International Journal of Pure and Applied Mathematics

Volume 2 No. 2 2002, 227-234

WEAKLY DEFECTIVE PROJECTIVE VARIETIES

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Abstract: Fix positive integers s and a_i, $1 \le i \le s$. Set $\alpha := (a_1, \dots, a_s)$. Let $X_i \subset \mathbf{P}^N$, $1 \le i \le s$, be irreducible subvarieties. The s-ple $(X_1, \dots, X_s)$ is called α-defective if the join of a_1 copies of X_i, a_2 copies of X_2, and so on until we take the join with a_s copies of X_s has not the expected dimension. Fix general a_i-ples $(P_{i,1}, \dots, P_{i,a_i}) \in X_i^{a_i}$ and let $\mathbf{X}(P_{1,1}, \dots, P_{s,a_s})$ be the linear span of all tangent spaces $T_{P_{i,x_i}} X_i$, $1 \le i \le s$, $1 \le x_i \le a_i$. $(X_1, \dots, X_s)$ is called weakly α-defective if there is a pair (i, j) such that for a general hyperplane H containing $\mathbf{X}(P_{1,1}, \dots, P_{s,a_s})$ there is a positive dimensional variety Φ such that $P_{i,j} \in \Phi \subseteq X_i$ and H is tangent to X_i at a general point of Φ. Here we prove that α-defectivity implies weakly α-defectivity and classify the weakly defective s-ples such that $\dim(X_s) = 1$.

AMS Subject Classification: 14N05
Key Words: projective varieties, joins, contact locus, defective projective varieties, weakly defective projective varieties

1. Introduction

We work over an algebraically closed field $\mathbf{K}$ with $\text{char}(\mathbf{K}) = 0$. Fix positive

Received: June 6, 2002

integers s and a_i, $1 \le i \le s$. Set $\alpha := (a_1, \dots, a_s)$. Let $X_i \subset \mathbf{P}^N$, $1 \le i \le s$, be irreducible subvarieties with $X_i \ne X_j$ for every pair (i,j) with $i \ne j$. We allow the case in which $X_i \subset X_j$ for some (i,j) with $i \ne j$. Set $\mathbf{X} := (X_1, \dots, X_s)$ and let $\mathbf{X}(\alpha) \subseteq \mathbf{P}^N$ be the join of a_1 copies of X_i, a_2 copies of X_2, and so on until we take the join with a_s copies of X_s. Fix general a_i-ples $(P_{i,1}, \dots, P_{i,a_i}) \in X_i^{a_i}$ and let $\mathbf{X}(P_{1,1}, \dots, P_{s,a_s})$ be the linear span of all tangent spaces $T_{P_{i,x_i}} X_i$, $1 \le i \le s$, $1 \le x_i \le a_i$. By Terracini lemma ([1], Cor. 1.11) we have $\dim(\mathbf{X}(\alpha)) = \dim(\mathbf{X}(P_{1,1}, \dots, P_{s,a_s}))$. Thus $\dim(\mathbf{X}(\alpha)) \le \min\{N, \sum_{i=1}^{s}(\dim(X_i)+1) - 1\}$. We will say that $\mathbf{X}$ is α-defective if $\dim(\mathbf{X}(\alpha)) < \min\{N, \sum_{i=1}^{s}(\dim(X_i)+1) - 1\}$. Generalizing the case $s = 1$ studied in [3] we will say that $\mathbf{X}$ is weakly α-defective if for general $(P_{i,1}, \dots, P_{i,a_i}) \in X_i^{a_i}$, $1 \le i \le s$, there is a pair (i,j) such that for a general hyperplane H containing $\mathbf{X}(P_{1,1}, \dots, P_{s,a_s})$ there is a positive dimensional variety Φ such that $P_{i,j} \in \Phi \subseteq X_i$ and H is tangent to X_i at a general point of Φ. The maximal dimension of any such subvariety Φ will be called the degree of weakly α-defectivity of the s-ple $(X_1, \dots, X_s)$. Fix an integer u with $1 \le u \le s$. We will say that $\mathbf{X}$ is weakly α-defective on the u-factor or on the factor u if for general $(P_{i,1}, \dots, P_{i,a_i}) \in X_i^{a_i}$, $1 \le i \le s$, there is an integer j with $1 \le j \le a_u$ such that a general hyperplane H containing $\mathbf{X}(P_{1,1}, \dots, P_{s,a_s})$ is tangent to X_u along a positive dimensional irreducible variety containing $P_{u,j}$. We chosed a normalization different from the one chosen in [3] for the case $s = 1$: our m-defective (respectively weakly m-defective) varieties are the $(m-1)$-defective (respectively weakly $(m-1)$-defective) varieties of [3]. We will say that $\mathbf{X}$ is sharply α-defective (respectively sharply weakly α-defective) if it is α-defective (respectively weakly α-defective and for all $\beta = (b_1, \dots, b_x)$ with $x \le s$, $b_i \le a_i$ for every i with $1 \le i \le x$ and $b_1 + \cdots + b_x < a_1 + \dots a_s$ it is not β-defective.

Here are our main results on this topic.

Theorem 1. *Let $\mathbf{X} = (X_1, \dots, X_s)$ be an α-defective s-ple of $\mathbf{P}^N$ such that each variety $X_1, \dots, X_s$ is non-degenerate. Then $\mathbf{X}$ is weakly α-defective*

Theorem 2. *Let $X_i \subset \mathbf{P}^N$, $1 \le i \le s$, be integral non-degenerate varieties with $\dim(X_s) = 1$. Fix positive integers a_i, $1 \le i \le s$, and set $\alpha := (a_1, \dots, a_s)$ and $\alpha(s) := (a_1, \dots, a_{s-1})$. Assume $\sum_{i=1}^{s} a_i(\dim(X_i)+1) < N$. If the s-ple $(X_1, \dots, X_s)$ is weakly α-defective, then the $(s-1)$-ple $(X_1, \dots, X_{s-1})$ is weakly $\alpha(s)$-defective.*

Theorem 3. *Fix positive integers s and a_i, $1 \le i \le s$ with $s \ge 2$. Set $\alpha := (a_1, \dots, a_s)$. Let $X_i \subset \mathbf{P}^N$, $1 \le i \le s$, be irreducible non-degenerate* *subvarieties. Assume $\sum_{i=1}^{s} a_i(\dim(X_i)+1) \le N$ and that no variety X_i is weakly a_i-defective. Then for a general $(g_1,\dots,g_{s-1}) \in Aut(\mathbf{P}^N)^{s-1}$ the s-ple $(X_1, g_1(X_2),\dots,g_{s-1}(X_s))$ is not weakly α-defective.*

In Section 3 we will generalize the notion of weak defectivity to the case of higher order osculating spaces and prove a generalization of Theorem 3.

2. The Proofs

Proof of Theorem 1. By induction on the integer $a_1+\cdots+a_s$ we may assume that $\mathbf{X}$ is sharply α-defective. Fix general $(P_{i,1},\dots,P_{i,a_i}) \in X_i^{a_i}$, $1 \le i \le s-1$, and, if $a_s \ge 2$, general $P_{s,j} \in X_s$, $1 \le j \le a_s-1$. Let V be the linear span of all tangent spaces of X_i at $P_{i,u}$, $1 \le i \le s-1$, $1 \le u \le a_i$, and X_s at $P_{s,j}$, $1 \le j \le a_s-1$. By Terracini lemma ([1], Cor. 1.11) and the sharp α-defectivity of $\mathbf{X}$ we have $\dim(V) = \sum_{i=1}^{s} a_i(\dim(X_i)+1) - \dim(X_s) - 2$. Let $f : \mathbf{P}^N \backslash V \to \mathbf{P}^m$, $m = N - \dim - 1$, be the linear projection from V. Set $g := f|(X_s \backslash X_s \cap V)$; g is a well-defined rational map because X_s is non-degenerate and hence it is not contained in V. By Terracini lemma ([1], Cor. 1.11) the α-defectivity of $\mathbf{X}$ implies that for a general $Q \in X_s$ we have $T_Q X_s \cap V \neq \emptyset$. This implies that for a general $Q \in X_s$ the rational map g has differential with non-zero kernel. Since char$(\mathbf{K}) = 0$, this implies that for a general $Q \in X_s$ the fiber $g^{-1}(g(Q))$ has a positive dimensional component, Φ, with $Q \in \Phi$ and $\dim(\Phi) = \sum_{i=1}^{s}(\dim(X_i)+1) - 1 - \dim(\mathbf{X}(\alpha)) > 0$. Take $P_{s,a_s} = Q$. A general hyperplane M of $\mathbf{P}^m$ tangent to $g(X_s \backslash X_s \cap V)$ at $f(P_{s,a_s})$ corresponds to a general hyperplane H of $\mathbf{P}^N$ tangent to X_i, $1 \le i \le s$ at all points $P_{i,j}$, $1 \le j \le a_i$. By construction H is tangent to X_s at a general point of Φ, proving the theorem.

Remark 1. Assume $\mathbf{X}$ sharply α-defective. The proof of Theorem 1 shows that for every integer i with $1 \le i \le s$ the s-ple $\mathbf{X}$ is weakly α-defective on the factor i.

Remark 2. Let $X \subseteq \mathbf{P}^N$ be an irreducible non-degenerate variety. Since char$(\mathbf{K}) = 0$, X is reflexive and a general hyperplane tangent to X is tangent to X along a linear subspace ([5], p. 173). Call $\delta \ge 0$ the dimension of this linear space. The integer $\delta + 1$ is the codimension of the dual variety X^* of X in $\mathbf{P}^{N*}$. If $Q \in X_{reg}$ and L is a linear space with $Q \in L \subseteq X$, then $L \subseteq T_Q X$. Thus X is weakly 1-defective if and only if $\delta > 0$ and in that case δ is the order of weak 1-defectivity of X. Quite often for a general $Q \in X$ there is a linear subspace V with $\dim(V) > \delta$ and $P \in V \subseteq X$ and hence $V \subseteq T_Q X$. For instance in the case of a smooth scroll we usually have $\dim(V) = 2\delta$. Notice that usually $T_Q X$ is not tangent to X along a positive dimensional subvariety of X even when $\delta > 0$. For instance, by Zak 's Tangency theorem ([6], Theorem 1.7) this is never the case for any Q if X is smooth.

We believe that the following invariant $z(\mathbf{X}, \alpha)$ is interesting, but we will not use it in this paper.

Definition 1. Fix positive integers s and a_i, $1 \le i \le s$. Set $\alpha := (a_1, \dots, a_s)$. Let $X_i \subset \mathbf{P}^N$, $1 \le i \le s$, be irreducible subvarieties such that $\langle X_1 \cup \cdots \cup X_s \rangle = \mathbf{P}^N$ and $X_i \neq X_j$ for every pair (i, j) with $i \neq j$. Set $\mathbf{X} := (X_1, \dots, X_s)$. Fix general a_i-ples $(P_{i,1}, \dots, P_{i,a_i}) \in X_i^{a_i}$ and let $\mathbf{X}(P_{1,1}, \dots, P_{s,a_s})$ be the linear span of all tangent spaces $T_{P_{i,x_i}} X_i$, $1 \le i \le s$, $1 \le x_i \le a_i$. Assume that $\mathbf{X}$ is not weakly α-defective and call $z(\mathbf{X}, \alpha)$ the codimension (in the linear space of all hyperplanes containing $\mathbf{X}(P_{1,1}, \dots, P_{s,a_s})$) of the set of all hyperplanes, H, containing $\mathbf{X}(P_{1,1}, \dots, P_{s,a_s})$ and for which there is a pair (i, j) such that H is tangent to X_i along a positive dimensional irreducible variety containing $P_{i,j}$.

Let $X, Y \subset \mathbf{P}^N$ be integral non-degenerate varieties such that Y is a curve. It is well-known (see for instace [2], Cor. 2.3, with respect to the data $i = 1$ and $\lambda = 1$) that $\dim([X; Y]) = \min\{N, \dim(X) + 2\}$. Hence, non-degenerate curves behave in the best possible way with respect to defectivity.

Remark 3. Let $X_i \subset \mathbf{P}^N$, $1 \le i \le s$, be integral varieties such that X_s is a non-degenerate curve. Fix positive integers s and a_i, $1 \le i \le s$, and set $\alpha := (a_1, \dots, a_s)$. Assume that the s-ple $\mathbf{X} := (X_1, \dots, X_s)$ is weakly α-defective. Fix general a_i-ples $(P_{i,1}, \dots, P_{i,a_i}) \in X_i^{a_i}$ and let $\mathbf{X}(P_{1,1}, \dots, P_{s,a_s})$ be the linear span of all tangent spaces $T_{P_{i,x_i}} X_i$, $1 \le i \le s$, $1 \le x_i \le a_i$. By assumption for a general hyperplane H containing $\mathbf{X}(P_{1,1}, \dots, P_{s,a_s})$ there are a pair (i, j) and an irreducible variety $\Phi \subseteq X_i$ such that $P_{i,j} \in \Phi$, $\dim(\Phi) > 0$ and H is tangent to X_i at every point of $(X_i)_{reg} \cap \Phi$. In particular $\Phi \subseteq H$. Since X_s is a non-degenerate curve, we have $i \neq s$.

Proof of Theorem 2. Fix general a_i-ples $(P_{i,1}, \dots, P_{i,a_i}) \in X_i^{a_i}$ and let $\mathbf{X}(P_{1,1}, \dots, P_{s,a_s})$ be the linear span of all tangent spaces $T_{P_{i,x_i}} X_i$, $1 \le i \le s$, $1 \le x_i \le a_i$. By assumption for a general hyperplane H containing $\mathbf{X}(P_{1,1}, \dots, P_{s,a_s})$

there are a pair (i,j) and an irreducible variety $\Phi \subseteq X_i$ such that $P_{i,j} \in \Phi$, dim$(\Phi) > 0$ and H is tangent to X_i at every point of $(X_i)_{reg} \cap \Phi$. By Remark 3, we have $i \neq s$. Notice that a linear projection of a non-degenerate variety is a non-degenerate variety. Call $\mathbf{Y}$ the linear span of all tangent spaces $T_{P_{u,x_u}}X_i$, $1 \leq u \leq s$, $1 \leq x_u \leq a_u$ and $(u,x_u) \neq (i,j)$. $\mathbf{Y}$ does not contain Φ by the generality of the points $P_{u,y}$. Hence, taking the linear projection from $\mathbf{Y}$ we reduce the proof of Theorem 2 to the particular case $s = 2, a_1 = a_2 = 1$ in a smaller projective space, say $\mathbf{P}^m$. By assumption (X_1,X_2) is weakly $(1,1)$-defective and X_2 is a non-degenerate curve. Assume that X_1 is not weakly 1-defective, i.e. assume that the dual variety $X_1^* \subseteq \mathbf{P}^{m*}$ is a hypersurface (Remark 1). Set $B := \{H \in X_1^*$: the contact locus of H intersects $(X_1)_{reg}$ in a positive dimensional set$\}$, $\Gamma := \{(Q,H) : Q \in (X_1)_{reg}, H \in \mathbf{P}^{m*}$ and $T_QX_1 \subset H\}$ and let $\pi_1 : \Gamma \to X_1$ and $\pi_2 : \Gamma \to \mathbf{P}^{m*}$ be the projections. Thus Γ is irreducible, dim$(\Gamma) = m-1$ and the closure of $\pi_2(\Gamma)$ is the dual variety X_1^*. Since X_1 is weakly 1-ordinary and char$(\mathbf{K}) = 0$, π_2 is birational onto its image. Hence for every codimension one subvariety D of X_1^* we have dim$(\pi_2^{-1}(D)) \leq m-2$. Notice that $\pi_2^{-1}(H)$ is positive-dimensional for every $H \in B$. Hence for any codimension one subvariety D of X_1^* the contact locus of a general $H \in D$ is finite. Thus dim$(B) \leq m-3$. Since X_2 is a curve and it is not a line, we have dim$(X_2^*) = m-1$. Thus $X_1^* \cap X_2^*$ cannot be contained in B, i.e. the pair (X_1,X_2) is not weakly $(1,1)$-defective, contradiction. □

Remark 4. Fix positive integers s and a_i, $1 \leq i \leq s$, with $s \geq 2$. Set $\alpha := (a_1,\dots,a_s)$. Let $X_i \subset \mathbf{P}^N$, $1 \leq i \leq s$, be irreducible non-degenerate subvarieties such that $X_i \neq X_j$ for every pair (i,j) with $i \neq j$. Set $\mathbf{X} := (X_1,\dots,X_s)$ and assume that $\mathbf{X}$ is weakly α-defective. Fix general a_i-ples $(P_{i,1},\dots,P_{i,a_i}) \in X_i^{a_i}$ and let $\mathbf{X}(P_{1,1},\dots,P_{s,a_s})$ be the linear span of all tangent spaces $T_{P_{i,x_i}}X_i$, $1 \leq i \leq s$, $1 \leq x_i \leq a_i$. Let H be a general hyperplane H containing $\mathbf{X}(P_{1,1},\dots,P_{s,a_s})$. By assumption there are integers i,j with $1 \leq i \leq s$ and $1 \leq j \leq a_i$ and an irreducible variety $\Phi \subseteq X_i$ such that $P_{i,j} \in \Phi$, $\dim(\Phi) > 0$ and H is tangent to X_i at every point of $(X_i)_{reg} \cap \Phi$. Assume $\dim(\Phi) = \dim(X_i) - 1$; with the terminology of [3], this would be the divisorial case. We claim that $\mathbf{X}(P_{1,1},\dots,P_{s,a_s})$ is tangent to Φ at each regular point of Φ. Indeed, since $\dim(\Phi) = \dim(X_i) - 1$ and X_i is non-degenerate, $\mathbf{X}(P_{1,1},\dots,P_{s,a_s}) \cap X_i$ contains only finitely many effective divisors of X_i, say Φ and $D_1,\dots,D_x$ with $x \geq 0$. Assume that $\mathbf{X}(P_{1,1},\dots,P_{s,a_s})$ is not everywhere tangent to Φ. Hence the general hyperplane, M, containing $\mathbf{X}(P_{1,1},\dots,P_{s,a_s})$ is not everywhere tangent to Φ. By assumption, there is a divisor D_j such that M is tangent to X_i at a general point of D_j. But since there are only finitely many divisors $D_1, \dots, D_x$ while the hyperplanes containing $\mathbf{X}(P_{1,1}, \dots, P_{s,a_s})$ are infinite, the trick just used (fix a general point of D_j and a general tangent vector to D_j) proves that M is not tangent to X_i at the general point of any of the divisors $\Phi, D_1, \dots, D_x$.

Proposition 1. *Fix positive integers s and a_i, $1 \le i \le s$ with $s \ge 2$. Set $\alpha := (a_1, \dots, a_s)$ and $\alpha(s) := (a_1, \dots, a_{s-1})$. Let $X_i \subset \mathbf{P}^N$, $1 \le i \le s$, be irreducible non-degenerate subvarieties such that $X_i \ne X_j$ for every pair (i,j) with $i \ne j$. Assume $\sum_{i=1}^s a_i(dim(X_i)+1) \le N$, that $(X_1, \dots, X_{s-1})$ is not weakly $\alpha(s)$-defective and that X_s is not weakly a_s-defective. Then for a general $g \in Aut(\mathbf{P}^N)$ the s-ple $(X_1, \dots, X_s)$ is not weakly α-defective.*

Proof. Fix general a_i-ples $(P_{i,1}, \dots, P_{i,a_i}) \in X_i^{a_i}$, $1 \le i \le s$, and let Y be the linear span of all tangent spaces $T_{P_{i,x_i}}X_i$, $1 \le i \le s-1$, $1 \le x_i \le a_i$. Set $A := \{H \in \mathbf{P}^{N*} : H$ contains $Y\}$ and $B := \{H \in A : H$ is tangent to $X_1 \cup \dots \cup X_{s-1}$ along a positive dimensional irreducible variety containing one of the points $P_{i,j}$ with $1 \le i \le s-1\}$. By assumption $B \ne A$. Let E be the linear span of $T_{P_{s,k}}X_s$, $1 \le k \le a_s$. Notice that for every $g \in Aut(\mathbf{P}^N)$ the linear space $g(E)$ is the linear span of the tangent spaces to $g(X_s)$ at the points $g(X_{s,k})$, $1 \le k \le a_s$. By the dimensional transversality of a general translate ([4]) we have $A \cap g(E) \ne B \cap g(E)$ for a general $g \in Aut(\mathbf{P}^N)$. Hence to prove Proposition 1 it is sufficient to show that for a general $g \in Aut(\mathbf{P}^N)$ a general hyperplane containing $Y \cup g(E)$ is not tangent to X_s along a positive dimensional subvariety containing one of the points $g(P_{s,k})$, $1 \le k \le a_s$. Set $F := \{H \in \mathbf{P}^{N*} : E \subset H\}$ and $G := \{H \in F : H$ is tangent to X_s along a positive dimensional irreducible variety containing one of the points $P_{s,k}$, $1 \le k \le a_s\}$. By assumption we have $G \ne F$. By the transversality of a general translate ([4]) we have $g^{-1}(A) \cap E \ne g^{-1}(A) \cap F$ for a general $g \in Aut(\mathbf{P}^N)$. Since $g(g^{-1}(A) \cap E) = A \cap g(E)$ and $g(g^{-1}(A) \cap F) = A \cap g(F)$, we conclude. □

Proof of Theorem 3. Use induction on s and apply Proposition 1.

3. Higher Order Osculating Spaces

Fix positive integers s and a_i, $1 \le i \le s$. For every pair (i,j), with $1 \le i \le s$ and $1 \le j \le a_i$, fix an integer $m_{i,j} > 0$. Set $\alpha := (a_1, \dots, a_s)$ and $\mu := (m_{1,1}, \dots, m_{1,a_1}; \dots; m_{s,1}, \dots, m_{s,a_s})$. Let $X_i \subset \mathbf{P}^N$, $1 \le i \le s$, be irreducible subvarieties with $\langle X_1 \cup \dots \cup X_s \rangle = \mathbf{P}^N$ and $X_i \ne X_j$ for every pair (i,j) with $i \ne j$. Set $\mathbf{X} := (X_1, \dots, X_s)$. Fix general a_i-ples $(P_{i,1}, \dots, P_{i,a_i}) \in X_i^{a_i}$ and let

$\mathbf{X}(P_{1,1},\dots,P_{s,a_s};\mu)$ be the linear span of the osculating spaces $Osc(m_{i,j})_{P_{i,x_i}}X_i$ of order $m_{i,j}$ to X_i at $P_{i,j}$.

We will say that $\mathbf{X}$ is weakly (α,μ)-defective if for general $(P_{i,1},\dots,P_{i,a_i}) \in X_i^{a_i}$, $1 \le i \le s$, there is a pair (i,j) and an irreducible variety Φ such that $\dim(\Phi) > 0$, $P_{i,j} \in \Phi \subseteq X_i$ and a general hyperplane H containing $\mathbf{X}(P_{1,1},\dots,P_{s,a_s};\mu)$ is tangent to X_i at a general point of Φ. The maximal dimension of any such variety Φ will be called the degree of weakly (α,μ)-defectivity of $\mathbf{X}$. Now take positive integers $x, b_1,\dots,b_x$ and for all pairs (i,j) with $1 \le i \le x$ and $1 \le j \le a_i$ fix integers $n_{i,j} > 0$. Set $\beta := (b_1,\dots,b_x)$ and $\nu := (n_{1,1},\dots,n_{1,a_1};\dots;n_{s,1},\dots,n_{s,a_s})$.

We will say that $(\beta,\nu) \le (\alpha,\mu)$ if $x \le s$, $b_i \le a_i$ for all integers i with $1 \le i \le x$ and $n_{i,j} \le m_{i,j}$ for all pairs (i,j) with $1 \le i \le x$ and $1 \le j \le b_i$.

We will say that $(\beta,\nu) < (\alpha,\mu)$ if $(\beta,\nu) \le (\alpha,\mu)$ and $(\beta,\nu) \neq (\alpha,\mu)$.

We will say that $\mathbf{X}$ is sharply weakly (α,μ)-defective if it is weakly (α,μ)-defective but it is not weakly weakly (β,ν)-defective for all pairs (β,ν) such that $(\beta,\nu) < (\alpha,\mu)$; when $x < s$ we also require that this is the case after permuting the indices $1,\dots,s$ and making a corresponding permutation of the s-ples α and β so that we allow to delete any $s-x$ of the members of the s-ple $(X_1,\dots,X_s)$. A possible generalization: fix an integer $n_{i,j}$ with $1 \le n_{i,j} \le m_{i,j}$ and require that the general hyperplane H containing $\mathbf{X}(P_{1,1},\dots,P_{s,a_s};\mu)$ is osculating of order at least $n_{i,j}$ to X_i at a general point of Φ.

For all positive integers x, m set $\Gamma(x,m) := \binom{m+x}{x} - 1$. Notice that $\Gamma(x,m)$ is the expected dimension of the osculating space of order m to a general point of an x-dimensional variety.

The proof of Theorem 3 gives the following result.

Theorem 4. *Fix positive integers s, a_i, $1 \le i \le s$, and $m_{i,j}$, $1 \le i \le s$ and $1 \le j \le a_i$ with $s \ge 2$. Set $\alpha := (a_1,\dots,a_s)$, and $\mu := (m_{1,1},\dots,m_{1,a_1};\dots;m_{s,1},\dots,m_{s,a_s})$, and $\mu_i := (m_{i,1},\dots,m_{i,a_i})$. Let $X_i \subset \mathbf{P}^N$, $1 \le i \le s$, be irreducible non-degenerate subvarieties such that $X_i \neq X_j$ for every pair (i,j) with $i \neq j$. Assume*

$$\sum_{i=1}^{s}\sum_{j=1}^{a_i}\Gamma(\dim(X_i),m_{i,j}) < N\,,$$

and that each variety X_i is not weakly (a_i,μ_i)-defective. Then for a general $(g_1,\dots,g_{s-1}) \in Aut(\mathbf{P}^N)^{s-1}$ the s-ple $(X_1,g_1(X_2),\dots,g_{s-1}(X_s))$ is not weakly (α,μ)-defective.

Acknowledgements

The author was partially supported by MURST and GNSAGA of INdAM (Italy).

References

[1] B. Ådlandsvik, Joins and higher secant varieties, *Math. Scand.*, **62** (1987), 213–222.

[2] B. Ådlandsvik, Varieties with an extremal number of degenerate higher secant varieties, *J. Reine Angew. Math.*, **392** (1988), 16–26.

[3] L. Chiantini and C. Ciliberto, Weakly defective varieties, *Trans. Amer. Math. Soc.*, **354** (2002), 151–178.

[4] S.L. Kleiman, The transversality of a general translate, *Compositio Math.*, **28** (1974), 287–297.

[5] S.L. Kleiman, Tangency and duality, In: *Proc. 1984 Vancouver Conference in Algebraic Geometry*, CMS - AMS Conference Proceedings, No. 6 (1985), 163–226.

[6] F.L. Zak, Tangents and Secants of Algebraic Varieties, *Transl. Math. Monograph*, **127**, AMS (1993).