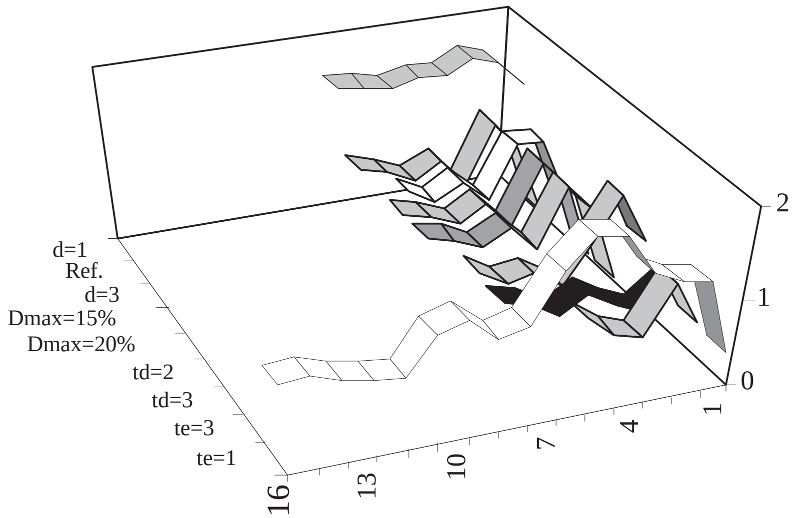

Figure 5.1: The Logistic Map Neural Model. Running Estimate of the Dominant Lyapunov Exponent for the RMS series

evolution path for both neural models. The resulting values of the exponent, positive in all cases, are shown in Figures 5.5 and 5.6.

For both models, the Dominant Lyapunov Exponent seems to retain positive values for every calculation effort. Numerical verification of the above assertion has been worked out for many more combinations of parameter values, than shown in 5.5 and 5.6.

The reference set offers the closest estimation to the mean value of the exponent. Thus, the values 0.8877 (bits/sec) and 0.5565 (bits/sec) can be considered as the representative ones for the RMS series produced during training of the Logistic and Henon neural models correspondingly.

5.2. Additional Training

Two main conclusions can be reached from the application of additional training to the neural models.

1. No significant improvement or worsening of the performance is observed. The range of the RMS series shrinks and the influence of the RMS fluctuation to the absolute error decays.

2. Lyapunov exponent calculation and its stability investigation point out that chaotic response of the RMS series is present in this case also.

In the following diagrams, the average and actual evolution paths show the

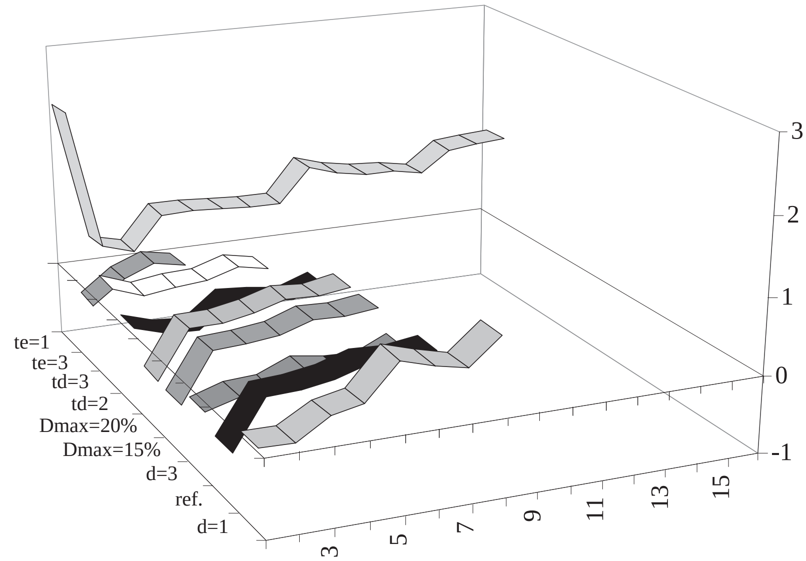

Figure 5.2: The Henon Map Neural Model. Running Estimate of the Dominant Lyapunov Exponent for the RMS series

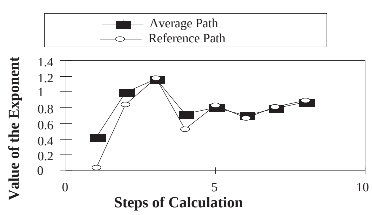

Fig.5.3: The Logistic Map Model. Average and reference evolution path of the Dominant Lyapunov Exponent for the RMS series.

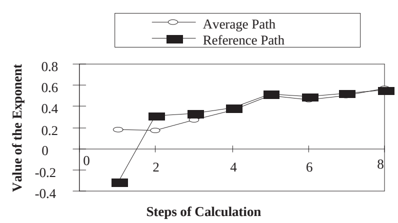

Fig.5.4: The Henon Map Model. Average and reference evolution path of the Dominant Lyapunov Exponent for the RMS series.

trend of the exponent. Additionally, the diagrams of the resulting values are helpful in selecting representative estimations for the evolution path and final value of the dominant Lyapunov exponent.

The average and reference evolution paths for the worst neural model of the logistic map are presented in Figure 5.7. The reference path can be considered representative as it evolves near the average evolution path and terminates very close to it (1.39 bits/sec). This resulting value (the nearest to the average) is the representative exponent estimation. As shown in Figure 5.9 the exponent remains remarkably positive for all the presented parameter sets.

Figure 5.8 refers to the best neural model of the logistic map and presents the evolution paths of the dominant Lyapunov exponent for the average, refer-

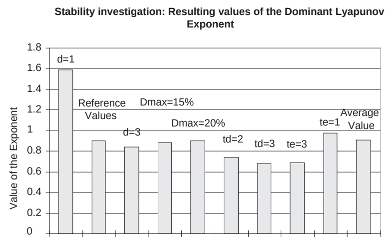

Fig.5.5: The Logistic Map Model. Resulting values of the dominant Lyapunov exponent.

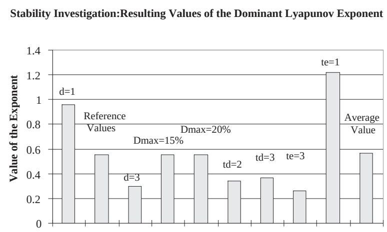

Fig.5.6: The Henon Map Model. Resulting values of the dominant Lyapunov exponent.

ence, and best resulting paths. The best resulting path offers the nearest estimation of the exponent to the average and is produced when $(D, t_d, t_e, d_{max}) = (2, 2, 2, 10\%)$ Although the estimated exponent (1.0793 bits/sec) of the best resulting path is the nearest to the average and can be considered as the representative estimation of the exponent, the reference path applies a more satisfactory picture of the evolution procedure.

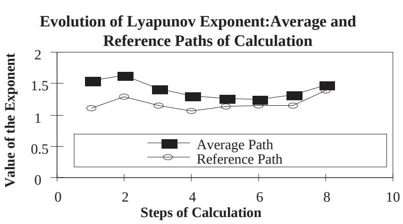

Fig.5.7:The Logistic Map worst Model. Average and reference evolution path of the Dominant Lyapunov Exponent for the RMS series produced during additional training.

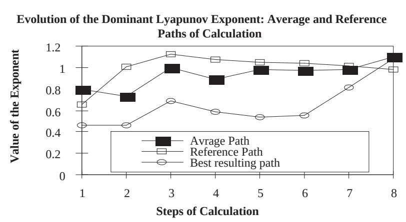

Fig.5.8:The Logistic Map best Model. Average, reference and best resulting evolution path of the Dominant Lyapunov Exponent for the RMS series produced during additional training..

The positive manner of the calculated exponent, for different parameter sets, is shown in Figures 5.9 and 5.10. Figures 5.11 and 5.12 present the evolution paths of the exponent for the best and worst models of the Henon map. The best evolving path of Figure 5.11 corresponds to $d_{max} = 15\%$ and the best resulting path to $t_d = 3$. The representative exponent value (1.2bits/sec) corresponds to the latter case.

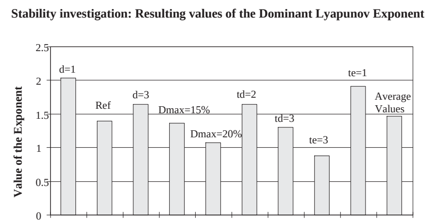

Fig.5.9: The Logistic Map Model-Additional training. Resulting values of the dominant Lyapunov exponent for the worst responding network.

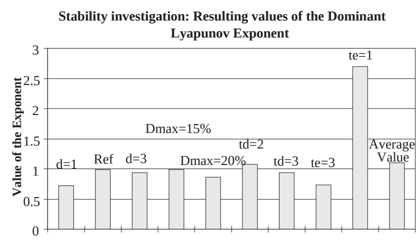

Fig.5.10: The Logistic Map Model-Additional training. Resulting values of the dominant Lyapunov exponent for the best responding network.

In 5.12 (worst model), the reference path evolves in parallel to the average and considered as representative while the best resulting path offers the representative value (0.9131 bits/sec) and corresponds to $t_d = 2$. Figures 13 and 14 present the resulting values of the exponent for the best and the worst models of Henon map.

5.3. Conclusions

The study presented above leads directly to the following inferences.

1. The RMS series produced during training of the neural networks modeling the chaotic attractors of the two representative models are sensitive to the initial weight vector. This fact complicates the decision process for the selection of training strategy, since there is no apriori knowledge for the best starting values of the vector.

2. The chaotic fluctuation of the RMS seriously affects the actual performance of the neural model, measured by the distribution of the absolute error over critical pre-selected error levels. This assertion should be taken into consideration, especially for the cases in which a fixed pre-chosen accuracy is required for a certain application (chaos control using neural networks [2], [18].

3. Additional training applied to the neural models no more improves or worsens the networks' response since the initial weight vector captures a certain

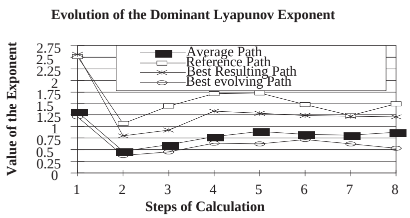

Fig.5.11 The Henon Map best Model. Average reference, best evolving and best resulting evolution path of the Dominant Lyapunov Exponent for the RMS series produced during additional training.

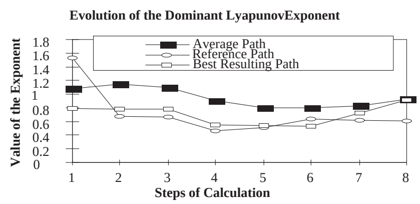

Fig.5.12: The Henon Map worst Model. Average, reference and best resulting evolution path of the Dominant Lyapunov Exponent for the RMS series produced during additional training.

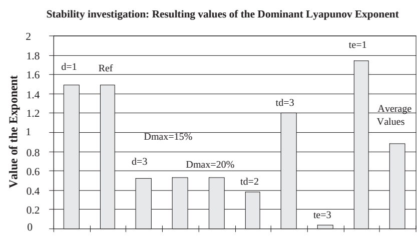

Fig.5.13: The Henon Map Model-Additional training. Resulting values of the dominant Lyapunov exponent for the best responding network.

kind of activity during the recall or testing phase. Although chaotic fluctuation of the RMS is still present, its effects on the actual performance of the model seem to decrease. In this case, what increases the complication of the training strategy selection is the possibility of a slight improvement of the neural model during additional training. In conclusion, a best responding initial weight vector together with an optimum number of training epochs should be considered for the creation of the best network.

A possible explanation of the chaotic features described in this paper should take into consideration the implications of chaotic response of the best learned input-output pairs asserted in [12] and [14]. In paper [12] the chaotic manner of these pairs is examined for simple autoasociative networks. Thus, the production of a general scheme considering such a response and the detailed investigation of the chaotic manner of the best learned pairs for networks with a more complicated structure requires further investigation.

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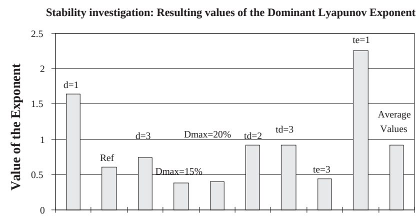

Fig.5.14: The Henon Map Model-Additional training. Resulting values of the dominant Lyapunov exponent for the worst responding network.

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