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LIKELIHOOD RATIO DETECTION OF RANDOM SIGNALS: THE CASE OF CAUSALLY FILTERED AND WEIGHTED WIENER AND POISSON NOISES

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Abstract: This paper contains first of all conditions for absolute continuity as well as a new likelihood ratio formula for detecting a random signal whose law is unknown, and which is obscured by noise that is modeled as the output of a causal filter of weighted Wiener and Poisson processes. Secondly, the derivation presented reveals, through its reproducing kernel Hilbert space framework, the reasons that make the method work, as well as its limitations. The tools used are the Cramér-Hida representation and stochastic calculus not assuming the "usual conditions."

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1. Problem Description

Underwater acoustics is about sonars, that is, techniques that use waves of mechanical vibration to transmit and receive information, under water, for such waves that are propagated most easily in that environment. Detection is about

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spotting the presence of information in a background of noise. One would thus expect that detection processes would transit through the wave equation whose prototype is given by $Dp = dN$, where p is pressure, N is noise, and D is an operator of the form

$$D = \Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{b}{c^2}\frac{\partial}{\partial t}\Delta,$$

where $b, c > 0$, and $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. b is viscosity, and c, velocity.

One would solve the equation and compute the law of the process. There would be a process for noise only, and a process for the signal obscured by noise. Computation of the likelihood ratio would follow. The crucial difficulty is that b and c are themselves stochastic fields of very complex structure and that "determination of the probability distributions of the wave fields in a stochastic medium (also in a weakly inhomogeneous stochastic medium) is impossible" (Sobczyk [21, p. 152]).

A second approach that has been attempted could be defined as phenomenological in that an acoustic field is defined to be a linear superposition of individual fields randomly produced by similar sources (Middleton [19]). The randomness is described by various analytically tractable processes. Unfortunately such methods produce explicit laws only for the first order, implying reliance on independent observations and consequently loss of information. They furthermore require estimation of parameters for which there seldom will be adequate information.

A third approach is basically statistical and it is that which is pursued here. It consists in positing a very general signal-in-noise model, then deriving a generic likelihood ratio function, since the likelihood ratio is the best detector in most practical instances (Helstrom [12]), and then discretizing the likelihood ratio to fit it to data. It should be noted that discretizing must follow derivation of the likelihood, as the law of the signal-in-noise process is typically unknown.

Indeed, "in the operation of a sonar system the operator is repeatedly faced with the problem of detecting a signal which is obscured by noise. This signal may be an echo resulting from a transmitted signal over which the operator has some control, or it may have its origin in some external source. These two modes of operation are commonly distinguished as active and passive sonar. Similar situations arise in radar surveillance and in seismic exploration ... Signals come in all shapes and forms. In active sonar systems one may use simple sinusoidal signals of fixed duration and modulations thereof. There are impulsive signals such as those made with explosions or thumpers. At the other extreme one may make use of pseudorandom signals. In passive systems, the signals whose detection are sought may be noise in the conventional meaning of the word;

noise produced by propellers or underwater swimmers, for example. ...

... When echoes are produced by extended targets such as submarines, there are two distinct ways in which the echo structure is affected. First, there is the interference between reflections from the different structural features on the hull of the submarine. This interference leads to a target strength that fluctuates rapidly with changes in the aspect. Secondly, there is the elongation of the composite echo due to the distribution of reflecting features along the submarine. ... A final source of pulse distortion is the Doppler shifts produced by the relative motions between the source, the medium, and the targets. Since the source, the medium and the target (or detector in passive listening) may each have a different vector velocity relative to the bottom, the variety of effects may be quite large" (Horton [14, 1.1]).

Previous work on this topic has mostly been a (difficult) mathematical travail with models whose bearing on really applied problems has been somewhat tenuous. A recent survey on what is available can be found in Kailath et al [16]. One will find below a method to obtain likelihood ratios for a fairly general and realistic family of noise models.

1.1. The Detection Problem

"Detection of stochastic signals in Gaussian or non-Gaussian noise is a valid model for many important signal detection problems. In some of these problems, the noise is very nonstationary and the signal cannot be represented as a set of narrowband components. Many examples can be given of applications where such problems arise; they abound in such areas as sonar and radar" (Baker et al [2, p. 1]).

The notation shall be S for the signal sent, N for the noise, and $S+N$ for the signal received when a signal was sent. This additive model can be a restriction, but it has been shown that this is not the case when the noise is purely Gaussian (Baker et al [1]).

There are four operations that one must perform successfully to be able to claim that a likelihood ratio detection problem is solved. To enumerate them, let P_N and P_{S+N} be, respectively, the probabilities induced on $L_2\,[0,T]$ by N and $S+N$. $[0,T]$ is the time span of observation for detection. One must then firstly ascertain that the likelihood ratio exists for the problem at hand, which is not always the case as Slepian [20] noticed already in 1958! Technically, one must check that P_{S+N} is absolutely continuous with respect to P_N.

Secondly, when the likelihood ratio exists, one must produce a functional

Λ that is defined, and thus computable without knowing which of the P_N or P_{S+N} regimes obtains, for every observable noise or signal in noise path. One must have in particular that, for measurable A,

$$P_{S+N}(A) = \int_A \Lambda(f) P_N(df).$$

The functional Λ being available, one must thirdly be able to solve for Λ_0, for every predefined probability of false alarm $\alpha \in]0,1[$, the equation[1]

$$\alpha = P_N\left(f \in \mathcal{L}[0,T] : \ \Lambda(f) > \Lambda_0\right),$$

and then compute the probability of detection $1-\beta$, where

$$\beta = P_{S+N}\left(f \in \mathcal{L}[0,T] : \ \Lambda(f) \leq \Lambda_0\right).$$

Fourthly, given observations of f at times $0 \leq t_1 < \cdots < t_n \leq T$, one must be able to obtain approximations Λ_n to Λ and $\Lambda_0^{(n)}$ to Λ_0 such that, simultaneously,

$$\begin{aligned} P_N\left(f \in \mathcal{L}[0,T] : \ \Lambda_n(f(t_1),\ldots,f(t_n)) > \Lambda_0^{(n)}\right) &\approx \alpha, \\ P_{S+N}\left(f \in \mathcal{L}[0,T] : \ \Lambda_n(f(t_1),\ldots,f(t_n)) > \Lambda_0^{(n)}\right) &\approx 1-\beta. \end{aligned}$$

1.2. Content of Paper

This paper contains the derivation, under minimal assumptions, of a likelihood detection formula for a random signal whose law is unknown, and which is blurred by a noise which is the output of a causal filters, that filters a weighted sum of Wiener and Poisson processes. The usefulness of such models is discussed at length in Baker et al [2], [3], [4].

Two remarks about the derivation should be made. As one is using stochastic calculus on $D[0,T]$, for processes which are adapted to the filtration generated by the evaluation maps, and defined simultaneously for couples of probability measures not known, *a priori*, to be mutually absolutely continuous, the usual assumption of the "usual conditions" of stochastic calculus being met is not warranted. That is why the reference used for stochastic calculus is Von Weizsäcker et al [23]. Secondly, most of the derivation is made under assumptions that are somewhat more general than those used in the end to obtain the likelihood ratio: the main reason for so doing is that one can thus better

[1] $f \in \mathcal{L}[0,T]$ means that $\int f^2 < \infty$.

see where one meets the limits of the method that is used. In particular, one eventually explicitly sees to what extent the chosen framework is essential for a "realistic" modelling of signal detection problems.

The authors of this paper are happy to aknowledge a debt to Dr. C.R. Baker, of UNC, Chapel Hill, who saw early on (in 1980 already) that, in order to obtain realistic detection models, one should couple the Cramér-Hida representation with stochastic calculus, and to Dr. J. Mémin, of Rennes 1, who was the first to obtain the proper form of the likelihood ratio for Wiener noise (Mémin [18]) and whose methods have proven of use here also.

2. The Model

2.1. The Noise N_α

The noise N_α is defined as the integral of a non-anticipative deterministic kernel with respect to a process with orthogonal increments, and may be looked at as a filtered white noise with independent, weighted Gaussian and Poisson components.

2.1.1. The Integrator

As usual, one assumes that $(\Omega, \mathcal{A}, P)$ is the reference probability space, and that all processes considered are defined on that space, and adapted to a filtration $\underline{\mathcal{A}}$ of $\mathcal{A}$, which satisfies the "usual conditions".

A generalized Brownian motion is a Brownian motion for which the variance function is a non-negative, monotone non-decreasing, and continuous function. It is the type of Brownian motion that emerges from the Cramér-Hida representation (Cramér [8] and Hida [13]). Its paths are almost surely continuous, and those that are not may be taken as being continuous to the right (Von Weizsäcker et al [23, 4.3.5, p. 71]). It is denoted B_1 in the sequel, and β_1 represents its variance function:

$$V\left[B_1(\cdot,t)\right] = E\left[B_1^2(\cdot,t)\right] = \beta_1(t)\,,\ 0 \le t \le T.$$

One has that (Von Weizsäcker [23, Examples, p. 148]), for fixed $t \in [0,T]$, almost surely, with respect to P,

$$\langle B_1\rangle(\omega,t) = \beta_1(t)\,.$$

B_2 denotes a Poisson process. Then $\beta_2(t)$, which stands for $E[B_2(\cdot,t)]$, is finite, and continuous for $t \geq 0$ (Todorovic [22, 2.4.1, p. 41]). Let

$$\tilde{B}_2(\omega,t) = B_2(\omega,t) - \beta_2(t),$$

where $\tilde{B}_2$ is a square integrable martingale. One has that (Von Weizsäcker [23, Examples, p. 148]), for fixed $t \in [0,T]$, almost surely, with respect to P,

$$\langle B_2\rangle(\omega,t) = \beta_2(t).$$

Furthermore, for fixed $t \in [0,T]$, almost surely, with respect to P,

$$[B_2](\omega,t) = B_2(\omega,t).$$

It is assumed that B_1 and B_2 are independent.

Let then $0 \leq \alpha \leq 1$, and set[2]:

$$\beta_\alpha(t) = \alpha\beta_1(t) + (1-\alpha)\beta_2(t),$$

$$B_\alpha(\omega,t) = \sqrt{\alpha}\,B_1(\omega,t) + \sqrt{1-\alpha}\,\tilde{B}_2(\omega,t).$$

B_α is then a square integrable martingale, and, for fixed $t \in [0,T]$, almost surely, with respect to P,

$$\langle B_\alpha\rangle(\omega,t) = \beta_\alpha(t),$$

$$[B_\alpha](\omega,t) = \alpha\beta_1(t) + (1-\alpha)B_2(\omega,t).$$

2.1.2. The Integrand

Let F denote a Borel measurable function over the rectangle $[0,T]\times[0,T]$ that has the following properties:

- for t and x in $[0,T]$, fixed, but arbitrary, such that $x > t$, $F(t,x) = 0$,
- for $t \in [0,T]$, fixed, but arbitrary, $\int_0^t F^2(t,x)\,\beta_\alpha(dx) < \infty$,
- the map $t \mapsto [F(t,\cdot)]_\alpha \in L_2[\beta_\alpha]$ is continuous (where $[F(t,\cdot)]_\alpha$ is the equivalence class of $F(t,\cdot)$ in $L_2[\beta_\alpha]$),
- $\{[F(t,\cdot)]_\alpha,\ t \in [0,T]\}$ generates $L_2[\beta_\alpha]$.

Remark. The conditions that F must satisfy are those that insure that N_α has a canonical representation of multiplicity one, in the sense of Cramér-Hida [8], [13]. A discussion of the nature of the restriction on the noise process that is thus introduced may be found in Ephremides [10].

[2] To have a weighted sum of the type $\tilde{\alpha}B_1 + (1-\tilde{\alpha})\tilde{B}_2$ one would need to divide B_α by $\sqrt{\alpha} + \sqrt{1-\alpha}$ but this would only add complexity to the notation.

2.1.3. Noise Model and Properties

One sets, as an integral with respect to a process with orthogonal increments (Todorovic [22, 7.4, p. 160]), and for $t \in [0,T]$ fixed, but arbitrary,

$$N_\alpha(\omega,t) = \int_0^t F(t,x)\, B_\alpha(\omega,dx).$$

Then, for $t \in [0,T]$ fixed, but arbitrary, $E[N_\alpha(\cdot,t)] = 0$, and the covariance C_{N_α} of N_α is, furthermore, given by the following equality:

$$C_{N_\alpha}(s,t) = \int_0^{s\wedge t} F(s,x)\, F(t,x)\, \beta_\alpha(dx), \ (s,t) \in [0,T] \times [0,T].$$

As a consequence of the assumptions on F, the function $t \mapsto C_{N_\alpha}(t,t)$ is continuous for $t \in \left]0,T\right[$. N_α is thus continuous in quadratic mean (Todorovic [22, 6.21, p. 133]) and its covariance is continuous (Todorovic [22, 6.2.2, p. 133]). Furthermore, the paths of N_α are, almost surely, with respect to P, in $\mathcal{L}_2[0,T]$.

Let $H(N_\alpha)$ denote the reproducing kernel Hilbert space (RKHS) of N_α. One has that (Grenander [11, p. 97])

$$H(N_\alpha) = \left\{ \tilde{f}(t) = \int_0^t F(t,x)\, f(x)\, \beta_\alpha(dx), \ f \in \mathcal{L}_2[\beta_\alpha] \right\}.$$

For the inner product $\langle \cdot , \cdot \rangle_{H(N_\alpha)}$ of $H(N_\alpha)$, one has furthermore that

$$\langle \tilde{f}, \tilde{g} \rangle_{H(N_\alpha)} = \langle f, g \rangle_{L_2[\beta_\alpha]}$$

whenever f and $g \in \mathcal{L}_2[\beta_\alpha]$, and

$$\tilde{f}(t) = \int_0^t F(t,x)\, f(x)\, \beta_\alpha(dx), \ \tilde{g}(t) = \int_0^t F(t,x)\, g(x)\, \beta_\alpha(dx).$$

The covariance operator $R_{N_\alpha} : L_2[0,T] \longrightarrow L_2[0,T]$ is computed using the formula:

$$R_{N_\alpha}\left([f]_{L_2[0,T]}\right) = \left[\int_0^T C_{N_\alpha}(\cdot,x)\, f(x)\, dx\right]_{L_2[0,T]}, \ f \in \mathcal{L}_2[0,T],$$

where square brackets denote again equivalence classes. This operator is non negative, self-adjoint, and continuous, with finite trace (Balakrishnan [5, p. 125]). It can thus be written as

$$R_{N_\alpha} = \sum_{i=1}^{\infty} \lambda_i [e_i \otimes e_i],$$

where, for an orthonormal family $\{e_n,\ n \in \mathbb{N}\}$,

$$R_{N_\alpha} e_n = \lambda_i e_n, \quad [e_n \otimes e_n] f = \langle f, e_n \rangle_{L_2[0,T]} e_n, \quad \lambda_i \geq 0, \quad \sum_{n=1}^{\infty} \lambda_n < \infty.$$

In an obvious way, one may write, in $L_2[P]$,

$$[N_\alpha(\cdot,t)]_{L_2[P]} = \left[N_\alpha^{(1)}(\cdot,t)\right]_{L_2[P]} + \left[N_\alpha^{(2)}(\cdot,t)\right]_{L_2[P]},$$

with

$$\begin{aligned} N_\alpha^{(1)}(\omega,t) &= \sqrt{\alpha} \int_0^t F(t,x)\, B_1(\omega,dx)\ , \\ N_\alpha^{(2)}(\omega,t) &= \sqrt{1-\alpha} \int_0^t F(t,x)\, \tilde{B}_2(\omega,dx)\, . \end{aligned}$$

$N_\alpha^{(1)}$ and $N_\alpha^{(2)}$ are independent, and thus, on $L_2[0,T]$,

$$P_{N_\alpha} = P_{N_\alpha^{(1)}} \star P_{N_\alpha^{(2)}},$$

where the star denotes convolution, and, for instance, P_{N_α} is the measure induced on $L_2[0,T]$ by P and the maps

$$\omega \mapsto \langle N_\alpha(\omega,\cdot), f \rangle_{L_2[0,T]},\ f \in \mathcal{L}_2[0,T].$$

If $\mathcal{S}_\alpha^{(1)}, \mathcal{S}_\alpha^{(2)}, \mathcal{S}_\alpha$ denote, respectively, the supports, in $L_2[0,T]$, of $P_{N_\alpha^{(1)}}$, $P_{N_\alpha^{(2)}}$, and P_{N_α}, one then has that

$$\mathcal{S}_\alpha = \overline{\mathcal{S}_\alpha^{(1)} + \mathcal{S}_\alpha^{(2)}}.$$

Remark. $\mathcal{R}$ denoting range, let

$$K_1 = \overline{\mathcal{R}\left(R^{\frac{1}{2}}_{N_\alpha^{(1)}}\right)},\ K = \overline{\mathcal{R}\left(R^{\frac{1}{2}}_{N_\alpha}\right)}.$$

The expression given above, relating the supports of $P_{N_\alpha^{(1)}}$, $P_{N_\alpha^{(2)}}$, and P_{N_α}, and a result of Itô [15] yield then that, whenever $K_1 = L_2[0,T]$, then

$$\mathcal{S}_\alpha = L_2[0,T].$$

If now $U : L_2[\beta_\alpha] \longrightarrow L_2[0,T]$ denotes the operator for which Uf is the equivalence class, in $L_2[0,T]$, of $\langle F(t,\cdot), f\rangle_{L_2[\beta_\alpha]}$, one furthermore has that

$$U = R_{N_\alpha}^{\frac{1}{2}} J^\star, \text{ where } J : L_2[0,T] \longrightarrow L_2[\beta_\alpha]$$

is a partial isometry onto $L_2[\beta_\alpha]$, with initial space K. The operator J is unitary as soon as $K = L_2[0,T]$. A sufficient condition is that K_1 be $L_2[0,T]$, that is, that $P_{N_\alpha^{(1)}}$ have full support.

Let finally K° be the range of $R_{N_\alpha}^{\frac{1}{2}}$, and define, on K°, the inner product

$$\langle R_{N_\alpha}^{\frac{1}{2}} f, R_{N_\alpha}^{\frac{1}{2}} g\rangle_{K^\circ} = \langle f, g\rangle_{L_2[0,T]}.$$

Then $L_2[\beta_\alpha]$ and K° are unitarily equivalent, and thus so are $H(N_\alpha)$ and K°. As a consequence, one has, *mutatis mutandis*,

$$K^\circ = K_1^\circ + K_2^\circ.$$

2.2. The Signal S

Let S denote a random signal, adapted to $\underline{\mathcal{A}}$. It is assumed that, almost surely, with respect to P,

$$S(\omega,\cdot) \in H\left(N_\alpha^{(1)}\right).$$

As it can be seen further on in the paper, the method used does work for $S(\omega,\cdot) \in H(N_\alpha)$ only when $\beta_1 = \beta_2$. Nevertheless the assumptions which are made, though less "natural" and elegant than the latter, for the problem at hand, cover that case also. They have however the advantage of unmasking the role of each assumption.

It can be shown (Baker et al [1, Theorem 3, Step 3, p. 170]) that the following representation obtains:

$$S(\omega,t) = \alpha \int_0^t F(t,x)\, s(\omega,x)\, \beta_1(dx),$$

for some predictable s, with paths in $\mathcal{L}_2[\beta_1]$. Thus one always has that

$$P\left(\omega \in \Omega : \ \|s(\omega,\cdot)\|^2_{L_2[\beta_1]} < \infty\right) = 1.$$

Also, in what follows, s will usually be progressively measurable, except when a predictability assumption is required, and then the assumption will be explicit. One sees here that the latter is not a restriction.

Finally, still in what follows, X_α will represent the process $S_\alpha + N_\alpha$ and Y_α a process such that, for $t \in [0,T]$, fixed, but arbitrary, almost surely, with respect to P,

$$Y_\alpha(\omega,t) = \alpha \int_0^t s(\omega,x)\,\beta_1(dx) + B_\alpha(\omega,t).$$

2.3. Summary List of Assumptions

Here is a list of recurrent assumptions which will be called upon in order to shorten the statement of many propositions. $\mathcal{D}$ denotes the σ-field of $D[0,T]$ generated by the evaluation maps

$$\{ev(f,t) = f(t),\ t \in [0,T],\ f \in D[0,T]\},$$

$\mathcal{D}_t$ that which is generated by the evaluation maps "up to time t," and $\underline{\mathcal{D}} = \{\mathcal{D}_t,\ t \in [0,T]\}$.

A0. The basic probability space is $(\Omega, \mathcal{A}, P)$, and the basic filtration is $\underline{\mathcal{A}}$. For $\underline{\mathcal{A}}$ the "usual assumptions" obtain.

A1. $B_\alpha^{(\cdot)}$ is a process, defined on an appropriate probability space, with respect to an appropriate filtration, represented by the symbol "$\cdot$" (which can be absent!). It has the following defining ingredients:

- $0 < \alpha < 1$;
- $B_\alpha^{(\cdot)} = \sqrt{\alpha}\, B_1^{(\cdot)} + \sqrt{1-\alpha}\, \tilde{B}_2^{(\cdot)}$;
- $B_1^{(\cdot)}$ is generalized Brownian motion with variance function β_1 : it has continuous paths, almost surely, and the non-continuous ones are continuous to the right β_1 is continuous non-decreasing;
- $B_2^{(\cdot)}$ is a Poisson process with expectation β_2 and $\tilde{B}_2^{(\cdot)} = B_2^{(\cdot)} - \beta_2$;
- $B_1^{(\cdot)}$ and $B_2^{(\cdot)}$ are independent;

A2. s is a process, progressively measurable for $\underline{\mathcal{A}}$, with the property[3]

$$P\left(\omega \in \Omega : \int_0^T s^2(\omega,x)\,\beta_1(dx) < \infty\right) = 1.$$

[3]This assumption is the consequence of the "requirement" that $S(\omega,\cdot) \in H\left(N_\alpha^{(1)}\right)$.

A3. Y_α is a process with paths in $D[0,T]$. It has the property that, for $t \in [0,T]$, fixed, but arbitrary, almost surely, with respect to P,

$$Y_\alpha(\omega,t) = \alpha \int_0^T s(\omega,x)\,\beta_1(dx) + B_\alpha(\omega,t).$$

A4. s is a process, progressively measurable for $\underline{\mathcal{D}}$, with the property that

$$P\left(\omega \in \Omega : \int_0^T s^2\left(Y_\alpha(\omega,\cdot),x\right)\beta_1(dx) < \infty\right) = 1.$$

A5. Y_α is a process with paths in $D[0,T]$. It has the property that, for $t \in [0,T]$, fixed, but arbitrary, almost surely, with respect to P,

$$Y_\alpha(\omega,t) = \alpha \int_0^T s\left(Y_\alpha(\omega,\cdot),x\right)\beta_1(dx) + B_\alpha(\omega,t).$$

A6. s is a process, progressively measurable for $\underline{\mathcal{D}}$, with the property that

$$P\left(\omega \in \Omega : \int_0^T s^2\left(B_\alpha(\omega,\cdot),x\right)\beta_1(dx) < \infty\right) = 1.$$

A7. For ϕ, a deterministic, strictly positive and measurable function such that

$$\int_0^T \phi(x)\,\beta_2(dx) < \infty,$$

one can and does define

$$\begin{aligned}\ln\{L_{\alpha,s,\phi}(\omega,t)\} &= -\sqrt{\alpha}\int_0^t s(\omega,x)\,B_1(dx)\\ &\quad - \frac{\alpha}{2}\int_0^t s^2(\omega,x)\,\beta_1(dx) + \int_0^t \ln[\phi(x)]\,B_2(\omega,dx)\\ &\quad + \int_0^t [1-\phi(x)]\,\beta_2(dx).\end{aligned}$$

Remark. The terms of $L_{\alpha,s,\phi}$ involving ϕ, B_2, and β_2 are basically those that yield the likelihood in the "pure" Poisson case (with deterministic intensity: see – Brémaud [6, T2, p. 165]).

A "likelihood ratio" L of the form

$$\ln[L] = -\int s\,dB_\alpha - \gamma\int s^2\,d[B_\alpha],$$

or

$$\ln[L] = -\int s\ dB_\alpha - \delta \int s^2\ d\langle B_\alpha \rangle,$$

would require, to progress along the "Girsanov's route," which is the one that shall be travelled, an s with uniformly bounded jumps and, in the first case, jumps strictly smaller than one (Kallenberg [17, Lemma 23.19, p. 449]). On the one hand, it is unlikely that such evidence would be readily available, and, on the other, the simpler form that has been chosen for the initial likelihood provides sufficient evidence to confirm the fact that the "effective" part of the likelihood is its Gaussian component.

A8.

$$E_P\left[L_{\alpha,s,\phi}(\cdot, T)\right] = 1.$$

The following is a useful lemma (Mémin [18, 2.8. – Théorème, p. 14]).

Lemma 1. *When, respectively,* **A0**, **A2**, **A4** *and* **A6** *obtain, it can then always be furthermore assumed, without the "usual assumptions," that the maps*

$$\begin{aligned}
t &\mapsto \int_0^t |\, s \,| (\omega, x)\, \beta_1(dx), \\
t &\mapsto \int_0^t s^2(\omega, x)\, \beta_1(dx), \\
t &\mapsto \int_0^t |\, s \,| \left(Y_\alpha(\omega, \cdot), x\right) \beta_1(dx), \\
t &\mapsto \int_0^t s^2\left(Y_\alpha(\omega, \cdot), x\right) \beta_1(dx)
\end{aligned}$$

are all continuous in the extended real line.

3. Absolute Continuity and Likelihood Ratio for P_{B_α} and P_{Y_α}

3.1. A Version of Girsanov's Theorem

The results in this section follow from repeated use of standard results of stochastic calculus, and in particular, of Ito's formula (Von Weizsäcker et al

[23, P. 194]). It is assumed that **A0**, **A1**, **A2**, and **A7** obtain. The process M is defined by the relation

$$M(\omega,t)=\int_0^t s(\omega,x)\,B_1(\omega,dx)\,.$$

Then

$$L_{\alpha,s,\phi}(\omega,t)=1-\sqrt{\alpha}\int_0^t L_{\alpha,s,\phi}(\omega,x-)\,M(\omega,dx)\\
-\int_0^t L_{\alpha,s,\phi}(\omega,x-)\,[1-\phi(x)]\,\tilde{B}_2(\omega,dx)\,.$$

$L_{\alpha,s,\phi}$ is thus a positive local martingale, and, consequently, a supermartingale. So $E[L_{\alpha,s,\phi}(\cdot,t)]\le 1,\ 0\le t\le T$.

To use the change of measure method, one must prove that the original signal-plus-noise process has, with respect to an explicitly defined, absolutely continuous probability measure, the same law as the original noise. It follows from the statements used below that one can only have, to achieve that $\phi\equiv 1$.

In what follows, it is furthermore assumed that **A8** obtains. This has the consequence that

$$E[L_{\alpha,s,\phi}(\cdot,t)]=1,\ 0\le t\le T,$$

which in turn allows one to define a probability measure $Q_{\alpha,s,\phi}$ by setting

$$Q_{\alpha,s,\phi}(A)=\int_A L_{\alpha,s,\phi}(\omega,T)\,P(d\omega)\,,\ A\in\mathcal{A}.$$

As an immediate consequence, one has that P and $Q_{\alpha,s,\phi}$, as defined above, are mutually absolutely continuous. Furthermore

$$\frac{dQ_{\alpha,s,\phi}}{dP}=L_{\alpha,s,\phi}(\cdot,T)\quad\text{and}\quad\frac{dP}{dQ_{\alpha,s,\phi}}=\frac{1}{L_{\alpha,s,\phi}(\cdot,T)}\,.$$

Let the process $Z_{\alpha,s,\phi}$ be defined as follows. Set

$$U_{\alpha,s,\phi}(\omega,t)=\sqrt{\alpha}\int_0^t s(\omega,x)\,\beta_1(dx)+B_1(\omega,t)\,,$$

and

$$V_{\alpha,s,\phi}(\omega,t)=B_2(\omega,t)-\int_0^t \phi(x)\,\beta_2(dx)\,.$$

Then,

$$Z_{\alpha,s,\phi}=\sqrt{\alpha}\,U_{\alpha,s,\phi}+\sqrt{1-\alpha}\,V_{\alpha,s,\phi},$$

and one subsequently has that

- the process $U_{\alpha,s,\phi}(\omega,t)$ is, with respect to $Q_{\alpha,s,\phi}$, a generalized Brownian motion such that

$$\langle U_{\alpha,s,\phi}\rangle^{Q_{\alpha,s,\phi}} = \beta_1,$$

  where the notation $\langle U_{\alpha,s,\phi}\rangle^{Q_{\alpha,s,\phi}}$ is chosen to remind one of the measure that prevails;

- the process B_2 is, with respect to $Q_{\alpha,s,\phi}$, a Poisson process such that

$$E\left[B_2(\cdot,t)\right] = \int_0^t \phi(x)\,\beta_2(dx);$$

- the process $Z_{\alpha,s,\phi}$ is, with respect to $Q_{\alpha,s,\phi}$, a martingale;

- with respect to $Q_{\alpha,s,\phi}$, $U_{\alpha,s,\phi}$ and B_2 are independent processes.

As a consequence of the above, one has the following proposition, which is a version of Girsanov's theorem. For a process X, $\underline{X}$ denotes the random element $\omega \mapsto X(\omega,\cdot)$.

Proposition 1. *It is assumed that* **A0**, **A1**, **A2**, **A7** *and* **A8** *obtain. Then, with respect to* $Q_{\alpha,s,1}$, $Z_{\alpha,f,1}$ *defined by*

$$Z_{\alpha,f,1} = \sqrt{\alpha}\,U_{\alpha,f,1} + \sqrt{1-\alpha}\,V_{\alpha,f,1}$$

is such that

$$Q_{\alpha,s,1} \circ \underline{Z}_{\alpha,s,1}^{-1} = P \circ \underline{B}_{\alpha}^{-1}.$$

Remark. In what follows, one will replace $Z_{\alpha,s,1}$ with the shorter Y_α.

3.2. Absolute Continuity and Radon-Nikodým Derivatives for P_{B_α} and P_{Y_α}

With the model assumed so far, only an implicit form of the likelihood ratio is available. That is, if P_{B_α} is the probability induced on $\mathcal{D}$ by P and $\underline{B}_\alpha$, and P_{Y_α} that induced by P and $\underline{Y}_\alpha$, obviously:

Proposition 2. *It is assumed that* **A0**, **A1**, **A2**, **A7** *and* **A8** *obtain. Then,* P_{Y_α} *and* P_{B_α} *are mutually absolutely continuous probability measures defined on* $\mathcal{D}$, *and, for* $f \in D[0,T]$,

– *almost surely, with respect to* P_{Y_α},

$$\frac{dP_{B_\alpha}}{dP_{Y_\alpha}}[f] = E_{P_{Y_\alpha}}\left[\ L_{\alpha,s,1}(\cdot,T)\ \middle|\ \underline{Y}_\alpha = f\ \right],$$

– *almost surely, with respect to* P_{B_α},

$$\frac{dP_{Y_\alpha}}{dP_{B_\alpha}}[f] = E_{P_{B_\alpha}}\left[\ \frac{1}{L_{\alpha,s,1}(\cdot,T)}\ \middle|\ \underline{Y}_\alpha = f\ \right].$$

It can be shown, as in the case for which $B_\alpha = B_1$, that the following corollary is valid.

Corollary. *It is assumed that* **A0**, **A1**, **A2**, **A7** *obtain. Then the following is true:* $E_P[L(\cdot,T)] < 1$, *and one has that* P_{Y_α} *is absolutely continuous with respect to* P_{B_α}.

3.3. Factorizations

Explicit expressions for the likelihood ratio require that the right-hand sides in the conclusion of Proposition 2 to be explicitly expressed as ordinary functions defined on $D[0,T]$. This is achieved through factorization by Y_α of the different components of $L_{\alpha,s,1}$.

When the evaluation maps, denoted ev, are taken as processes with respect to "lifted‘" probabilities of the form P_U, one will use the notation ev^{P_U} for ev. $\sigma_t^\circ(Y_\alpha)$ is the σ-field generated by $\{Y_\alpha(\cdot,s),\ s \le t\}$, and $\sigma_t(Y_\alpha)$ is $\sigma_t^\circ(Y_\alpha)$ completed with the sets of measure zero, with respect to P, belonging to $\mathcal{A}_t$. $\underline{\sigma}^\circ(Y_\alpha)$ and $\underline{\sigma}(Y_\alpha)$ denote the resulting filtrations.

Proposition 3. *It is assumed* **A0** *and* **A1** *obtain. Let* Y_α *denote a process with paths in* $D[0,T]$. *If* B_α *is adapted to* $\underline{\sigma}(Y_\alpha)$, *there exist, defined on* $(D[0,T],\mathcal{D},P_{Y_\alpha})$, *processes* $B_1^{Y_\alpha}$, $B_2^{Y_\alpha}$, *and* $B_\alpha^{Y_\alpha}$, *adapted to* $\underline{\mathcal{D}}$, *with paths in* $D[0,T]$, *such that, for*[4]

$$B_\alpha^{Y_\alpha} = \sqrt{\alpha}\ B_1^{Y_\alpha} + \sqrt{1-\alpha}\ \tilde{B}_2^{Y_\alpha},$$

$$\begin{aligned} P_{Y_\alpha} \circ \left[\underline{B}_1^{Y_\alpha}\right]^{-1} &= P \circ \underline{B}_1^{-1}, \\ P_{Y_\alpha} \circ \left[\underline{B}_2^{Y_\alpha}\right]^{-1} &= P \circ \underline{B}_2^{-1}, \\ P_{Y_\alpha} \circ \left[\underline{B}_\alpha^{Y_\alpha}\right]^{-1} &= P \circ \underline{B}_\alpha^{-1}, \end{aligned}$$

[4] $\tilde{B}_2^{Y_\alpha} = B_2^{Y_\alpha} - \beta_2$.

and, for $t \in [0,T]$, fixed, but arbitrary, almost surely, with respect to P,

$$\begin{aligned} B_1(\omega,t) &= B_1^{Y_\alpha}(Y_\alpha(\omega,\cdot),t), \\ B_2(\omega,t) &= B_2^{Y_\alpha}(Y_\alpha(\omega,\cdot),t), \\ B_\alpha(\omega,t) &= B_\alpha^{Y_\alpha}(Y_\alpha(\omega,\cdot),t). \end{aligned}$$

Proof. The notation used is as follows. For any process U such that $U(\omega,t-)$ makes sense,

$$\{\Delta U\}(\omega,t) = U(\omega,t) - U(\omega,t-).$$

The process

$$U^s(\omega,t) = \sum_{x\le t}\{\Delta U\}(\omega,x)$$

is then called the process of the jumps of U. The process U^c is subsequently defined as

$$U^c(\omega,t) = U(\omega,t) - U^s(\omega,t).$$

Then

$$\begin{aligned} \{\Delta B_\alpha\}(\omega,t) &= \sqrt{1-\alpha}\,\{\Delta B_2\}(\omega,t), \\ B_\alpha^s(\omega,t) &= \sqrt{1-\alpha}\;B_2(\omega,t), \\ B_\alpha^c(\omega,t) &= \sqrt{\alpha}\;B_1(\omega,t) - \sqrt{1-\alpha}\;\beta_2(t). \end{aligned}$$

Consequently, as B_α is adapted to $\underline{\sigma}(Y_\alpha)$, so are then B_α^s, and B_α^c, and thus B_1 and B_2. Since $D[0,T]$ is a metric space, it can be checked that, as in the purely Gaussian case (Mémin [18, 2.4. – Lemme, p. 8]), there is, defined on $(D[0,T],\mathcal{D},P_{Y_\alpha})$, a process $B_1^{Y_\alpha}$, adapted to $\underline{\mathcal{D}}$, with paths in $C[0,T]$, such that, for $t \in [0,T]$, fixed, almost surely, with respect to P,

$$B_1(\omega,t) = B_1^{Y_\alpha}(Y_\alpha(\omega,\cdot),t).$$

It thus suffices to obtain the analogous result for B_2.

As in the Gaussian case, there exists, for $t \in [0,T]$, fixed, but arbitrary, a modification $\overline{B}_2(\cdot,t)$, of $B_2(\cdot,t)$, which is adapted to $\sigma_t^\circ(Y_\alpha)$, and for which one has that

$$\overline{B}_2(\omega,t) = B_2^{Y_\alpha}(Y_\alpha(\omega,\cdot),t),$$

for some $B_2^{Y_\alpha}(\cdot,t)$ adapted to $\mathcal{D}_t$.

Let now $t_i^{(n)}$ denote the fraction $\frac{i}{2^n}T,\ 1 \leq i \leq 2^n$, and $\mathcal{T}_D^{(n)}$ the set

$$\left\{t_i^{(n)},\ 1 \leq i \leq 2^n\right\}.$$

One has that

$$\mathcal{T}_D^{(n)} \subset \mathcal{T}_D^{(n+1)},$$

and that $\mathcal{T}_D$, defined by

$$\mathcal{T}_D = \bigcup_{n=1}^{\infty} \mathcal{T}_D^{(n)},$$

is a dense subset of $[0,T]$. By construction, the paths of $B_2^{Y_\alpha}$, restricted to $\mathcal{T}_D$, are, almost surely, with respect to P_{Y_α}, restrictions of paths of B_2, a counting process associated with a Poisson process. So, given $n \in I\!N$, and $f \in D\,[0,T]$, one defines the set $\mathcal{T}_n^{Y_\alpha}[f]$ as follows:

$$\mathcal{T}_n^{Y_\alpha}[f] = \left\{t \in \mathcal{T}_D : B_2^{Y_\alpha}(f,t) \geq n\right\}.$$

The next step requires the following definitions:

$$T_n^{Y_\alpha}[f] = \begin{cases} T\,, & \text{if} \quad \mathcal{T}_n^{Y_\alpha}[f] = \emptyset\,, \\ \inf \mathcal{T}_n^{Y_\alpha}[f]\,, & \text{if} \quad \mathcal{T}_n^{Y_\alpha}[f] \neq \emptyset \end{cases}$$

and, for $t \in [0,T]$,

$$\hat{B}_2^{Y_\alpha}(f,t) = \sum_{n=1}^{\infty} I_{[\![T_n^{Y_\alpha},T]\!]}(f,t).$$

One has that

$$\left\{f \in D\,[0,T] : \hat{B}_2^{Y_\alpha}(f,t) = n\right\} = \left\{f \in D\,[0,T] : T_n^{Y_\alpha}[f] \leq t < T_{n+1}^{Y_\alpha}[f]\right\},$$

and thus that

$$\left\{\omega \in \Omega : \hat{B}_2^{Y_\alpha}(Y_\alpha(\omega,\cdot),t) = n\right\} = \left\{\omega \in \Omega : T_n^{Y_\alpha}[Y_\alpha(\omega,\cdot)] \leq t < T_{n+1}^{Y_\alpha}[Y_\alpha(\omega,\cdot)]\right\}.$$

Let

$$A = \left\{\omega \in \Omega : B_2^{Y_\alpha}(Y_\alpha(\omega,\cdot),s) < n,\ s \in \mathcal{T}_D\right\}.$$

As

$$\left\{\omega \in \Omega : B_2^{Y_\alpha}\left(Y_\alpha(\omega,\cdot), s\right) \geq n,\ s \in \mathcal{T}_D\right\}$$
$$= \left\{\omega \in \Omega : \overline{B}_2(\omega, s) \geq n,\ s \in \mathcal{T}_D\right\},$$

it follows that

$$T_n^{Y_\alpha}\left[Y_\alpha(\omega,\cdot)\right]$$
$$= I_A(\omega)\, T + I_{A^c}(\omega) \inf\left\{s \in \mathcal{T}_D : B_2^{Y_\alpha}\left(Y_\alpha(\omega,\cdot), s\right) \geq n\right\}$$
$$= I_A(\omega)\, T + I_{A^c}(\omega) \inf\left\{s \in \mathcal{T}_D : \overline{B}_2(\omega, s) \geq n\right\}.$$

Let N denote a measurable set of measure zero, with respect to P, such that, for $\omega \in N^c$,

$$\overline{B}_2(\omega, s) = B_2(\omega, s),\ s \in \mathcal{T}_D.$$

Then, for $\omega \in N^c$,

$$T_n^{Y_\alpha}\left[Y_\alpha(\omega,\cdot)\right] = I_{A\cap N^c}(\omega)\, T + I_{A^c\cap N^c}(\omega) \inf\left\{s \in \mathcal{T}_D : B_2(\omega, s) \geq n\right\}.$$

The process B_2, being separable and continuous in probability, every dense subset is a separant, so that

$$\inf\left\{s \in \mathcal{T}_D : B_2(\omega, s) \geq n\right\} = \inf\left\{t \in [0,T] : B_2(\omega, t) \geq n\right\}.$$

Define thus

$$\tilde{T}_n[\omega] = \begin{cases} T, & \text{if } \ B_2(\omega, T) < n, \\ \inf\left\{t \in [0,T] : B_2(\omega, t) \geq n\right\}, & \text{if } \ B_2(\omega, T) \geq n. \end{cases}$$

Then, almost surely, with respect to P,

$$T_n^{Y_\alpha}\left[Y_\alpha(\omega, t)\right] = \tilde{T}_n[\omega],$$

and, consequently, for $t \in [0,T]$, fixed, but arbitrary,

$$\hat{B}_2^{Y_\alpha}\left(Y_\alpha(\omega,\cdot), t\right) = B_2(\omega, t).$$

Thus, with respect to P_{Y_α}, $\hat{B}_2^{Y_\alpha}$ is a Poisson process restricted to $[0,T]$, and $T_n^{Y_\alpha}$, being one of the times of discontinuity of $\hat{B}_2^{Y_\alpha}$, is a stopping time for $\underline{\mathcal{D}}$. In the sequel, $\hat{B}_2^{Y_\alpha}$ will be dentoted $B_2^{Y_\alpha}$, and $\tilde{B}_2^{Y_\alpha}$ will be the Poisson martingale

$$\left\{B_2^{Y_\alpha}(\omega, t) - \beta_2(t),\ (\omega, t) \in \Omega \times [0,T]\right\}.$$

Corollary. *Let* $\sigma_t^{Y_\alpha}(B_\alpha)$ *be the* σ*-field generated by* $\sigma_t^\circ(B_\alpha)$ *and the sets of* $\sigma_t^\circ(Y_\alpha)$ *which have measure zero for* P*. Similarly, let* $\sigma_t^{Y_\alpha}\left(B_\alpha^{Y_\alpha}\right)$ *be the* σ*-field generated by* $\sigma_t^\circ\left(B_\alpha^{Y_\alpha}\right)$ *and the sets of* $\mathcal{D}_t$ *which have measure zero for* P_{Y_α}*. Then*

$$\sigma_t^{Y_\alpha}(B_\alpha) = \underline{Y}_\alpha^{-1}\left\{\sigma_t^{Y_\alpha}\left(B_\alpha^{Y_\alpha}\right)\right\}.$$

Proposition 4. *It is assumed that* **A0***,* **A1***,* **A4** *and* **A5** *obtain. There is then a process* $B_\alpha^{Y_\alpha}$*, defined on the base* $(D[0,T], \mathcal{D}, P_{Y_\alpha})$*, adapted to* $\underline{\mathcal{D}}$*, such that, for* $t \in [0,T]$*, fixed, but arbitrary, almost surely, with respect to* P_{Y_α}*,*

$$ev^{P_{Y_\alpha}}(f,t) = \alpha \int_0^t s(f,x)\,\beta_1(dx) + B_\alpha^{Y_\alpha}(f,t),$$

with **A1** *true for* $B_\alpha^{Y_\alpha}$*.*

Proof. Define $B_\alpha^{Y_\alpha}$ as

$$B_\alpha^{Y_\alpha}(f,t) = ev^{P_{Y_\alpha}}(f,t) - \alpha \int_0^t s(f,x)\,\beta_1(dx).$$

By definition one thus has that the map

$$t \mapsto B_\alpha^{Y_\alpha}(f,t)$$

is, almost surely, with respect to P_{Y_α}, in $D[0,T]$. But the paths of $B_\alpha^{Y_\alpha}$ that are not in $D[0,T]$ can be taken as continuous to the right, thanks to Lemma 1. It is furthermore adapted to $\underline{\mathcal{D}}$. Finally, for $t \in [0,T]$, fixed, but arbitrary, almost surely, with respect to P,

$$B_\alpha^{Y_\alpha}(Y_\alpha(\omega,\cdot),t) = Y_\alpha(\omega,t) - \alpha \int_0^t s(Y_\alpha(\omega,\cdot),x)\,\beta_1(dx) = B_\alpha(\omega,t).$$

Thus, with respect to P_{Y_α}, $B_\alpha^{Y_\alpha}$ is a Lévy process. But

$$\left\{\Delta B_\alpha^{Y_\alpha}\right\}(Y_\alpha(\omega,\cdot),t) = \{\Delta B_\alpha\}(\omega,t) = \sqrt{1-\alpha}\,\{\Delta B_2\}(\omega,t),$$

so that the jump process of $B_\alpha^{Y_\alpha}$ is, with respect to P_{Y_α}, a Poisson process. Consequently, its continuous part is a generalized Brownian motion. □

Proposition 5. *It is assumed that* **A0***,* **A1***,* **A4** *and* **A5** *obtain. Let then the process* M *be defined on the base* $(\Omega, \mathcal{A}, P)$*, and for the filtration* $\underline{\sigma}^\circ(Y_\alpha)$*, as*

$$M(\omega,t) = \int_0^t s(Y_\alpha(\omega,\cdot),x)\,B_1(\omega,dx).$$

One can then find a process M^{Y_α}, *defined on the base* $(D[0,T], \mathcal{D}, P_{Y_\alpha})$, *and adapted to the filtration* $\underline{\mathcal{D}}$, *with the following properties: its paths are continuous to the right and belong, almost surely, with respect to* P_{Y_α}, *to* $C[0,T]$. *Furthermore, for a generalized Brownian motion* $B_1^{Y_\alpha}$, *with variance* β_1, *defined on* $(D[0,T], \mathcal{D}, P_{Y_\alpha})$, *and adapted to* $\underline{\mathcal{D}}$, *for* $t \in [0,T]$, *fixed, but arbitrary, almost surely with respect to* P_{Y_α},

$$M^{Y_\alpha}(f,t) = \int_0^t s(f,x)\, B_1^{Y_\alpha}(f,dx),$$

and

$$M^{Y_\alpha}(Y_\alpha(\omega,\cdot),t) = M(\omega,t).$$

Proof. One first considers simple processes s of the form

$$s(f,t) = I_A(f)\, I_{]u,v]}(t),\ u < v,\ A \in \mathcal{D}_u.$$

If one sets $B = \underline{Y}_\alpha^{-1}[A]$, B then belongs to $\sigma_u^\circ(Y_\alpha)$, and, by definition,

$$\int_0^t s(Y_\alpha(\omega,\cdot),x)\, B_1(\omega,dx) = I_A(Y_\alpha(\omega,\cdot))\{B_1(\omega,t\wedge v) - B_1(\omega,t\wedge u)\}.$$

Now, from Proposition 3, one has that $B_1(\omega,t) = B_1^{Y_\alpha}(Y_\alpha(\omega,\cdot),t)$, so that, setting

$$M^{Y_\alpha}(f,t) = \int_0^t s(f,x)\, B_1^{Y_\alpha}(f,dx),$$

one has the result for simple processes which are products of the appropriate indicators I_A and $I_{]u,v]}$.

Let now $\mathcal{S}$ denote the class of processes s, defined on $D[0,T] \times [0,T]$, which are progressively measurable for $\underline{\mathcal{D}}$, bounded and such that[5]

$$\{s \circ Y_\alpha\} \cdot B_1 = \left\{s \cdot B_1^{Y_\alpha}\right\} \circ Y_\alpha,$$

as stated. $\mathcal{S}$ is a vector space containing all constants. It is closed for uniform and monotone convergence. If $\mathcal{S}_f$ denotes the subspace of $\mathcal{S}$ made of finite linear combinations of simple processes of the form

$$s(f,t) = I_A(f)\, I_{]u,v]}(t),\ u < v,\ A \in \mathcal{D}_u,$$

[5] $\circ$ denotes composition and $\cdot$ stochastic integration.

one gets a subspace which is stable for multiplication. The monotone class theorem then yields that $\mathcal{S}$ contains all bounded predictable processes, and thus all elementary processes in the sense of Von Weizsäcker [23, P. 72]. The properties of the stochastic integral suffice then to claim that the proposition's assertion is true. □

Remark. The same proof yields, *mutatis mutandis*, the same result with B_1 replaced with B_α, and $B_1^{Y_\alpha}$ replaced with $B_\alpha^{Y_\alpha}$.

3.4. Likelihood Ratios for P_{B_α} and P_{Y_α}

This section contains the likelihood ratio formulae for the detection of the signal in Y_α when the noise is B_α. They only depend, as it should be, on the signal sent, the statistics of the noise, and the received waveform. Checking that the formulae are correct is routine, given the preceding factorizations (Propositions 3 to 5).

Theorem 1. *It is assumed that* **A0***,* **A1***,* **A4***,* **A5***,* **A7** *with* $\phi = 1$*, and* **A8** *obtain. Then:*

- P_{Y_α} *and* P_{B_α} *are mutually absolutely continuous;*

- *for almost every* $f \in D[0,T]$*, with respect to* P_{Y_α}*,*

$$\ln\left[\frac{dP_{B_\alpha}}{dP_{Y_\alpha}}\right](f) = -\sqrt{\alpha}\int_0^T s(f,x)\,B_1^{Y_\alpha}(f,dx) - \frac{\alpha}{2}\int_0^T s^2(f,x)\,\beta_1(dx);$$

- *for almost every* $f \in D[0,T]$*, with respect to* P_{Y_α}*,*

$$-\ln\left[\frac{dP_{B_\alpha}}{dP_{Y_\alpha}}\right](f) = \int_0^T s(f,x)\,ev^{P_{Y_\alpha}}(f,dx) - \frac{\alpha}{2}\int_0^T s^2(f,x)\,\beta_1(dx) - \sqrt{1-\alpha}\int_0^T s(f,x)\,\tilde{B}_2^{Y_\alpha}(f,dx) = \ln\left[\frac{dP_{Y_\alpha}}{dP_{B_\alpha}}\right](f);$$

– *for almost every $f \in D[0,T]$, with respect to P_{B_α},*

$$-\ln\left[\frac{dP_{B_\alpha}}{dP_{Y_\alpha}}\right](f) = \int_0^T s(f,x)\, ev^{P_{B_\alpha}}(f,dx)$$
$$-\frac{\alpha}{2}\int_0^T s^2(f,x)\,\beta_1(dx) - \sqrt{1-\alpha}\int_0^T s(f,x)\,\tilde{B}_2^{Y_\alpha,B_\alpha}(f,dx)$$
$$= \ln\left[\frac{dP_{Y_\alpha}}{dP_{B_\alpha}}\right](f),$$

where $\tilde{B}_2^{Y_\alpha,B_\alpha}$ is the representation of $\tilde{B}_2^{Y_\alpha}$ with respect to P_{B_α}. It can be checked that the two processes are identical.

4. Path Requirements for Absolute and Mutual Absolute Continuity

Mutual absolute continuity has been obtained under two conditions, namely that the random variable $L_{\alpha,s,1}(\cdot,T)$ has expectation one, and that the "signal-plus-noise" process is the solution of a stochastic differential equation. The first condition is not, given the context, "natural" in that "natural" conditions are in terms of the finiteness of the signal's energy, i.e., in the chosen context, that of the signal's RKHS norm. As such, it is then a "path condition," rather than an "expectation condition" This section is thus devoted firstly to the investigation of mutual absolute continuity in terms of such path conditions. In the second part of the section, one studies innovation representations of "signal-plus-noise" models which are the usual way to transform the received signal into the solution of a stochastic differential equation.

4.1. Sufficient path Conditions for Absolute and Mutual Absolute Continuity

In what follows one keeps the same assumptions that prevailed to this point. The first result is the next proposition (Proposition 6) which will be proved as a sequence of lemmas: it determines conditions for mutual absolute continuity in terms of square integrability of the "derivative" of the signal paths. As only assumption **A4**, and not assumptions **A4** and **A6**, do follow from the RKHS requirement, Proposition 6 must be "weakened," and that leads to Proposition 7 which however "calls on" Proposition 6. Proposition 6 requires assumptions that one is unlikely to be able to check, but its corollary says that the Cramér-Hida framework is sufficient to insure that these assumptions obtain.

Proposition 6. *It is assumed that* **A0**, **A1**, **A4**, **A5** *and* **A6** *obtain, and furthermore that s is predictable and that both*[6]

$$P_{B_\alpha}\left(f \in D\left[0,T\right]: \int_0^T |s|\left(f,x\right)\beta_2\left(dx\right) < \infty\right) = 1,$$

and

$$P_{Y_\alpha}\left(f \in D\left[0,T\right]: \int_0^T |s|\left(f,x\right)\beta_2\left(dx\right) < \infty\right) = 1$$

obtain. Then Theorem 1 is valid.

Proof. It shall be presented in a sequence of lemmas (Lemma 5 to Lemma 9), followed by a short conclusion (Epilogue to Proposition 6).

Remark. From Proposition 3, given the assumptions **A0**, **A1**, **A4** and **A5** of the present proposition, one has that there exist, on $\left(D\left[0,T\right], \mathcal{D}, P_{Y_\alpha}\right)$,

– a generalized Brownian motion $B_1^{Y_\alpha}$, adapted to $\underline{\mathcal{D}}$, with

$$V\left[B_1^{Y_\alpha}\left(\cdot,t\right)\right] = \beta_1\left(t\right),$$

– a Poisson process $B_2^{Y_\alpha}$, adapted to $\underline{\mathcal{D}}$, independent of $B_1^{Y_\alpha}$, for which

$$E\left[B_2^{Y_\alpha}\left(\cdot,t\right)\right] = \beta_2\left(t\right),$$

such that, for $t \in \left[0,T\right]$, fixed, but arbitrary, for almost every $f \in D\left[0,T\right]$, with respect to P_{Y_α},

$$ev^{P_{Y_\alpha}}\left(f,t\right) = \alpha \int_0^t s\left(f,x\right)\beta_1\left(dx\right) + B_\alpha^{Y_\alpha}\left(f,t\right),$$

with $B_\alpha^{Y_\alpha}\left(f,t\right) = \sqrt{\alpha}\, B_1^{Y_\alpha}\left(f,t\right) + \sqrt{1-\alpha}\, \tilde{B}_\alpha^{Y_\alpha}\left(f,t\right)$, and $\tilde{B}_2^{Y_\alpha} = B_2^{Y_\alpha} - \beta_2$.

Furthermore, from Lemma 1, one has that s can be replaced by $\tilde{s}$, for which one has the further following properties:

– the map $\tilde{\nu}\left(f,t\right) = \int_0^t \tilde{s}^2\left(f,x\right)\beta_1\left(dx\right)$ is continuous in $\overline{I\!R}_+$,

[6]These assumptions on s are needed in the proof of Lemma 8.

– one has that

$$P_{B_\alpha}\left(f \in D[0,T] : \|\tilde{s}(f,\cdot)\|^2_{L_2[\beta_1]} < \infty\right) = 1,$$

and that

$$P_{Y_\alpha}\left(f \in D[0,T] : \|\tilde{s}(f,\cdot)\|^2_{L_2[\beta_1]} < \infty\right) = 1,$$

– and, for $t \in [0,T]$, fixed, but arbitrary, for almost every $f \in D[0,T]$, with respect to P_{Y_α},

$$ev^{P_{Y_\alpha}}(f,t) = \alpha \int_0^t \tilde{s}(f,x)\,\beta_1(dx) + B_\alpha^{Y_\alpha}(f,t).$$

The following steps restrict the problem to paths $f \in D[0,T]$ for which $\tilde{\nu}(f,T) < \infty$. The (strict) stopping time $T_n : D[0,T] \longrightarrow [0,T]$ is defined by the equality

$$T_n(f) = \begin{cases} T, & \text{if } \{t \in [0,T] : \tilde{\nu}(f,t) \geq n\} = \emptyset, \\ \inf\{t \in [0,T] : \tilde{\nu}(f,t) \geq n\}, & \text{if } \{t \in [0,T] : \tilde{\nu}(f,t) \geq n\} \neq \emptyset. \end{cases}$$

It should be noted that $\lim_{n\uparrow\infty} T_n(f) = T$ if and only if $t < T$ implies $\tilde{\nu}(f,t) < \infty$.

Further definitions are needed, as follows:

$$\begin{aligned} \tilde{D}[0,T] &= \{f \in D[0,T] : \tilde{\nu}(f,T) < \infty\}, \\ \tilde{D} &= D \cap \tilde{D}[0,T],\ D \in \mathcal{D}, \\ \tilde{P}_{Y_\alpha}\left(\tilde{D}\right) &= P_{Y_\alpha}\left(D \cap \tilde{D}[0,T]\right), \\ \tilde{\mathcal{D}} &= \mathcal{D} \cap \tilde{D}[0,T], \\ \underline{\tilde{\mathcal{D}}} &= \underline{\mathcal{D}} \cap \tilde{D}[0,T]. \end{aligned}$$

The process $\widetilde{ev}_n^{P_{Y_\alpha}}$ is subsequently defined on the base $\left(\tilde{D}[0,T], \tilde{\mathcal{D}}, \tilde{P}_{Y_\alpha}\right)$, with respect to the filtration $\underline{\tilde{\mathcal{D}}}$, as

$$\widetilde{ev}_n^{P_{Y_\alpha}}(f,t) = \begin{cases} ev^{P_{Y_\alpha}}(f,t), & \text{if } (f,t) \in [\![0,T_n[\![\,. \\ ev^{P_{Y_\alpha}}(f,t) - \alpha\int_{T_n}^t \tilde{s}(f,x)\,\beta_1(dx), & \text{if } (f,t) \in [\![T_n,T]\!]\,. \end{cases}$$

This process can be rewritten as

$$\widetilde{ev}_n^{P_{Y_\alpha}}(f,t) = f(t) - I_{[\![T_n,T]\!]}(f,t)\left\{\alpha\int_0^t I_{[\![T_n,T]\!]}(f,x)\,\tilde{s}(f,x)\,\beta_1(dx)\right\},$$

and this shows first that $\left\{\widetilde{ev}_n^{P_{Y_\alpha}}(f,t),\ t\in[0,T]\right\}\in D[0,T]$, as $f\in\tilde{D}[0,T]$, and then that, on $[\![0,T_n]\!]$,

$$\widetilde{ev}_n^{P_{Y_\alpha}}(f,t) = f(t) = ev(f,t).$$

One last definition yields the progressively measurable, bounded process $\tilde{s}_n$, given by the relation

$$\tilde{s}_n(f,t) = I_{[\![0,T_n]\!]}(f,t)\,\tilde{s}(f,t).$$

Let $J:\tilde{D}[0,T]\longrightarrow D[0,T]$ be the (injection) map defined by the relation $J(f)=f$. If E is a Borel set of $I\!R$,

$$\begin{aligned}[ev(\cdot,t)\circ J]^{-1}(E) &= \left\{f\in\tilde{D}[0,T]:\ ev(J(f),t)\in E\right\}\\ &= \tilde{D}[0,T]\cap\{f\in D[0,T]:\ ev(f,t)\in E\}\\ &\in \widetilde{\mathcal{D}}_t.\end{aligned}$$

The restriction of $\tilde{s}_n$ to $\tilde{D}[0,T]$ has thus the measurability properties of $\tilde{s}_n$ as defined on $D[0,T]$, and it will not then be necessary to introduce one more notation to distinguish one situation from the other. In particular, an integral of the form

$$\int_0^t \tilde{s}_n\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot),x\right)\beta_1(dx)$$

will be well defined for $f\in\tilde{D}[0,T]$.

Define now $\tilde{B}_\alpha^{Y_\alpha}$ as the restriction of $B_\alpha^{Y_\alpha}$ to $\tilde{D}[0,T]$. One can check that

$$\tilde{P}_{Y_\alpha}\circ\left[\underline{\tilde{B}}_\alpha^{Y_\alpha}\right]^{-1} = P_{Y_\alpha}\circ\left[\underline{B}_\alpha^{Y_\alpha}\right]^{-1}.$$

One may then state:

Lemma 2. *For every* $f\in\tilde{D}[0,T]$,

$$T_n\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot)\right) = T_n(f),$$

and, for $t \in [0,T]$, fixed, but arbitrary, almost surely, with respect to $\tilde{P}_{Y_\alpha}$,

$$\widetilde{ev}_n^{P_{Y_\alpha}}(f,t) = \alpha \int_0^t \tilde{s}_n\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot),x\right)\beta_1(dx) + \tilde{B}_\alpha^{Y_\alpha}(f,t).$$

Proof. Let $D_t^{(n)} = \{f \in D[0,T] : T_n(f) = t\} \in \mathcal{D}_t$. The function $I_{D_t^{(n)}}$ has then the representation

$$I_{D_t^{(n)}}(f) = F(ev(f,t_i),\ 0 \le t_i \le t,\ i \in I\!N),$$

the map $F : I\!R^\infty \longrightarrow I\!R$ being measurable. But, as $T_n(f) = t$, as seen above, for $i \in I\!N$,

$$\widetilde{ev}_n^{P_{Y_\alpha}}(f,t_i) = f(t_i) = ev(f,t_i),$$

so that

$$F(ev(f,t_i),\ 0 \le t_i \le t,\ i \in I\!N) = F\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,t_i),\ 0 \le t_i \le t,\ i \in I\!N\right),$$

and consequently that

$$I_{D_t^{(n)}}(f) = I_{D_t^{(n)}}\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot)\right),$$

which proves the first assertion of the lemma.

The same reason (and the definition of $\tilde{s}_n$) yields that

$$\tilde{s}_n(f,t) = \tilde{s}_n\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot),t\right).$$

Finally, when $t < T_n(f)$, and $f \in \tilde{D}[0,T]$,

$$\begin{aligned}
\widetilde{ev}_n^{P_{Y_\alpha}}(f,t) &= ev^{P_{Y_\alpha}}(f,t) \\
&= \alpha \int_0^t \tilde{s}(f,x)\beta_1(dx) + \tilde{B}_\alpha^{Y_\alpha}(f,t) \\
&= \alpha \int_0^t \tilde{s}_n(f,x)\beta_1(dx) + \tilde{B}_\alpha^{Y_\alpha}(f,t) \\
&= \alpha \int_0^t \tilde{s}_n\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot),x\right)\beta_1(dx) + \tilde{B}_\alpha^{Y_\alpha}(f,t),
\end{aligned}$$

and, when $t \geq T_n(f)$,

$$\begin{aligned}
\widetilde{ev}_n^{P_{Y_\alpha}}(f,t) &= ev^{P_{Y_\alpha}}(f,t) - \alpha \int_{T_n}^{t} \tilde{s}(f,x)\,\beta_1(dx) \\
&= \alpha \int_0^{T_n} \tilde{s}(f,x)\,\beta_1(dx) + \tilde{B}_\alpha^{Y_\alpha}(f,t) \\
&= \alpha \int_0^{t} \tilde{s}_n(f,x)\,\beta_1(dx) + \tilde{B}_\alpha^{Y_\alpha}(f,t) \\
&= \alpha \int_0^{t} \tilde{s}_n\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot),x\right)\beta_1(dx) + \tilde{B}_\alpha^{Y_\alpha}(f,t).
\end{aligned}$$

Define now $\widetilde{\Psi}_n : \tilde{D}[0,T] \longrightarrow I\!R$ by the relation

$$\begin{aligned}
\ln\left[\widetilde{\Psi}_n(f)\right] &= -\sqrt{\alpha}\int_0^{T} \tilde{s}_n\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot),x\right)\tilde{B}_1^{Y_\alpha}(f,dx) \\
&\quad -\frac{\alpha}{2}\int_0^{T} \tilde{s}_n^2\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot),x\right)\beta_1(dx).
\end{aligned}$$

One then has, since, by definition, $\int_0^T \tilde{s}_n^2(f,x)\,\beta_1(dx) \leq n$, □

Lemma 3. $E_{\tilde{P}_{Y_\alpha}}\left[\widetilde{\Psi}_n\right] = 1.$

Lemma 4. *For $f \in D[0,T]$, let Ψ be defined by*

$$\ln\left[\Psi(f)\right] = -\sqrt{\alpha}\int_0^T s(f,x)\,B_1^{Y_\alpha}(f,dx) - \frac{\alpha}{2}\int_0^T s^2(f,x)\,\beta_1(dx),$$

and let $\widetilde{\Psi}$ denote the restriction of Ψ to $\tilde{D}[0,T]$ $\left(\widetilde{\Psi} = \Psi \circ J\right)$. Then,

$$\lim_{n\to\infty} \widetilde{\Psi}_n(f) = \widetilde{\Psi}(f),$$

in probability, with respect to $\tilde{P}_{Y_\alpha}$.

Proof. For $(f,t) \in [\![0,T_n]\!]$,

$$\widetilde{ev}_n^{P_{Y_\alpha}}(f,t) = f(t),$$

so that, by monotone convergence, for almost every $f \in \tilde{D}[0,T]$, with respect to $\tilde{P}_{Y_\alpha}$,

$$\begin{aligned} &\lim_{n\to\infty} \int_0^T \tilde{s}_n^2 \left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot), x \right) \beta_1(dx) \\ &= \lim_{n\to\infty} \int_0^T I_{[\![0,T_n]\!]}(f,x)\, \tilde{s}^2(f,x)\, \beta_1(dx) \\ = &\int_0^T \tilde{s}^2(f,x)\, \beta_1(dx). \end{aligned}$$

Furthermore, for almost every $f \in \tilde{D}[0,T]$, with respect to $\tilde{P}_{Y_\alpha}$, for n large enough, $T_n(f) = T$, so that, for that same f, for n large enough,

$$\sup_{0\le t\le T} \left\{ |\tilde{s}(f,t)|\, I_{]\!]T_n,T]\!]}(f,t) \right\} = 0.$$

Consequently

$$\lim_n \tilde{P}_{Y_\alpha}\left(f \in \tilde{D}[0,T] : \sup_{0\le t\le T} \left\{ |\tilde{s}(f,t)|\, I_{]\!]T_n,T]\!]}(f,x) \right\} > \epsilon \right) = 0,$$

and thus (continuity of the integral: see – Von Weizsäcker et al [23, 5.5.3, p. 98]), if

$$\mathcal{J}_n(f,t) = \int_0^t \tilde{s}_n(f,x)\, \tilde{B}_1^{Y_\alpha}(f,dx) - \int_0^t \tilde{s}(f,t)\, \tilde{B}_1^{Y_\alpha}(f,dx),$$

then

$$\lim_n \tilde{P}_{Y_\alpha}\left(f \in \tilde{D}[0,T] : \sup_{0\le t\le T} |\mathcal{J}_n(f,t)| > \epsilon \right) = 0. \qquad \square$$

Lemma 5. *Let the probability measure $\tilde{Q}_n^{Y_\alpha}$ be defined on $\mathcal{D}$ by the following relation:*

$$\tilde{Q}_n^{Y_\alpha} = \tilde{P}_{Y_\alpha} \circ \left[\widetilde{ev}_n^{P_{Y_\alpha}} \right]^{-1}.$$

Then, for $A \in \mathcal{D}_{T_n}$,

$$\tilde{Q}_n^{Y_\alpha}(A) = \tilde{P}_{Y_\alpha}\left(\tilde{D}[0,T] \cap A \right) = P_{Y_\alpha}(A).$$

Proof. First, one can show, as in the continuous case (Von Weizsäcker et al [23, 2.2.6, page 35]), that $\mathcal{D}_{T_n} = \sigma\left(ev^{T_n}(\cdot,t),\ t\in[0,T]\right)$. Indeed, let $\theta_n : D[0,T] \longrightarrow D[0,T]$ be defined by the relation

$$ev(\theta_n(f),t) = ev^{T_n}(f,t) = f(t\wedge T_n(f)).$$

Let then $f_0 \in D[0,T]$ be fixed, but arbitrary, and set $t_0 = T_n(f_0)$. If $t \le t_0$, then

$$ev(f_0,t) = f_0(t) = f_0(t\wedge T_n(f_0)) = ev^{T_n}(f_0,t) = ev(\theta_n(f_0),t).$$

Thus, for every ψ adapted to $\mathcal{D}_{t_0}$, $\psi(f_0) = \psi(\theta_n(f_0))$. In particular,

$$I_{\{T_n\le t_0\}}(\theta_n(f_0)) = I_{\{T_n\le t_0\}}(f_0) = 1.$$

Consequently, for every ϕ adapted to $\mathcal{D}_{T_n}$,

$$\phi(\theta_n(f_0)) = \phi(\theta_n(f_0))\, I_{\{T_n\le t_0\}}(\theta_n(f_0)).$$

But $\phi I_{\{T_n\le t_0\}}$ is adapted to $\mathcal{D}_{t_0}$, so that

$$\phi(\theta_n(f_0)) = \phi(f_0)\, I_{\{T_n\le t_0\}}(f_0) = \phi(f_0).$$

As ϕ is adapted to $\mathcal{D}$, it has, for fixed, measurable F and $t_i \in [0,T]$, $i \in I\!N$, the following representation:

$$\phi(f) = F(ev(f,t_i),\ 0\le t_i\le T,\ i\in I\!N).$$

Using the relation $\phi(f) = \phi(\theta_n(f))$, valid for $f \in \mathcal{D}_{T_n}$, one has then that

$$\begin{aligned}\phi(f) = \phi(\theta_n(f)) &= F(ev(f\circ\theta_n,t_i),\ 0\le t_i\le T,\ i\in I\!N)\\ &= F\left(ev^{T_n}(f,t_i),\ 0\le t_i\le T,\ i\in I\!N\right)\end{aligned}$$

which is adapted to $\sigma\left(ev^{T_n}(\cdot,t),\ t\in[0,T]\right)$. This establishes that $\mathcal{D}_{T_n}$ is contained in $\sigma\left(ev^{T_n}(\cdot,t),\ t\in[0,T]\right)$.

The reverse inclusion is obtained by noticing that ev is continuous to the right, so that (Von Weizsäcker et al [23, p. 41]) $ev^{T_n}(\cdot,t)$ is adapted to $\mathcal{D}_{t\wedge T_n}$, and thus that

$$\sigma\left(ev^{T_n}(\cdot,t),\ t\in[0,T]\right) \subseteq \sigma\left(\cup_{t\in[0,T]}\mathcal{D}_{t\wedge T_n}\right) \subseteq \mathcal{D}_{T_n}.$$

Finally, for B Borel in $I\!R$, and $A = \left\{f \in D[0,T] : \ ev^{T_n}(f,t) \in B\right\}$,

$$\begin{aligned}
\tilde{Q}_n^{Y_\alpha}(A) &= \tilde{P}_{Y_\alpha}\left(f \in \tilde{D}[0,T] : \ \widetilde{ev}_n^{Y_\alpha}(f,\cdot) \in A\right) \\
&= \tilde{P}_{Y_\alpha}\left(f \in \tilde{D}[0,T] : \ \widetilde{ev}_n^{Y_\alpha}(f, t \wedge T_n(f)) \in B\right) \\
&= \tilde{P}_{Y_\alpha}\left(f \in \tilde{D}[0,T] : \ ev_n^{T}(f,t) \in B\right) \\
&= P_{Y_\alpha}(A).
\end{aligned}$$

One then finishes the proof with a monotone class argument. □

Lemma 6. *The assumptions are those of Proposition 6. The sequence* $\left\{\tilde{\Psi}_n, \ n \in I\!N\right\}$ *is then uniformly integrable for* $\tilde{P}_{Y_\alpha}$.

Proof. Let $\sigma_t^{\tilde{P}_{Y_\alpha}}\left(\widetilde{ev}_n^{P_{Y_\alpha}}\right)$ denote the σ-field generated by $\left\{\widetilde{ev}_n^{P_{Y_\alpha}}(\cdot, s), \ s \leq t\right\}$, and the sets of $\widetilde{\mathcal{D}}_t$ which have measure zero for $\tilde{P}_{Y_\alpha}$. By Lemma 2, the following obtains, almost surely, with respect to $\tilde{P}_{Y_\alpha}$, for $t \in [0,T]$, fixed, but arbitrary:

$$\widetilde{ev}_n^{P_{Y_\alpha}}(f,t) = \alpha \int_0^t \tilde{s}_n\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot), x\right) \beta_1(dx) + \tilde{B}_\alpha^{Y_\alpha}(f,t).$$

$\tilde{B}_\alpha^{Y_\alpha}(\cdot,t)$ is thus adapted to $\sigma_t^{\tilde{P}_{Y_\alpha}}\left(\widetilde{ev}_n^{P_{Y_\alpha}}\right)$.

The setup is now as follows. The underlying probability is $\tilde{P}_{Y_\alpha}$. $\widetilde{ev}_n^{P_{Y_\alpha}}$ is a process with paths in $D[0,T]$. $\tilde{B}_\alpha^{Y_\alpha}$ is a process for which **A1** obtains. As $\tilde{B}_\alpha^{Y_\alpha}(\cdot,t)$ is adapted to $\sigma_t^{\tilde{P}_{Y_\alpha}}\left(\widetilde{ev}_n^{P_{Y_\alpha}}\right)$, it follows from Proposition 3 that there is a process $\widetilde{B}_{\alpha,n}^{Y_\alpha}$ which factors $\tilde{B}_\alpha^{Y_\alpha}$ through $\widetilde{ev}_n^{P_{Y_\alpha}}$:

$$\tilde{B}_\alpha^{Y_\alpha}(f,t) = \widetilde{B}_{\alpha,n}^{Y_\alpha}\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot), t\right),$$

almost surely, with respect to $\tilde{P}^{Y_\alpha}$, and for which , with respect to the probability measure $\tilde{Q}_n^{Y_\alpha}$, defined in Lemma 5, **A1** obtains. One then sets, for $f \in D[0,T]$, almost surely, with respect to $\tilde{Q}_n^{Y_\alpha}$,

$$\ln\left[\Phi_n(f)\right] = -\sqrt{\alpha}\int_0^T \tilde{s}_n(f,x)\,\widetilde{B}_{1,n}^{Y_\alpha}(f,dx) - \frac{\alpha}{2}\int_0^T \tilde{s}_n^2(f,x)\,\beta_1(dx).$$

By Proposition 5, almost surely, with respect to $\tilde{P}_{Y_\alpha}$,

$$\tilde{\Psi}_n(f) = \Phi_n\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot)\right).$$

Furthermore, the equation

$$\widetilde{ev}_n^{P_{Y_\alpha}}(f,t) = \alpha \int_0^t \tilde{s}_n\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot),x\right)\beta_1(dx) + \tilde{B}_\alpha^{Y_\alpha}(f,t),$$

can be rewritten as

$$\widetilde{ev}_n^{P_{Y_\alpha}}(f,t) = \alpha \int_0^t \tilde{s}_n\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot),x\right)\beta_1(dx) + \widetilde{B}_{\alpha,n}^{Y_\alpha}\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot),t\right),$$

which yields

$$ev^{\tilde{Q}_n^{Y_\alpha}}(f,t) = \alpha \int_0^t \tilde{s}_n(f,x)\beta_1(dx) + \widetilde{B}_{\alpha,n}^{Y_\alpha}(f,t),$$

almost surely, with respect to $\tilde{Q}_n^{Y_\alpha}$. Applying Lemma 3, one has that

$$E_{\tilde{Q}_n^{Y_\alpha}}[\Phi_n] = E_{\tilde{P}_{Y_\alpha}}\left[\Phi_n \circ \widetilde{ev}_n^{P_{Y_\alpha}}\right] = E_{\tilde{P}_{Y_\alpha}}\left[\tilde{\Psi}_n\right] = 1.$$

The two relations

$$ev^{\tilde{Q}_n^{Y_\alpha}}(f,t) = \alpha \int_0^t \tilde{s}_n(f,x)\beta_1(dx) + \widetilde{B}_{\alpha,n}^{Y_\alpha}(f,t),$$

$$E_{\tilde{Q}_n^{Y_\alpha}}[\Phi_n] = 1,$$

together with Proposition 2, insure that $\tilde{Q}_n^{Y_\alpha}$ and P_{B_α} are mutually absolutely continuous and that, almost surely with respect to $\tilde{Q}_n^{Y_\alpha}$,

$$\frac{dP_{B_\alpha}}{d\tilde{Q}_n^{Y_\alpha}}(f) = E_{\tilde{Q}_n^{Y_\alpha}}\left[\Phi_n \mid \underline{ev}^{\tilde{Q}_n^{Y_\alpha}} = f\right] = \Phi_n(f).$$

But, according to Theorem 1 (fourth item), Φ_n has, with respect to P_{B_α}, and for some Poisson process $B_2^{Y_\alpha,B_\alpha}$, the following equivalent representation:

$$\begin{aligned}
\ln[\Phi_n](f) &= -\int_0^T \tilde{s}_n(f,x)\, ev^{P_{B_\alpha}}(f,dx) \\
&+ \frac{\alpha}{2}\int_0^T \tilde{s}_n^2(f,x)\beta_1(dx) \\
&+ \sqrt{1-\alpha}\int_0^T \tilde{s}_n(f,x)\tilde{B}_2^{Y_\alpha,B_\alpha}(f,dx).
\end{aligned}$$

Define

$$
\begin{aligned}
M_n(f,t) &= \int_0^T \tilde{s}_n(f,x)\, ev^{P_{B_\alpha}}(f,dx), \\
N_n(f,t) &= \int_0^T \tilde{s}_n(f,x)\, \tilde{B}_2^{Y_\alpha,B_\alpha}(f,dx), \\
V_n(f,t) &= \int_0^T \tilde{s}_n^2(f,x)\, \beta_1(dx), \\
W_n(f,t) &= -M_n(f,t) + \sqrt{1-\alpha}\; N_n(f,t) + \frac{\alpha}{2}\; V_n(f,t),
\end{aligned}
$$

and let $K > 0$ denote an arbitrary constant. One then has that

$$
\int_{\{\tilde{\Psi}_n > K\}} \tilde{\Psi}_n(f)\, \tilde{P}_{Y_\alpha}(df)
= \int_{\{\Phi_n > K\}} \Phi_n(f)\, \tilde{Q}_n^{Y_\alpha}(df) = P_{B_\alpha}(\Phi_n > K).
$$

But

$$
\begin{aligned}
P_{B_\alpha}(\Phi_n > K) &= P_{B_\alpha}(f \in D[0,T] :\ W_n(f,T) > \ln[K]) \\
&\le P_{B_\alpha}\left(f \in D[0,T] :\ |M_n(f,T)| > \frac{\ln[K]}{3}\right) \\
&+ P_{B_\alpha}\left(f \in D[0,T] :\ |N_n(f,T)| > \frac{\ln[K]}{3\sqrt{1-\alpha}}\right) \\
&+ P_{B_\alpha}\left(f \in D[0,T] :\ V_n(f,T) > \frac{2\ln[K]}{3\alpha}\right).
\end{aligned}
$$

Now,

$$P_{B_\alpha}\left(f \in D[0,T]: \ |M_n(f,T)| > \frac{\ln[K]}{3}\right)$$

$$\begin{aligned}
&= P\left(\omega \in \Omega: \ \left|\int_0^T \tilde{s}_n(B_\alpha(\omega,\cdot),x)\, B_\alpha(\omega,dx)\right| > \frac{\ln[K]}{3}\right) \\
&\leq P\left(\omega \in \Omega: \ \sqrt{\alpha}\left|\int_0^T \tilde{s}_n(B_\alpha(\omega,\cdot),x)\, B_1(\omega,dx)\right| > \frac{\ln[K]}{6}\right) \\
&+ P\left(\omega \in \Omega: \ \sqrt{1-\alpha}\int_0^T |\tilde{s}_n|(B_\alpha(\omega,\cdot),x)\, B_2(\omega,dx) > \frac{\ln[K]}{12}\right) \\
&+ P\left(\omega \in \Omega: \ \sqrt{1-\alpha}\int_0^T |\tilde{s}_n|(B_\alpha(\omega,\cdot),x)\, \beta_2(dx) > \frac{\ln[K]}{12}\right)
\end{aligned}$$

and, since, for a continuous local martingale M, and constants $\alpha > 0$, and $K > 0$, (Mémin [18, 2.83. Lemme, p. 19][7])

$$P(\omega \in \Omega: \ |M(\omega,t)| > \alpha) \leq P(\omega \in \Omega: \ \langle M\rangle(\omega,t) > K) + 2e^{-\frac{\alpha^2}{2K}},$$

one has, for $L > 0$,

$$\begin{aligned}
&P\left(\omega \in \Omega: \ \sqrt{\alpha}\left|\int_0^T \tilde{s}_n(B_\alpha(\omega,\cdot),x)\, B_1(\omega,dx)\right| > \frac{\ln[K]}{6}\right) \\
&\leq P\left(\omega \in \Omega: \ \int_0^T \tilde{s}_n^2(B_\alpha(\omega,\cdot),x)\, \beta_1(dx) > L\right) \\
&+ 2\exp\left\{-\frac{\left[\frac{\ln[K]}{6}\right]^2}{2L}\right\}.
\end{aligned}$$

Choosing $L = \ln[K]$, the latter exponential term becomes $K^{-\frac{1}{72}}$. Furthermore, as

$$P\left(\omega \in \Omega: \ \int_0^T \tilde{s}^2(B_\alpha(\omega,\cdot),x)\, \beta_1(dx) < \infty\right) = 1,$$

[7]See also the remark that follows.

it follows that

$$\lim_{K\uparrow\infty} P\left(\omega \in \Omega : \ \sqrt{\alpha}\left|\int_0^T \tilde{s}_n\left(B_\alpha\left(\omega,\cdot\right),x\right)B_1\left(\omega,dx\right)\right| > \frac{\ln\left[K\right]}{6}\right) = 0,$$

independently of n. Now, if τ_p's denotes the time at which jump number p of the Poisson process B_2 occurs, since $|\ \{p \in I\!N : \ \tau_p(\omega) \leq T\}\ | < \infty$, whatever $\omega \in \Omega$, one has that

$$\int_0^T |\tilde{s}|\left(B_\alpha\left(\omega,\cdot\right),x\right)B_2\left(\omega,dx\right) = \sum_{\tau_p \leq T} |\tilde{s}|\left(B_\alpha\left(\omega,\cdot\right),\tau_p\left(\omega\right)\right) < \infty,$$

from which it follows that

$$\lim_{K\uparrow\infty} P\left(\omega \in \Omega : \ \sqrt{1-\alpha}\int_0^T |\tilde{s}_n|\left(B_\alpha\left(\omega,\cdot\right),x\right)B_2\left(\omega,dx\right) > \frac{\ln\left[K\right]}{12}\right) = 0,$$

independently of n. Finally, by assumption,

$$P\left(\omega \in \Omega : \ \int_0^T |\tilde{s}|\left(B_\alpha\left(\omega,\cdot\right),x\right)\beta_2\left(dx\right) < \infty\right) = 1,$$

so that

$$\lim_{K\uparrow\infty} P\left(\omega \in \Omega : \ \sqrt{1-\alpha}\int_0^T |\tilde{s}_n|\left(B_\alpha\left(\omega,\cdot\right),x\right)\beta_2\left(dx\right) > \frac{\ln\left[K\right]}{12}\right) = 0,$$

independently of n. Consequently,

$$\lim_{K\uparrow\infty} P_{B_\alpha}\left(f \in D\left[0,T\right] : \ |M_n\left(f,T\right)| > \frac{\ln\left[K\right]}{3}\right) = 0,$$

independently of n. Since, $\tilde{Q}_n^{Y_\alpha}$ and P_{B_α} are mutually absolutely continuous, stochastic integrals with respect to these probabilities are indistinguishable (Von Weizsäcker et al [23, P. 245]) and thus

$$P_{B_\alpha}\left(f \in D\left[0,T\right] : \ |N_n\left(f,T\right)| > \frac{\ln\left[K\right]}{3\sqrt{1-\alpha}}\right)$$

$$= \tilde{Q}_n^{Y_\alpha}\left(f \in D\left[0,T\right] : \ |N_n\left(f,T\right)| > \frac{\ln\left[K\right]}{3\sqrt{1-\alpha}}\right).$$

As N_n is adapted to $\mathcal{D}_{T_n}$, on has, by Lemma 5, that

$$\begin{aligned}
&\tilde{Q}_n^{Y_\alpha}\left(f \in D\left[0,T\right]:\ \left|N_n\left(f,T\right)\right| > \frac{\ln\left[K\right]}{3\sqrt{1-\alpha}}\right) \\
= \ & P_{Y_\alpha}\left(f \in D\left[0,T\right]:\ \left|N_n\left(f,T\right)\right| > \frac{\ln\left[K\right]}{3\sqrt{1-\alpha}}\right) \\
= \ & P\left(\omega \in \Omega:\ \left|\int_0^T \tilde{s}_n\left(Y_\alpha\left(\omega,\cdot\right),x\right)\tilde{B}_2\left(\omega,dx\right)\right| > \frac{\ln\left[K\right]}{3\sqrt{1-\alpha}}\right) \\
\leq \ & P\left(\omega \in \Omega:\ \int_0^T \left|\tilde{s}\right|\left(Y_\alpha\left(\omega,\cdot\right),x\right)B_2\left(\omega,dx\right) > \frac{\ln\left[K\right]}{6\sqrt{1-\alpha}}\right) \\
+ \ & P\left(\omega \in \Omega:\ \int_0^T \left|\tilde{s}\right|\left(Y_\alpha\left(\omega,\cdot\right),x\right)\beta_2\left(\omega,dx\right) > \frac{\ln\left[K\right]}{6\sqrt{1-\alpha}}\right).
\end{aligned}$$

Consequently, as above,

$$\lim_{K\uparrow\infty} P_{B_\alpha}\left(f \in D\left[0,T\right]:\ \left|N_n\left(f,T\right)\right| > \frac{\ln\left[K\right]}{3\sqrt{1-\alpha}}\right) = 0,$$

independently of n. The term containing V_n has similarly a limit that vanishes. Lemma 6 is thus proved. □

Remark. If M is a local martingale, null at the origin, and such that its jumps are, almost surely, uniformly bounded ($|\Delta M| \leq \mu < \infty$), almost surely, then[8]

$$P\left(\left|M_t\right| > K\right) \leq P\left(2\varphi\left(\mu\frac{K}{L}\right)\left[M\right]_t > L\right) + 2e^{-\frac{K^2}{2L}},$$

where

$$\varphi\left(x\right) = -\frac{x + \ln\left(1-x\right)_+}{x^2}\ .$$

When M is continuous, $\mu = 0$, and this inequality allows one to bypass the assumptions on the integrability of s, with respect to β_2. Thus, even for s's with bounded jumps, there is no obvious extension of the method that works for the continuous case.

The following lemma is elementary, but useful.

Lemma 7. *Let* $(\Omega, \mathcal{A}, P)$ *and* $(\Omega, \mathcal{A}, Q)$ *denote two probability spaces, and assume that* $\Omega_0 \in \mathcal{A}$ *is such that* $P(\Omega_0) = Q(\Omega_0) = 1$. *Define then*

$$\mathcal{A}_0 = \mathcal{A} \cap \Omega_0, \text{ and, for } A \in \mathcal{A},\ A_0 = A \cap \Omega_0.$$

[8]The proof for the continuous case ($\mu = 0$), can be found in Mémin [18, 2.83, Lemme, p. 19].

Set finally

$$P_0(A_0) = P(A \cap \Omega_0), \text{ and } Q_0(A_0) = Q(A \cap \Omega_0).$$

Then, whenever P_0 and Q_0 are mutually absolutely continuous, so are P and Q, and furthermore, almost surely, with respect to P and Q,

$$\frac{dQ}{dP}(\omega) = \begin{cases} \frac{dQ_0}{dP_0}(\omega) & \text{if} \quad \omega \in \Omega_0 \\ 0 & \text{if} \quad \omega \notin \Omega_0 \end{cases}.$$

Epilogue to Proposition 6.
Lemmas 4 and 6 yield that

$$\lim_{n\to\infty} \widetilde{\Psi}_n(f) = \widetilde{\Psi}(f),$$

in $L_1\left[\tilde{P}_{Y_\alpha}\right]$. From Lemma 3 one then has that $E_{\tilde{P}_{Y_\alpha}}\left[\widetilde{\Psi}\right] = 1$. But then (Proposition 2), if $\tilde{P}_{B_\alpha}$ is the restriction of P_{B_α} to $\tilde{D}[0,T]$, also produced by $\tilde{B}_\alpha^{Y_\alpha}$, $\tilde{P}_{Y_\alpha}$ and $\tilde{P}_{B_\alpha}$ are mutually absolutely continuous. So, by Lemma 7, one has that P_{Y_α} and P_{B_α} are mutually absolutely continuous. Furthermore $E_{P_{Y_\alpha}}[\Psi] = 1$.

Corollary. *If $\beta_2 = \beta_1$, or if, almost surely, $S(\omega,\cdot) \in H(N_\alpha)$, Lemma 6 is true without the integrability conditions on s with respect to β_2, since then, for $i = 1, 2$,*

$$\left\{\int_0^t |s(x)|\, \beta_i(dx)\right\}^2 \leq \beta_i([0,T]) \int_0^T s^2(x)\, \beta_i(dx).$$

But then, to be true, Proposition 6 does not require those same conditions either.

Proposition 7. *It is assumed that* **A0**, **A1**, **A4**, *and* **A5** *obtain, that s is predictable and that both*

$$P_{B_\alpha}\left(f \in D[0,T] : \int_0^T |s|(f,x)\, \beta_2(dx) < \infty\right) = 1,$$

and

$$P_{Y_\alpha}\left(f \in D[0,T] : \int_0^T |s|(f,x)\, \beta_2(dx) < \infty\right) = 1$$

obtain also.

Then P_{Y_α} is absolutely continuous with respect to P_{B_α}, and, almost surely, with respect to P_{Y_α},

$$\begin{aligned}\ln\left[\frac{dP_{Y_\alpha}}{dP_{B_\alpha}}(f)\right] &= \int_0^T \tilde{s}(f,x)\, ev^{P_{Y_\alpha}}(f,dx) \\ &- \frac{\alpha}{2}\int_0^T \tilde{s}^2(f,x)\,\beta_1(dx) \\ &- \sqrt{1-\alpha}\int_0^T \tilde{s}(f,x)\,\tilde{B}_2^{Y_\alpha}(f,dx).\end{aligned}$$

Proof. Absolute continuity comes from the Corollary to Proposition 2. Let then f belong to $D[0,T]$, and

$$\begin{aligned}\ln[\Phi_n(f)] &= \int_0^T \tilde{s}_n(f,x)\, ev^{P_{B_\alpha}}(f,dx) \\ &- \frac{\alpha}{2}\int_0^T \tilde{s}_n^2(f,x)\,\beta_1(dx) \\ &- \sqrt{1-\alpha}\int_0^T \tilde{s}_n(f,x)\,\tilde{B}_2^{Y_\alpha,B_\alpha}(f,dx).\end{aligned}$$

Let T_n be the stopping time of the previous proposition, and set

$$C_n = \{f \in D[0,T] : \ T_n(f) = T\}.$$

Then, for $A \in \mathcal{D}$, $A \cap C_n$ belongs to $\mathcal{D}_{T_n}$, (Dellacherie et al [9, 56 Theoreme, 56.1, p. 189]), and, by Lemma 8,

$$\tilde{Q}_n^{Y_\alpha}(A \cap C_n) = \tilde{P}_{Y_\alpha}(A \cap C_n).$$

As $P_{Y_\alpha}\left(\tilde{D}[0,T]\right) = 1$, $\lim_n P_{Y_\alpha}(C_n) = 1$, $\tilde{Q}_n^{Y_\alpha}$ and P_{B_α} are mutually absolutely continuous, and, almost surely, with respect to P_{B_α},

$$\frac{d\tilde{Q}_n^{Y_\alpha}}{dP_{B_\alpha}} = \Phi_n,$$

one may write, using what precedes:

$$\begin{aligned} P_{Y_\alpha}(A) &= \lim_n P_{Y_\alpha}(A \cap C_n) = \lim_n P_{Y_\alpha}\left(A \cap \tilde{D}[0,T] \cap C_n\right) \\ &= \lim_n \tilde{P}_{Y_\alpha}(A \cap C_n) = \lim_n \tilde{Q}_n^{Y_\alpha}(A \cap C_n) \\ &= \lim_n \int_{A \cap C_n}(f) \frac{d\tilde{Q}_n^{Y_\alpha}}{dP_{B_\alpha}}(f) P_{B_\alpha}(df) = \lim_n \int_A I_{C_n} \Phi_n(f) P_{B_\alpha}(df). \end{aligned}$$

Let now

$$\begin{aligned} \ln[\Phi(f)] &= \int_0^T \tilde{s}(f,x)\, ev^{P_{Y_\alpha}}(f,dx) \\ &- \frac{\alpha}{2}\int_0^T \tilde{s}^2(f,x)\,\beta_1(dx) \\ &- \sqrt{1-\alpha}\int_0^T \tilde{s}(f,x)\,\tilde{B}_2^{Y_\alpha}(f,dx). \end{aligned}$$

When $T_n(f) = T$, $\int_0^T \tilde{s}^2(f,x)\,\beta_1(dx) \leq n$. Consequently, letting $C = \tilde{D}[0,T]$, one may write, as, on C_n, $I_C = 1$:

$$I_{C_n}(f)\,\Phi_n(f) = I_{C_n}(f)\, e^{I_C(f)\ln[\Phi_n(f)]}.$$

The following definitions will shorten some unwieldly expressions:

$$\begin{aligned} M_n(f,t) &= \int_0^t \tilde{s}_n(f,x)\, ev^{P_{B_\alpha}}(f,dx), \\ \tilde{s}_{n,p}(f,t) &= \tilde{s}_n(f,t) - \tilde{s}_{n+p}(f,t), \\ M_{n,p}(f,t) &= \int_0^t \tilde{s}_{n,p}(f,x)\, ev^{P_{B_\alpha}}(f,dx), \\ M_{n,p}^{(1)}(\omega,t) &= \int_0^t \tilde{s}_{n,p}(B_\alpha(\omega,\cdot),x)\, B_1(\omega,dx), \\ M_{n,p}^{(2)}(\omega,t) &= \int_0^t |\tilde{s}_{n,p}(B_\alpha(\omega,\cdot),x)|\, B_2(\omega,dx), \\ M_{n,p}^{(3)}(\omega,t) &= \int_0^t |\tilde{s}_{n,p}(B_\alpha(\omega,\cdot),x)|\, \beta_2(dx). \end{aligned}$$

One has that

$$
\begin{aligned}
& P_{B_\alpha}\left(f \in D\left[0,T\right] : I_C\left(f\right)\left|M_n\left(f,T\right) - M_{n+p}\left(f,T\right)\right| > K\right) \\
= \;& P_{B_\alpha}\left(f \in D\left[0,T\right] : I_C\left(f\right)\left|M_{n,p}\left(f,T\right)\right| > K\right) \\
\leq \;& P\left(\omega \in \Omega :\ \sqrt{\alpha}\ I_C\left(B_\alpha\left(\omega,\cdot\right)\right)\left|M_{n,p}^{(1)}\left(\omega,T\right)\right| > \frac{K}{3}\right) \\
+ \;& P\left(\omega \in \Omega :\ \sqrt{1-\alpha}\ I_C\left(B_\alpha\left(\omega,\cdot\right)\right) M_{n,p}^{(2)}\left(\omega,T\right) > \frac{K}{3}\right) \\
+ \;& P\left(\omega \in \Omega :\ \sqrt{1-\alpha}\ I_C\left(B_\alpha\left(\omega,\cdot\right)\right) M_{n,p}^{(3)}\left(\omega,T\right) > \frac{K}{3}\right).
\end{aligned}
$$

Because of Remark 2.8 Mémin [18, P. 19],

$$
P\left(\omega \in \Omega :\ \sqrt{\alpha}\ I_C\left(B_\alpha\left(\omega,\cdot\right)\right)\left|M_{n,p}^{(1)}\left(\omega,T\right)\right| > \frac{K}{3}\right)
$$

is dominated by

$$
P\left(\omega \in \Omega :\ \alpha\ I_C\left(B_\alpha\left(\omega,\cdot\right)\right)\langle M_{n,p}^{(1)}\rangle\left(\omega,T\right) > L\right) + 2\exp\left\{-\frac{K^2}{18L}\right\}.
$$

But, with respect to P,

$$
\begin{aligned}
& \langle M_{n,p}^{(1)}\rangle\left(\omega,T\right) \\
& \qquad = \int_0^T I_{\rrbracket T_n\left(B_\alpha(\omega,\cdot)\right),T_{n+p}\left(B_\alpha(\omega,\cdot)\right)\rrbracket}\left(\omega,x\right)\tilde{s}^2\left(B_\alpha\left(\omega,\cdot\right),x\right)\beta_1\left(dx\right),
\end{aligned}
$$

and, since, for $B_\alpha\left(\omega,\cdot\right) \in C = \tilde{D}\left[0,T\right]$,

$$
\int_0^T \tilde{s}^2\left(B_\alpha\left(\omega,\cdot\right),x\right)\beta_1\left(dx\right) < \infty,
$$

then

$$
\lim_{n,p\uparrow\infty} P\left(\omega \in \Omega :\ \sqrt{\alpha}\ I_C\left(B_\alpha\left(\omega,\cdot\right)\right)\left|M_{n,p}^{(1)}\left(\omega,T\right)\right| > \frac{K}{3}\right) = 0.
$$

Given the assumptions on the integrability of $|s|$, a similar argument yields that

$$
\lim_{n,p\uparrow\infty} P\left(\omega \in \Omega :\ \sqrt{1-\alpha}\ I_C\left(B_\alpha\left(\omega,\cdot\right)\right) M_{n,p}^{(2)}\left(\omega,T\right) > \frac{K}{3}\right) = 0,
$$

and that

$$\lim_{n,p\uparrow\infty} P\left(\omega \in \Omega : \ \sqrt{1-\alpha}\ I_C\left(B_\alpha\left(\omega,\cdot\right)\right) M_{n,p}^{(3)}\left(\omega, T\right) > \frac{K}{3}\right) = 0.$$

Thus, with respect to P_{B_α}, the sequence

$$\left\{ I_C\left(f\right) \int_0^T \tilde{s}_n\left(f,x\right) ev^{P_{B_\alpha}}\left(f,dx\right),\ n \in I\!N \right\}$$

has a limit in probability, which will be denoted $J_{B_\alpha}\left(f\right)$.

Now, for $f \in \tilde{D}\left[0,T\right]$,

$$\lim_{n\uparrow\infty} \int_0^T \tilde{s}_n^2\left(f,x\right)\beta_1\left(dx\right) = \int_0^T \tilde{s}^2\left(f,x\right)\beta_1\left(dx\right) < \infty,$$

and, for $I_C\left(f\right)\int_0^T \tilde{s}_n\left(f,x\right)\tilde{B}_2^{Y_\alpha,B_\alpha}\left(f,dx\right)$, one basically repeats the arguments already given. As, trivially, almost surely, with respect to P_{B_α}, $\lim_n I_{C_n} = I_C$,

$$P_{B_\alpha} - \lim_n \left\{ I_{C_n}\left(f\right)\ln\left[\Phi_n\left(f\right)\right]\right\} = J_{B_\alpha}\left(f\right)$$
$$-\frac{\alpha}{2}\ I_C\left(f\right)\int_0^T \tilde{s}^2\left(f,x\right)\beta_1\left(dx\right) - \sqrt{1-\alpha}\ I_C\left(f\right)\int_0^T \tilde{s}\left(f,x\right)\tilde{B}_2^{Y_\alpha,B_\alpha}\left(f,dx\right).$$

The exponential of this limit shall be denoted $\Phi^{P_{B_\alpha}}$.

As P_{Y_α} is absolutely continuous with respect to P_{B_α}, on one hand,

$$\int_0^T \tilde{s}_n\left(f,x\right) ev^{P_{Y_\alpha}}\left(f,dx\right) = \int_0^T \tilde{s}_n\left(f,x\right) ev^{P_{B_\alpha}}\left(f,dx\right),$$

and, on the other,

$$\int_0^T \tilde{s}_n\left(f,x\right)\tilde{B}_2^{Y_\alpha,B_\alpha}\left(f,dx\right) = \int_0^T \tilde{s}_n\left(f,x\right)\tilde{B}_2^{Y_\alpha}\left(f,dx\right),$$

so that, with respect to P_{Y_α}, Φ_n has the following representation:

$$\ln\left[\Phi_n\left(f\right)\right] = \int_0^T \tilde{s}_n\left(f,x\right) ev^{P_{Y_\alpha}}\left(f,dx\right)$$
$$-\frac{\alpha}{2}\int_0^T \tilde{s}_n^2\left(f,x\right)\beta_1\left(dx\right) - \sqrt{1-\alpha}\ \int_0^T \tilde{s}_n\left(f,x\right)\tilde{B}_2^{Y_\alpha}\left(f,dx\right).$$

But the assumptions made and in particular **A4** now imply that the limit in probability, with respect to P_{Y_α}, of the sequence $\{\Phi_n,\ n \in I\!N\}$ is Φ. So, with respect to P_{Y_α}, $\Phi^{P_{B_\alpha}} = \Phi$.

To finish the proof, one must still check that the sequence $\{I_{C_n}\Phi_n,\ n \in I\!N\}$ is uniformly integrable with respect to P_{B_α}, which insures that

$$\lim_{n\uparrow\infty} E_{P_{B_\alpha}}\left[I_{C_n}\Phi_n\right] = E_{P_{B_\alpha}}\left[\Phi^{P_{B_\alpha}}\right].$$

But, since $\tilde{Q}_n^{Y_\alpha}$ and P_{B_α} are mutually absolutely continuous, as seen in the proof of Lemma 6, and that $\Phi_n = \frac{d\tilde{Q}_n^{Y_\alpha}}{dP_{B_\alpha}}$ is one of the Radon-Nikodým derivatives, setting

$$\begin{aligned} \tilde{D}_n &= \{f \in D\,[0,T] :\ I_{C_n}(f)\,\Phi_n(f) > K\}, \\ D_n &= \{f \in D\,[0,T] :\ \Phi_n(f) > K\}, \end{aligned}$$

one has that

$$\begin{aligned} &\int_{\tilde{D}_n} I_{C_n}(f)\,\Phi_n(f)\,P_{B_\alpha}(df) \le \int_{D_n} \Phi_n(f)\,P_{B_\alpha}(df) = \tilde{Q}_n^{Y_\alpha}(D_n) \\ &\quad = \tilde{P}_{Y_\alpha} \circ \left[\underline{\tilde{ev}}_n^{P_{Y_\alpha}}\right]^{-1}(D_n) = \tilde{P}_{Y_\alpha}\left(f \in \tilde{D}\,[0,T] :\ \Phi_n \circ \underline{\tilde{ev}}_n^{P_{Y_\alpha}}(f) > K\right). \end{aligned}$$

But, on $[\![0,T_n]\!]$, $\tilde{ev}_n^{P_{Y_\alpha}} = ev$, so that

$$\begin{aligned} &\tilde{P}_{Y_\alpha}\left(f \in \tilde{D}\,[0,T] :\ \Phi_n \circ \underline{\tilde{ev}}_n^{P_{Y_\alpha}}(f) > K\right) \\ &\qquad = \tilde{P}_{Y_\alpha}\left(f \in \tilde{D}\,[0,T] :\ \Phi_n(f) > K\right) \\ &= P_{Y_\alpha}\left(f \in D\,[0,T] :\ \{f \in D\,[0,T] :\ \Phi_n(f) > K\} \cap \tilde{D}\,[0,T]\right). \end{aligned}$$

Now, assumptions **A0, A1**, **A4** and **A5** yield Proposition 4, which allows one to write

$$ev^{P_{Y_\alpha}}(f,t) = \alpha \int_0^t s(f,x)\,\beta_1(dx) + B_\alpha^{Y_\alpha}(f,t).$$

So, using the representation of Φ_n with respect to P_{Y_α}, one may then legitimately write:

$$\begin{aligned} & \int_{\tilde{D}_n} I_{C_n}(f)\,\Phi_n(f)\,P_{B_\alpha}(df) \\ \leq\ & \tilde{P}_{Y_\alpha}\left(\sqrt{\alpha}\int_0^T \tilde{s}_n(f,x)\,\tilde{B}_1^{Y_\alpha}(f,dx)\right. \\ +\ & \left.\frac{\alpha}{2}\int_0^T \tilde{s}_n^2(f,x)\,\beta_1(dx) > \ln[K]\right). \end{aligned}$$

The latter goes to zero as in previous arguments. □

Corollary 1. *When $\beta_2 = \beta_1$, or when, almost surely, $S(\omega,\cdot) \in H(N_\alpha)$, the integrability assumptions on s with respect to β_2 of Proposition 7 are no longer necessary, as the argument given in the Corollary to Proposition 6 is still valid.*

Corollary 2. *Given assumptions* **A0***,* **A1***,* **A4***, and* **A5***, assumption* **A6** *is necessary and sufficient for mutual absolute continuity of P_{B_α} and P_{Y_α}.*

4.2. Weak Solution of a Stochastic Differential Equation

The innovations representation of the "signal-plus-noise process," within the adopted RKHS framework, requires the seemingly unrelated, preliminary results that follow. Their reason for being will thus emerge later in the paper.

A weak solution of equation

$$Y_\alpha(\omega,t) = \alpha\int_0^t s(Y_\alpha(\omega,\cdot),x)\,\beta_1(dx) + B_\alpha(\omega,t)$$

is a triple $\{B_1^w, B_2^w, P^w\}$ such that

1. P^w is a probability measure on $\mathcal{D}$, such that, with respect to it,
   (a) B_1^w is a generalized Brownian motion, adapted to $\underline{\mathcal{D}}$, with variance $V_{P^w}[B_1^w(\cdot,t)] = \beta_1(t)$,
   (b) B_2^w is a Poisson process, adapted to $\underline{\mathcal{D}}$, for which $E_{P^w}[B_2^w(\cdot,t)] = \beta_2(t)$,
   (c) B_1^w and B_2^w are independent,
2. and, for fixed $t \in [0,T]$, almost surely, with respect to P^w,

$$ev^{P^w}(f,t) = \alpha\int_0^T s(f,x)\,\beta_1(dx) + B_\alpha^w(f,t),$$

where

$$\begin{aligned} B_{\alpha}^{w}(f,t) &= \sqrt{\alpha}\, B_{1}^{w}(f,t) + \sqrt{1-\alpha}\, \tilde{B}_{2}^{w}(f,t), \\ \tilde{B}_{2}^{w}(f,t) &= B_{2}^{w}(f,t) - \beta_{2}(t). \end{aligned}$$

Lemma 8. *Let B_α be a process satisfying* **A1**. *The process $ev^{P_{B_\alpha}}$ has then (with respect to P_{B_α}) the representation*

$$ev^{P_{B_\alpha}} = \sqrt{\alpha}\, B_1^{ev} + \sqrt{1-\alpha}\, \tilde{B}_2^{ev},$$

where

$$P_{B_\alpha} \circ [\underline{B}_1^{ev}]^{-1} = P \circ \underline{B}_1^{-1}, \text{ and } P_{B_\alpha} \circ [\underline{B}_2^{ev}]^{-1} = P \circ \underline{B}_2^{-1},$$

and $\tilde{B}_2^{ev} = B_2^{ev} - \beta_2$, for some probability space $(\Omega, \mathcal{A}, P)$.

Proof. First, given P_{B_α}, it can always, without restriction, be assumed that it is a measure induced from a $(\Omega, \mathcal{A}, P)$ space by a generalized Brownian motion B_1 and an independent Poisson process B_2 "summing" to the process $B_\alpha = \sqrt{\alpha} B_1 + \sqrt{1-\alpha} \tilde{B}_2$, as in assumption **A1**.The process B_2^{ev} is then defined by the equality:

$$B_2^{ev}(f,t) = \frac{1}{\sqrt{1-\alpha}} \sum_{u \le t} \left\{ \Delta ev^{P_{B_\alpha}} \right\}(f,u).$$

For fixed $0 \le t_1 < \cdots < t_n \le T$, and a Borel set $G \in \mathbb{R}^n$, let

$$G_D = \left\{ f \in D[0,T] : \left(B_2^{ev}(f,t_1), \ldots, B_2^{ev}(f,t_n) \right) \in G \right\}.$$

Let $G_D^\Omega = \underline{B}_\alpha^{-1}(G_D)$. If $\omega \in G_D^\Omega$, then

$$\left(\frac{1}{\sqrt{1-\alpha}} \sum_{u \le t_1} \{\Delta B_\alpha\}(\omega,u), \ldots, \frac{1}{\sqrt{1-\alpha}} \sum_{u \le t_n} \{\Delta B_\alpha\}(\omega,u) \right) \in G,$$

that is

$$\left(B_2(\omega,t_1), \ldots, B_2(\omega,t_n) \right) \in G.$$

B_2^{ev} is thus, with respect to P_{B_α}, a Poisson process such that

$$E_{P_{B_\alpha}}\left[B_2^{ev}(\cdot,t) \right] = \beta_2(t).$$

One shows similary that, with respect to P_{B_α}, B_1^{ev}, defined by

$$B_1^{ev} = \frac{1}{\sqrt{\alpha}} \left\{ ev^{P_{B_\alpha}} - \sqrt{1-\alpha}\ \left(B_2^{ev} - \beta_2\right)\right\},$$

is a generalized Brownian motion such that

$$E_{P_{B_\alpha}}\left[B_2^{ev}(\cdot,t)\right] = \beta_2(t). \qquad \square$$

Corollary. $\sigma_t^\circ\left(ev^{P_{B_\alpha}}\right) = \sigma_t^\circ\left(B_1^{ev}\right) \vee \sigma_t^\circ\left(B_2^{ev}\right)$

Proposition 8. *Let s be progressively measurable for $\mathcal{D}$, and assume that one has, for every $f \in D[0,T]$,*

$$\int_0^T s^2(f,x)\,\beta_1(dx) < \infty, \ \textit{and}\ \int_0^T |s|(f,x)\,\beta_2(dx) < \infty.$$

With the notation of Lemma 8, define, for almost every $f \in D[0,T]$, with respect to P_{B_α},

$$\ln[\Phi(f)] = \sqrt{\alpha}\int_0^T s(f,x)\,B_1^{ev}(f,dx) - \frac{\alpha}{2}\int_0^T s^2(f,x)\,\beta_1(dx).$$

Then,

$$Y_\alpha(\omega,t) = \alpha\int_0^t s\left(Y_\alpha(\omega,\cdot),x\right)\beta_1(dx) + B_\alpha(\omega,t)$$

has a weak solution if, and only if $E_{P_{B_\alpha}}[\Phi] = 1$, in which case the solution is unique.

Proof. Suppose first that $E_{P_{B_\alpha}}[\Phi] = 1$. Let then P^w be defined, as a probability, by the relation $dP^w = \Phi dP_{B_\alpha}$. Define also B_α^w as follows:

$$B_\alpha^w(f,t) = \alpha\int_0^t \{-s\}(f,x)\,\beta_1(dx) + ev^{P_{B_\alpha}}(f,t).$$

As Φ can be written in the form

$$\ln[\Phi(f)]$$
$$= -\sqrt{\alpha}\int_0^T \{-s\}(f,x)\,B_1^{ev}(f,dx) - \frac{\alpha}{2}\int_0^T \{-s\}^2(f,x)\,\beta_1(dx),$$

one may apply Girsanov's theorem (Proposition 1) to obtain that

$$P^w \circ \left[\underline{B_\alpha^w}\right]^{-1} = P_{B_\alpha} \circ \left[\underline{ev}^{P_{B_\alpha}}\right]^{-1} = P_{B_\alpha}.$$

As, furthermore,

$$ev^{P^w}(f,t) = \alpha \int_0^t s(f,x)\,\beta_1(dx) + B_\alpha^w(f,t),$$

one has then a weak solution, since, for instance, using Lemma 8, for G, a Borel set of $I\!R^n$, $0 \le t_1 < t_2 < t_3 < \cdots < t_n \le T$,

$$G_D = \{f \in D[0,T] : (f(t_1), f(t_2), f(t_3), \ldots, f(t_n)) \in G\},$$

and

$$B_2^w(f,t) = \sum_{u \le t} \{\Delta B_\alpha^w(f,t)\},$$

$$\begin{aligned} &P^w\left(f \in D[0,T] :\ B_2^w(f,\cdot) \in G_D\right) \\ &\qquad\qquad = P_{B_\alpha}\left(f \in D[0,T] :\ B_2^{ev}(f,\cdot) \in G_D\right). \end{aligned}$$

Suppose now that a weak solution exists. One has then, by definition,

$$ev^{P^w}(f,t) = \alpha \int_0^t s(f,x)\,\beta_1(dx) + B_\alpha^w(f,t),$$

which can be rewritten in the form

$$ev^{P^w}(f,t) = \alpha \int_0^t s\left(ev^{P^w}(f,\cdot), x\right)\beta_1(dx) + B_\alpha^w(f,t).$$

One can then apply Proposition 6 to get that P^w and $P_{B_\alpha^w}$ are mutually absolutely continuous, and that, almost surely, with respect to $P_{B_\alpha^w}$,

$$\begin{aligned} \ln\left[\frac{dP^w}{dP_{B_\alpha^w}}\right](f) &= \int_0^T s(f,x)\,ev^{P_{B_\alpha^w}}(f,dx) \\ &- \frac{\alpha}{2}\int_0^T s^2(f,x)\,\beta_1(dx) \\ &- \sqrt{1-\alpha}\int_0^T s(f,x)\,\tilde{B}_2^{ev^{P^w},B_\alpha^w}(f,dx), \end{aligned}$$

where $\tilde{B}_2^{ev^{P^w},B_\alpha^w}$ is the representation of $\tilde{B}_2^w$, with respect to $P_{B_\alpha^w}(\equiv P_{B_\alpha})$. Furthermore, with respect to $P_{B_\alpha^w}$, one has that (Lemma 8)

$$ev^{P_{B_\alpha^w}} = \sqrt{\alpha}B_1^{ev} + \sqrt{1-\alpha}\tilde{B}_2^{ev}.$$

Consequently

$$E_{P_{B_\alpha^w}}\left[\ln\left[\frac{dP^w}{dP_{B_\alpha^w}}\right]-\ln[\Phi]\right]^2$$

$$=(1-\alpha)\,E_{P_{B_\alpha^w}}\left[\int_0^T s(f,x)\,B_2^{ev}(f,dx)-\int_0^T s(f,x)\,B_2^{ev^{P^w},B_\alpha^w}(f,dx)\right]^2.$$

Now the evaluation map ev is a semimartingale with respect to P^w as well as with respect to $P_{B_\alpha^w}$. These two probability measures being mutually absolutely continuous, $[ev]^{P^w}=[ev]^{P_{B_\alpha^w}}$. As $[ev]^{P^w}=B_2^w$ and $[ev]^{P_{B_\alpha^w}}=B_2^{ev}$, and taking into account the fact that $B_2^w=\tilde{B}_2^{ev^{P^w},B_\alpha^w}$, one has that $E_{P_{B_\alpha}}[\Phi]=1$.

Suppose now that a second solution $\left(B_1^{\tilde{w}},B_2^{\tilde{w}},P^{\tilde{w}}\right)$ exists. As one must have

$$\frac{dP^{\tilde{w}}}{dP_{B_\alpha}}=\Phi,$$

$P^{\tilde{w}}=P^w$. □

Corollary 1. *Proposition 8 will be true whenever* $\beta_1=\beta_2$, *or* $S(\omega,\cdot)\in H(N_\alpha)$, *for every* $\omega\in\Omega$.

Corollary 2. *If one only assumes, in Proposition 8, that*

$$P_{B_\alpha}\left(f\in D[0,T]:\ \int_0^T s^2(f,x)\,\beta_1(dx)<\infty\right)=1,$$

and that

$$P_{B_\alpha}\left(f\in D[0,T]:\ \int_0^T |s|(f,x)\,\beta_2(dx)<\infty\right)=1,$$

one still has a solution, but it cannot any longer be claimed that it is unique.

4.3. Necessary Path Conditions for Absolute and Mutual Absolute Continuity of P_{Y_α} and P_{B_α}

To proceed one needs an observation that is stated as the following lemma.

Lemma 9. *Let* $(\Omega,\mathcal{A},P)$ *be a probability space, and let* $\underline{\mathcal{B}}^{(1)}$ *and* $\underline{\mathcal{B}}^{(2)}$ *be, with respect to* P, *two independent filtrations of* $\mathcal{A}$. *Set*

$$\mathcal{B}_t=\mathcal{B}_t^{(1)}\vee\mathcal{B}_t^{(2)} \text{ and } \underline{\mathcal{B}}=\{\mathcal{B}_t,\ t\in[0,T]\}.$$

Then, if M *is a martingale for* $\underline{\mathcal{B}}^{(1)}$, *it is also a martingale for* $\underline{\mathcal{B}}$.

The proposition that follows "embodies" the need of the RKHS assumption, but also the fact that an explicit expression for the likelihood ratio requires mutual absolute continuity rather than simply absolute continuity.

Proposition 9. *Suppose Y_α is a process, defined on $(\Omega, \mathcal{A}, P)$, adapted to $\underline{\mathcal{A}}$, with paths in $D[0,T]$, such that P_{Y_α} and P_{B_α} are mutually absolutely continuous. When $\beta_1 = \beta_2 \equiv \beta$, one can then find*

- *a process s, defined on $(D[0,T], \mathcal{D}, P_{Y_\alpha})$, predictable for $\underline{\mathcal{D}}$, and*
- *a zero-mean, generalized Brownian motion B_1 and a generalized Poisson process B_2, defined on $(\Omega, \mathcal{A}, P)$, adapted to $\underline{\sigma}^\circ(Y_\alpha)$, with*

$$V[B_1(\cdot,t)] = \beta(t) \text{ and } E[B_2(\cdot,t)] = \beta(t),$$

such that, for $B_\alpha = \sqrt{\alpha}B_1 + \sqrt{1-\alpha}\tilde{B}_2$, and, for $t \in [0,T]$, fixed, but arbitrary, almost surely, with respect to P,

$$Y_\alpha(\omega,t) = \alpha \int_0^t s(Y_\alpha(\omega,\cdot),x)\,\beta(dx) + B_\alpha(\omega,t),$$

with

$$P_{B_\alpha}\left(f \in D[0,T] : \int_0^T s^2(f,x)\,\beta(dx) < \infty\right) = 1,$$

and

$$P_{Y_\alpha}\left(f \in D[0,T] : \int_0^T s^2(f,x)\,\beta(dx) < \infty\right) = 1.$$

Proof. By Lemma 8, $ev^{P_{B_\alpha}} = \sqrt{\alpha}\, B_1^{ev} + \sqrt{1-\alpha}\, \tilde{B}_2^{ev}$. Let

$$\mathcal{B}_t^{(1)} = \sigma_t^\circ(B_1^{ev}), \text{ and } \underline{\mathcal{B}}^{(1)} = \left\{\mathcal{B}_t^{(1)},\ t \in [0,T]\right\}.$$

$\underline{\mathcal{B}}^{(1)}$ is a Brownian filtration.

Consider now the martingale L defined for $\underline{\mathcal{B}}^{(1)}$ as

$$L(f,t) = E_{P_{B_\alpha}}\left[\frac{dP_{Y_\alpha}}{dP_{B_\alpha}} \mid \mathcal{B}_t^{(1)}\right].$$

It has a modification $\tilde{L}$ (Von Weizsäcker et al [23, 9.7.5, p. 241]) which is continuous to the right and has continuous paths, almost surely, with respect to P_{B_α}. $\tilde{L}$ has then the representation (Von Weizsäcker [23, 9.7.4, p. 239])

$$\tilde{L}(f,t) = 1 + \sqrt{\alpha}\int_0^t s(f,x)\,B_1^{ev}(f,dx),$$

where s is predictable for $\underline{\mathcal{B}}^{(1)}$. Furthermore

$$P_{B_\alpha}\left(f \in D[0,T] : \int_0^T s^2(f,x)\,\beta(dx) < \infty\right) = 1.$$

Let

$$\tilde{T}(f) = \inf\left\{t \in [0,T] : \left[\tilde{L}(f,t) = 0\right] \text{ or } \left[\tilde{L}(f,t-) = 0\right]\right\}.$$

On $\left[\left[\tilde{T}, T\right]\right]$, the paths of $\tilde{L}$ are, almost surely, with respect to P_{B_α}, equal to zero. However, because P_{B_α} and P_{Y_α} are mutually absolutely continuous, $\tilde{L}(f,T) > 0$, almost surely, with respect to P_{B_α}. Consequently,

$$P_{B_\alpha}\left(f \in D[0,T] : \inf_{t\in[0,T]} \tilde{L}(f,t) > 0\right) = 1.$$

The expression $\ln\left[\tilde{L}(f,t)\right]$ does thus make sense, almost surely, with respect to P_{B_α}, and Itô's formula then yields:

$$\ln\left[\tilde{L}(f,t)\right] = \sqrt{\alpha}\int_0^t \frac{s(f,x)}{\tilde{L}(f,x)} B_1^{ev}(f,dx) - \frac{\alpha}{2}\int_0^t \left(\frac{s(f,x)}{\tilde{L}(f,x)}\right)^2 \beta(dx),$$

that is

$$\tilde{L}(f,t) = e^{\sqrt{\alpha}\int_0^t \frac{s(f,x)}{\tilde{L}(f,x)} B_1^{ev}(f,dx) - \frac{\alpha}{2}\int_0^t \left(\frac{s(f,x)}{\tilde{L}(f,x)}\right)^2 \beta(dx)}.$$

Set then

$$\tilde{s}(f,t) = \frac{s(f,x)}{\tilde{L}(f,x)}.$$

Since

$$\begin{aligned}\int_0^T \tilde{s}^2(f,x)\,\beta(dx) &= \int_0^T \left(\frac{s(f,x)}{\tilde{L}(f,x)}\right)^2 \beta(dx) \\ &\leq \frac{1}{\inf_{t\in[0,T]} \tilde{L}^2(f,t)} \int_0^T s^2(f,x)\,\beta_1(dx),\end{aligned}$$

one has that

$$P_{B_\alpha}\left(f \in D[0,T] : \int_0^T \tilde{s}^2(f,x)\,\beta(dx) < \infty\right) = 1,$$

so that also

$$P_{Y_\alpha}\left(f \in D[0,T] : \int_0^T \tilde{s}^2(f,x)\,\beta(dx) < \infty\right) = 1.$$

Finally, $E_{P_{B_\alpha}}\left[\tilde{L}(\cdot,T)\right]=1$. Consequently, there exists a weak solution to the "formal"[9] equation

$$Y_\alpha(\omega,t)=\alpha\int_0^T\tilde{s}\left(Y_\alpha(\omega,\cdot),x\right)\beta(dx)+B_\alpha(\omega,t).$$

By the Corollary to Lemma 8 and Lemma 9, $\tilde{L}$ is, with respect to P_{B_α}, a martingale for $\underline{\mathcal{D}}$. One then defines, on $(D[0,T],\mathcal{D})$, and for the filtration $\underline{\mathcal{D}}$,

$$P_\alpha^w(df)=\tilde{L}(f,T)\,P_{B_\alpha}(df),$$

and, with respect to P_α^w,

$$B_\alpha^w(f,t)=-\alpha\int_0^t\tilde{s}(f,x)\,\beta(dx)+ev^{P_\alpha^w}(f,t).$$

By Girsanov's theorem, one has that

$$P_\alpha^w\circ[\underline{B}_\alpha^w]^{-1}=P_{B_\alpha}.$$

Finally, $\tilde{L}(\cdot,T)$, being a martingale for $\underline{\mathcal{D}}$ (Lemma 9), is a version of $\frac{dP_{Y_\alpha}}{dP_{B_\alpha}}$. Consequently $P_\alpha^w=P_{Y_\alpha}$. One must then set

$$B_\alpha^{Y_\alpha}=B_\alpha^w\circ\underline{Y}_\alpha^{-1}.$$ □

Proposition 10. *Suppose Y_α is a process, defined on $(\Omega,\mathcal{A},P)$, adapted to $\underline{\mathcal{A}}$, with paths in $D[0,T]$, such that P_{Y_α} is absolutely continuous with respect to P_{B_α}. When $\beta_1=\beta_2\equiv\beta$, one can then find*

- *a process s, defined on $(D[0,T],\mathcal{D},P_{Y_\alpha})$, progressively measurable for $\underline{\mathcal{D}}$, and*

- *a zero-mean, generalized Brownian motion B_1 and a generalized Poisson process B_2, defined on $(\Omega,\mathcal{A},P)$, adapted to $\underline{\sigma}^\circ(Y_\alpha)$, with*

$$V[B_1(\cdot,t)]=\beta(t)\ \ \text{and}\ E[B_2(\cdot,t)]=\beta(t),$$

[9] One should not, in particular, take the B_α of the "formal" equation as the B_α of the proposition's conclusion!

such that, for $B_\alpha = \sqrt{\alpha} B_1 + \sqrt{1-\alpha}\tilde{B}_2$, *and, for* $t \in [0,T]$, *fixed, but arbitrary, almost surely, with respect to* P,

$$Y_\alpha(\omega,t) = \alpha \int_0^t s\left(Y_\alpha(\omega,\cdot), x\right)\beta(dx) + B_\alpha(\omega,t),$$

with

$$P_{Y_\alpha}\left(f \in D[0,T]: \int_0^T s^2(f,x)\beta(dx) < \infty\right) = 1.$$

Proof. One has again, as in Proposition 9, that

$$\tilde{L}(f,t) = 1 + \sqrt{\alpha}\int_0^t s(f,x) B_1^{ev}(f,dx),$$

with

$$P_{B_\alpha}\left(f \in D[0,T]: \int_0^T s^2(f,x)\beta(dx) < \infty\right) = 1.$$

But now $\tilde{L}$ can be equal to zero, and that is why one defines

$$T_n(f) = \begin{cases} \inf\left\{t \in [0,T]: \tilde{L}(f,t) < \frac{1}{n}\right\}, & \text{if } \left\{t \in [0,T]: \tilde{L}(f,t) < \frac{1}{n}\right\} \neq \emptyset, \\ T, & \text{if } \left\{t \in [0,T]: \tilde{L}(f,t) < \frac{1}{n}\right\} = \emptyset. \end{cases}$$

If $B^{(1)} \in \mathcal{B}_{t\wedge T_n}^{(1)}$, then

$$\begin{aligned} P_{Y_\alpha}\left(B^{(1)}\right) &= \int_{B^{(1)}} \tilde{L}(f,T) P_{B_\alpha}(df) \\ &= \int_{B^{(1)}} E\left[\tilde{L}(\cdot,T) \mid \mathcal{B}_{t\wedge T_n}^{(1)}\right] P_{B_\alpha}(df) = \int_{B^{(1)}} \tilde{L}(f, t\wedge T_n) P_{B_\alpha}(df). \end{aligned}$$

Thus one has, on $\mathcal{B}_{t\wedge T_n}^{(1)}$, $P_{Y_\alpha}(df) = \tilde{L}(f,t\wedge T_n) P_{B_\alpha}(df)$. But, as $\tilde{L}(\cdot, t\wedge T_n) \geq \frac{1}{n}$, one has also, still on $\mathcal{B}_{t\wedge T_n}^{(1)}$, that

$$P_{B_\alpha}(df) = \frac{P_{Y_\alpha}(df)}{\tilde{L}(f, t\wedge T_n)},$$

so that, since $D[0,T]$ belongs to $\mathcal{B}_{t\wedge T_n}^{(1)}$,

$$E_{P_{Y_\alpha}}\left[\frac{1}{\tilde{L}(\cdot, t\wedge T_n)}\right] = P_{B_\alpha}(D[0,T]) = 1.$$

The sequence $\{T_n,\ n \in I\!N\}$ is increasing and bounded. It thus has a limit, denoted $\lim_n T_n$, which is a stopping time. As $\tilde{L}$ is continuous, almost surely, with respect to P_{Y_α}, one has, by Fatou's lemma, that

$$\begin{aligned} E_{P_{Y_\alpha}}\left[\frac{1}{\tilde{L}\left(\cdot, t \wedge \lim_n T_n\right)}\right] &= E_{P_{Y_\alpha}}\left[\liminf_n \left\{\frac{1}{\tilde{L}\left(\cdot, t \wedge T_n\right)}\right\}\right] \\ &\leq \liminf_n E_{P_{Y_\alpha}}\left[\frac{1}{\tilde{L}\left(\cdot, t \wedge T_n\right)}\right] = 1, \end{aligned}$$

that is,

$$E_{P_{Y_\alpha}}\left[\frac{1}{\tilde{L}\left(\cdot, t \wedge \lim_n T_n\right)}\right] \leq 1.$$

As $\tilde{L}\left(\cdot, \lim_n T_n\right) = 0$, almost surely, with respect to P_{Y_α}, one must have $\lim_n T_n = T$, almost surely, with respect to P_{Y_α}. Furthermore, as

$$\int_0^{T_n} \left\{\frac{s\left(f,x\right)}{\tilde{L}\left(f,x\right)}\right\}^2 \beta\left(dx\right) \leq n^2 \|s\left(f,\cdot\right)\|^2_{L_2[\beta]},$$

it follows that

$$\begin{aligned} 1 &= P_{Y_\alpha}\Big(f \in D\left[0,T\right]:\ \|s\left(f,\cdot\right)\|^2_{L_2[\beta]} < \infty\Big) \\ &\leq P_{Y_\alpha}\left(f \in D\left[0,T\right]:\ \int_0^{T_n} \left\{\frac{s\left(f,x\right)}{\tilde{L}\left(f,x\right)}\right\}^2 \beta\left(dx\right) < \infty\right). \end{aligned}$$

Consequently,

$$P_{Y_\alpha}\left(f \in D\left[0,T\right]:\ \int_0^{T} \left\{\frac{s\left(f,x\right)}{\tilde{L}\left(f,x\right)}\right\}^2 \beta\left(dx\right) < \infty\right) = 1.$$

As $I_{[\![0,T_n]\!]}\frac{s}{\tilde{L}}$ is in $L_2\left[\beta\right]$, almost surely, with respect to P_{B_α}, one can legitimately define, on $\left(D\left[0,T\right], \mathcal{D}, P_{B_\alpha}\right)$, and for the filtration $\underline{\mathcal{D}}$, the process $\widetilde{B}_{\alpha,n}$, by the following relation:

$$\widetilde{B}_{\alpha,n}\left(f,t\right) = -\alpha \int_0^t I_{[\![0,T_n]\!]}\left(f,x\right)\ \frac{s\left(f,x\right)}{\tilde{L}\left(f,x\right)}\beta\left(dx\right) + ev^{P_{B_\alpha}}\left(f,t\right).$$

$\tilde{L}^{T_n}$ being a martingale for the filtration $\underline{\mathcal{B}}^{(1)}$, and thus, by Lemma 9, for the filtration $\underline{\mathcal{D}}$, one thus defines, on $\mathcal{D}_t$, a probability Q_n when setting

$$Q_n\left(df\right) = \tilde{L}\left(f, t \wedge T_n\right) P_{B_\alpha}\left(df\right).$$

One must then show that, on $(D[0,T], \mathcal{D}, Q_n)$, $\widetilde{B}_{\alpha,n}$ is martingale for $\underline{\mathcal{D}}$ such that

$$Q_n \circ \underline{\widetilde{B}}_{\alpha,n}^{-1} = P_{B_\alpha}.$$

But, almost surely, with respect to P_{B_α},

$$\tilde{L}(f, t \wedge T_n) \geq \frac{1}{n},$$

so that one can compute $\ln\left[\tilde{L}(f, t \wedge T_n)\right]$, and consequetly apply Itô's formula to obtain, almost surely, with respect to P_{B_α}, the following equality:

$$\begin{aligned} \ln\left[\tilde{L}(f, t \wedge T_n)\right] &= \sqrt{\alpha} \int_0^t I_{[\![0,T_n]\!]}(f,x) \, \frac{s(f,x)}{\tilde{L}(f,x)} B_1(f,dx) \\ &- \frac{\alpha}{2} \int_0^t I_{[\![0,T_n]\!]}(f,x) \left\{ \frac{s(f,x)}{\tilde{L}(f,x)} \right\}^2 \beta(dx). \end{aligned}$$

But

$$E_{P_{B_\alpha}}\left[\tilde{L}(\cdot, t \wedge T_n)\right] = 1,$$

because of the martingale property of $\tilde{L}(\cdot, t \wedge T_n)$, for $\underline{\mathcal{D}}$, and P_{B_α}. One can thus invoke Girsanov's theorem to assert that, the base being $(D[0,T], \mathcal{D}, Q_n)$, and the filtration $\underline{\mathcal{D}}$,

$$Q_n \circ \underline{\widetilde{B}}_{\alpha,n}^{-1} = P_{B_\alpha}.$$

Now $\widetilde{B}_{\alpha,n+1}^{T_n} = \widetilde{B}_{\alpha,n}$, so that one can legitimately define, again for the base $(D[0,T], \mathcal{D}, P_{B_\alpha})$, and the filtration $\underline{\mathcal{D}}$, the process

$$\widetilde{B}_\alpha(f,t) = -\alpha \int_0^t I_{[\![0,\lim_n T_n]\!]}(f,x) \, \frac{s(f,x)}{\tilde{L}(f,x)} \beta(dx) + ev^{P_{B_\alpha}}(f,t).$$

Since $\lim_n T_n = T$, almost surely, with respect to P_{Y_α}, and that the latter is absolutely continuous with respect to P_{B_α}, one has, almost surely, with respect to P_{Y_α},

$$\widetilde{B}_\alpha(f,t) = -\alpha \int_0^t \frac{s(f,x)}{\tilde{L}(f,x)} \beta(dx) + ev^{P_{Y_\alpha}}(f,t).$$

One must then finally check that, for the base $(D[0,T], \mathcal{D}, P_{Y_\alpha})$ and the filtration $\underline{\mathcal{D}}$,

$$P_{Y_\alpha} \circ \underline{\widetilde{B}}_\alpha^{-1} = P_{B_\alpha}.$$

To that end, one recognizes that

$$\tilde{L}(f, t\wedge T_n)\ \widetilde{B}_\alpha(f, t\wedge T_n) = \tilde{L}(f, t\wedge T_n)\ \widetilde{B}_{\alpha,n}(f, t\wedge T_n).$$

But, on the base $(D[0,T], \mathcal{D}, Q_n)$, and for the filtration $\underline{\mathcal{D}}$, $\widetilde{B}_{\alpha,n}$ is a martingale, and $Q_n(df) = \tilde{L}(f, t\wedge T_n) P_{B_\alpha}(df)$, so that $\tilde{L}(\cdot, \cdot\wedge T_n)$ $\widetilde{B}_\alpha(\cdot, \cdot\wedge T_n)$ is a martingale on the base $(D[0,T], \mathcal{D}, P_{B_\alpha})$, and for the filtration $\underline{\mathcal{D}}$, and consequently a martingale on the base $(D[0,T), \mathcal{D}, Q_n)$, for the same the filtration. Since $\underline{\mathcal{B}}^{(1)} \subseteq \underline{\mathcal{D}}$, T_n is a stopping time for $\underline{\mathcal{D}}$, and thus, since, on one hand, $\mathcal{D}_{t\wedge T_n} = \mathcal{B}^{(1)}_{t\wedge T_n} \vee \mathcal{B}^{(2)}_{t\wedge T_n}$, and, on the other, $Q_{n|\mathcal{D}_{t\wedge T_n}} = P_{Y_\alpha|\mathcal{D}_{t\wedge T_n}}$,

$$P_{Y_\alpha}(df) = \tilde{L}(f, t\wedge T_n) P_{B_\alpha}(df).$$

$\widetilde{B}_\alpha$ is thus a local martingale on the base $(D[0,T], \mathcal{D}, P_{Y_\alpha})$, for the filtration $\underline{\mathcal{D}}$.

One must also check that $\widetilde{B}_\alpha$ has, with respect to P_{Y_α} the same law as B_α with respect to P. But, for scalars $\theta_1, \ \ldots\ , \theta_p$, forming the vector $\underline{\theta}_p$, and times $0 \leq t_1, \ \ldots\ , t_p \leq T$,

$$\begin{aligned} E_{P_{Y_\alpha}}\left[e^{i\langle \underline{\theta}_p, \underline{\widetilde{B}}^{(p)}_\alpha\rangle_{I\!R^p}}\right] &= \lim_n E_{P_{Y_\alpha}}\left[e^{i\langle \underline{\theta}_p, \underline{\widetilde{B}}^{(p)}_{\alpha,n}\rangle_{I\!R^p}}\right] \\ &= \lim_n E_{Q_n}\left[e^{i\langle \underline{\theta}_p, \underline{\widetilde{B}}^{(p)}_{\alpha,n}\rangle_{I\!R^p}}\right] \\ &= E_P\left[e^{i\langle \underline{\theta}_p, \underline{\widetilde{B}}^{(p)}\rangle_{I\!R^p}}\right], \end{aligned}$$

where $\underline{\widetilde{B}}^{(p)}_\alpha$, $\underline{\widetilde{B}}^{(p)}_{\alpha,n}$, and $\underline{\widetilde{B}}^{(p)}$ are vectors with respective components $\widetilde{B}_\alpha(\cdot, t_i)$, $\widetilde{B}_{\alpha,n}(\cdot, t_i)$, and $B_\alpha(t_i)$, for $0 \leq t_i \leq T,\ 1 \leq i \leq p$. □

Corollary. *It is assumed that* **A0**, **A1** *and* **A2** *obtain. One then has the following "innovations representation," for* $t \in [0,T]$, *almost surely, with respect to* P*:*

$$Y_\alpha(\omega, t) = \int_0^t \tilde{s}(Y_\alpha(\omega, \cdot), x)\, \beta_1(dx) + B^{Y_\alpha}_\alpha(\omega, t),$$

where

- $\tilde{s}$ *is defined on* $(D[0,T], \mathcal{D}, P_{Y_\alpha})$, *and is progressively measurable for* $\underline{\mathcal{D}}$,
- $B^{Y_\alpha}_\alpha$ *is defined on* $(\Omega, \mathcal{A}, P)$, *adapted to* $\underline{\sigma}^\circ(Y_\alpha)$, *and*
- $P \circ \left[\underline{B}^{Y_\alpha}_\alpha\right]^{-1} = P_{B_\alpha}$.

5. Absolute Continuity and Likelihood Ratio for P_{N_α} and P_{X_α}.

5.1. The "Inversion" Process M

The Cramér-Hida representation "says" intuitively that the paths of B_α and N_α are, probabilistically, in one-to-one correspondence. The mathematical expression for this intuition is the M process whose definition and properties follow.

Terms whose definition is omitted are those of Section 2. $I_{[0,t]}$ denotes the indicator of the interval $[0,t]$. The basic probability space is

$$(L_2[0,T], \mathcal{B}(L_2[0,T]), P_{N_\alpha}).$$

For $t \in [0,T]$, fixed, but arbitrary, the following variables indexed on i are considered on $L_2[0,T] \times [0,T]$:

$$M_i(f,t) = \frac{1}{\lambda_i} \langle U\left[I_{[0,t]}\right], e_i\rangle_{L_2[0,T]} \langle f, e_i\rangle_{L_2[0,T]} .$$

One has that $E_{P_{N_\alpha}}[M_i(\cdot,t)] = 0$, and that

$$\begin{aligned} E_{P_{N_\alpha}}[M_i(\cdot,t) M_j(\cdot,t)] &= \frac{1}{\lambda_i\lambda_j} \langle U\left[I_{[0,t]}\right], e_i\rangle_{L_2[0,T]} \langle U\left[I_{[0,t]}\right], e_j\rangle_{L_2[0,T]} \\ &\times E_{P_{N_\alpha}}\left[\langle f, e_i\rangle_{L_2[0,T]} \langle f, e_j\rangle_{L_2[0,T]}\right] \\ &= \delta_{i,j} \langle I_{[0,t]}, J[e_i]\rangle_{L_2[\beta_\alpha]} \langle I_{[0,t]}, J[e_j]\rangle_{L_2[\beta_\alpha]}. \end{aligned}$$

One needs the following facts which result from simple calculations.

- The family $\{J[e_i],\ i \in \mathbb{N}\}$ is a complete orthonormal set in $L_2[\beta_\alpha]$.
- $\sum_{i=1}^{\infty} E_{P_{N_\alpha}}\left[M_i^2(\cdot,t)\right] = \left\|I_{[0,t]}\right\|^2_{L_2[\beta_\alpha]} = \beta_\alpha(t)$.
- For $t \in [0,T]$, fixed, but arbitrary, the series $\sum_{i=1}^{\infty} M_i(f,t)$ converges almost surely, with respect to P_{N_α}, and in $L_2[P_{N_\alpha}]$.

The inversion process M is defined by the following relation:

$$M(f,t) = \sum_{i=1}^{\infty} M_i(f,t).$$

Then one can sequentially check that the statements of the following list obtain.

– For $(i,t) \in I\!N \times [0,T]$, fixed, but arbitrary,

$$E\left[B_\alpha(\cdot,t)\ \langle N_\alpha(\cdot,\cdot), e_i\rangle_{L_2[0,T]}\right] = \langle U\left[I_{[0,t]}\right], e_i\rangle_{L_2[0,T]}.$$

– For $t \in [0,T]$, fixed, but arbitrary,

$$E\left[\{M(N_\alpha(\cdot,\cdot),t) - B_\alpha(\cdot,t)\}^2\right] = 0.$$

– Let $0 < t_1 < \cdots < t_n \leq T$, and $\theta_1,\ \ldots,\ \theta_n$, be arbitrary constants. Then,

$$E_{P_{N_\alpha}}\left[e^{i\sum_{j=1}^n \theta_j M(\cdot,t_j)}\right] = E\left[e^{i\sum_{j=1}^n \theta_j B_\alpha(\cdot,t_j)}\right].$$

– M has, with respect to P_{N_α}, independent increments.

– For $0 < s < t \leq T$, $E_{P_{N_\alpha}}[M(\cdot,s)\,M(\cdot,t)] = \beta_\alpha(s \wedge t)$.

– For $0 < s < t \leq T$, $E_{P_{N_\alpha}}\left[\{M(\cdot,t) - M(\cdot,s)\}^2\right] = \beta_\alpha(t) - \beta_\alpha(s)$.

– Let $t \in [0,T]$, be fixed, but arbitrary, and let $\mathcal{M}_t^\circ$ be the σ-algebra generated by $\{M(\cdot,s),\ s \leq t\}$, on $L_2[0,T]$. Then, with respect to P_{N_α}, M is a square integrable martingale for $\underline{\mathcal{M}}^\circ = \{\mathcal{M}_t^\circ,\ t \in [0,T]\}$.

– The process M is separable.

– With respect to P_{N_α}, the paths of M almost surely belong to $D[0,T]$.

5.2. The Conditional Law of B_α given N_α

This conditional law is given by the following proposition.

Proposition 11. *The assumptions being those of Section 2.1, B_α has, with respect to N_α, a regular conditional law which is a point mass located at M.*

Proof. Let $F \subseteq D[0,T]$, and $G \subseteq L_2[0,T]$ be measurable subsets. One has that

$$\begin{aligned} & P(\omega \in \Omega\ : B_\alpha(\omega,\cdot) \in F,\ N_\alpha(\omega,\cdot) \in G) \\ = \ & P(\omega \in \Omega :\ M(N_\alpha(\omega,\cdot),\cdot) \in F,\ N_\alpha(\omega,\cdot) \in G). \end{aligned}$$

Indeed, as, in $L_2[P]$, for $t \in [0,T]$, fixed, but arbitrary,

$$B_\alpha(\cdot,t) = M(N_\alpha(\cdot,\cdot),t),$$

this equality is obviously true whenever

$$
\begin{gathered}
0 \leq t_1 < \cdots < t_p \leq T, \\
B_i \in \mathcal{B}[\mathbb{R}], \ 1 \leq i \leq p, \\
F = \left\{ f \in D[0,T] : \ ev_{t_1}(f) \in B_1, \ \ldots \ , ev_{t_p}(f) \in B_p \right\}, \\
\{g_1, \ \ldots \ , g_q\} \subseteq L_2[0,T], \\
\tilde{B}_j \in \mathcal{B}[\mathbb{R}], \ 1 \leq j \leq q, \\
G = \left\{ g \in L_2[0,T] : \ \langle g, g_1 \rangle_{L_2[0,T]} \in \tilde{B}_1, \ \ldots \ , \langle g, g_q \rangle_{L_2[0,T]} \in \tilde{B}_q \right\}.
\end{gathered}
$$

As such sets generate the corresponding σ-algebras, the equality is true in general. But then

$$
\begin{aligned}
& P\left(\omega \in \Omega : \ M\left(N_\alpha(\omega,\cdot),\cdot\right) \in F, \ N_\alpha(\omega,\cdot) \in G\right) \\
= & \int_G P_{N_\alpha}(dg)\, P\left(M \circ N_\alpha \in F \mid N_\alpha = g\right) \\
= & \int_G P_{N_\alpha}(dg)\, E\left[I_F\left(M \circ N_\alpha\right) \mid N_\alpha = g\right] \\
= & \int_G P_{N_\alpha}(dg)\, I_F\left(M(g)\right).
\end{aligned}
$$

Corollary. $E_{P_{N_\alpha}}\left[\frac{dP_{Y_\alpha}}{dP_{B_\alpha}} \mid N_\alpha = g\right] = \frac{dP_{Y_\alpha}}{dP_{B_\alpha}}\left(M(g,\cdot)\right).$

5.3. Existence and Form of the Likelihood Ratio

One has the following theorem, which in particular shows that there is no need of robust versions of the likelihood ratio in the sense of Clark [7]:

Theorem 2. *One writes B for B_α, N for N_α, and Y for Y_α. Other notation is as already encountered. Suppose then that*

$$
N(\omega,t) = \int_0^T F(t,x)\, B(\omega,dx),
$$

where

- *assumptions* **A0** *and* **A1** *are valid for B with $\beta_1 = \beta_2 = \beta$,*
- *F is a non-anticipative ($F(t,x) = 0$, for $x > t$), measurable function, defined on $[0,T] \times [0,T]$, whose equivalence classes generate $L_2[\beta]$,*

– *$S(\omega,\cdot) \in H(N)$, almost surely, with respect to P.*

The following statements are then valid.

– *P_{S+N} is absolutely continuous with respect to P_N.*

– *One has that*

$$\frac{dP_{S+N}}{dP_N}(f) = \tilde{\Lambda} \circ M(f),$$

where, for $f \in \mathcal{L}_2[0,T]$, M is the process

$$M(f,t) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \langle UI_{[0,t]}, e_k \rangle_{L_2[0,T]} \langle f, e_k \rangle_{L_2[0,T]}.$$

– *With respect to P_Y, and for $f \in D[0,T]$, $\tilde{\Lambda}$ has the representation*

$$\begin{aligned} \ln\left[\tilde{\Lambda}(f)\right] &= \int_0^T s(f,x)\, ev^{P_Y}(f,dx) \\ &- \frac{\alpha}{2}\int_0^T s^2(f,x)\,\beta(dx) \\ &- \sqrt{1-\alpha}\int_0^T s(f,x)\,\tilde{B}_2^Y(f,dx), \end{aligned}$$

with $\tilde{B}_2^Y$, a Poisson martingale, independent of B_1^Y, and s, the predictable process resulting from the RKHS condition of assumption 3.

– *With respect to P_B, $\tilde{\Lambda}$ can be approximated by the sequence $I_{C_n}\Phi_n$, where $C_n = \{f \in D[0,T] :\ T_n(f) = T\}$, T_n is the stopping time of Proposition 6, and Φ_n is given by the following expression, which must be interpreted as that of 3.:*

$$\begin{aligned} \ln\left[\Phi_n(f)\right] &= \int_0^T \tilde{s}_n(f,x)\, ev^{P_B}(f,dx) \\ &- \frac{\alpha}{2}\int_0^T \tilde{s}_n^2(f,x)\,\beta(dx) \\ &- \sqrt{1-\alpha}\ \int_0^T \tilde{s}_n(f,x)\,\tilde{B}_2^{Y,B}. \end{aligned}$$

– *If one can assume that*

$$P_N\left(f \in D\,[0,T]:\ \int_0^T s^2\left(M\left(f,\cdot\right),x\right)\beta\left(dx\right) < \infty\right) = 1,$$

P_{S+N} and P_N are then mutually absolutely continuous, and, mutatis mutandis, *the likelihood formula of 3. obtains with respect to $P_B = P_{\underline{N}\circ\underline{M}^{-1}}$. A sufficient condition for the latter, in terms of S, is*

$$E\left[\exp\left\{\frac{1}{2}\|S\left(\cdot,\cdot\right)\|^2_{H(N)}\right\}\right] < \infty.$$

Proof. Assumption 3, in conjunction with Baker et al [1, Theorem 3, Step 3, p. 170] means that, for some appropriate s,

$$P\left(\omega \in \Omega:\ \int_0^T s^2\left(\omega,x\right)\beta\left(dx\right) < \infty\right) = 1.$$

The Corollary to Proposition 7 yields then that P_Y is absolutely continuous with respect to P_B, and then, from Proposition 10, one has that Y has a stochastic integral representation. The specific form of the likelihood follows then from Proposition 7.

Now, as, *mutatis mutandis* (Baker et al [1, Theorem 1, p. 163]),

$$\underline{N} = \Phi \circ \underline{B}, \text{ and } \underline{S+N} = \Phi \circ \underline{Y},$$

for any Borel set A of $L_2\,[0,T]$, it follows, using the Corollary to Proposition 11, that

$$P_{S+N}\left(A\right) = \int_A \frac{dP_Y}{dP_B}\left(\underline{M}\left(f\right)\right)P_N\left(df\right).$$

The other statements repeat earlier relevant results. □

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