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LINEAR FRACTIONAL TRANSFORMATION OF CONTINUED FRACTIONS WITH BOUNDED PARTIAL QUOTIENTS: ARITHMETICAL AND GEOMETRICAL POINT OF VIEW

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Abstract: In this paper I will give new proofs of some simple theorems concerning continued fractions. Like J.O. Shallit said, "the proofs in the literature seem to be missing, incomplete, or hard to locate". In paticular, I will give two proofs of the following "*folk theorem*": if α is an irrational number whose continued fraction has bounded partial quotients, then any non-trivial linear fractional transformation of α also has bounded partial quotients. The first proof is based upon arithmetics arguments and the second one upon the geometrical interpretation of the best approximations to α. The result is a consequence of the following inequality due to Lagarias and Shallit [3]:

$$\frac{1}{|ad-bc|}L(\alpha) \leq L\left(\frac{a\alpha+b}{c\alpha+d}\right) \leq |ad-bc|L(\alpha)\,,$$

where $a, b, c, d \in \mathbb{Z}$, with $|ad-bc| \neq 0$ and $L(\alpha)$ is the Lagrange constant of α.

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1. The Lagrange Constant

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and

$$\alpha := [a_0, a_1, \ldots]$$

be the continuous fraction decomposition of α. Recall that it is obtained as follows; put $a_0 := [\alpha]$, where $[.]$ denotes the integer part. Then $\alpha = a_1 + \frac{1}{\alpha_1}$ with $\alpha_1 > 1$, and we set $a_1 := [\alpha_1]$. If $a_0, a_1, \ldots, a_{n-1}$ and $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are known, then $\alpha_{n-1} = a_{n-1} + \frac{1}{\alpha_n}$, with $\alpha_n > 1$ and we set $a_n := [\alpha_n]$. It can be shown [4] that this process does not terminate if and only if α is irrational. The integers $a_0, a_1, \ldots$ are the partial quotients of α, the numbers $\alpha_1, \alpha_2, \ldots$ are the complete quotients of α and the rationals

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n] = a_0 + \frac{1}{a_1 +}\frac{1}{a_2 +} \ldots + \frac{1}{a_n},$$

with p_n, q_n relatively prime integers, are the *convergents* of α and are such that $p_n/q_n \to \alpha$ as $n \to \infty$. It is well known that the p_n, q_n are recursively defined by the relations

$$p_0 := a_0,\ q_0 := 1,\ p_1 := a_0a_1 + 1,\ q_1 := a_1,$$

$$p_n := a_np_{n-1} + p_{n-2},\ \ q_n := a_nq_{n-1} + q_{n-2}.$$

The following Lemma is useful to define the Lagrange constant of α.

Lemma 1. *To each irrational number α corresponds a unique (extended) number $L(\alpha) \in [\sqrt{5}, \infty]$ (the Lagrange constant of α) having the following properties*

(i) *For each positive number $\mu < L(\alpha)$ there exist infinitely many pairs (p_i, q_i) with $q_i \neq 0$, such that*
$$\left|\alpha - \frac{p_i}{q_i}\right| \leq \frac{1}{\mu q_i^2}.$$

(ii) *If $L(\alpha)$ is finite, then, for each $\mu > L(\alpha)$, there are only finitely many pairs (p_i, q_i) satisfying the inequality*

$$\left|\alpha - \frac{p_i}{q_i}\right| \leq \frac{1}{\mu q_i^2}.$$

Proof. Let

$$\mu_i := q_i^{-2} \left| \alpha - \frac{p_i}{q_i} \right|^{-1} = q_i^{-1} |\alpha q_i - p_i|^{-1}, \quad i \geq 1,$$

$$L(\alpha) := \limsup_{i \to \infty} \mu_i \in \mathbb{R} \cup \{+\infty\}.$$

It then follows from the elementary properties of the upper limit that $L(\alpha)$ satisfies the conditions of the lemma, with the exception of the estimate $L(\alpha) \geq \sqrt{5}$. But a well known theorem of Hurwitz [4] asserts that for infinitely many pairs (p_i, q_i) one has

$$\left| \alpha - \frac{p_i}{q_i} \right| < \frac{1}{\sqrt{5} q_i^2},$$

so that the proof is complete. □

If we set

$$\mathcal{L}(\alpha) := \left\{ L \in \mathbb{R}_0^+ : \text{ infinitely many } (p_i, q_i) \text{ satisfy } \left| \alpha - \frac{p_i}{q_i} \right| \leq \frac{1}{L q_i^2} \right\},$$

then the above lemma clearly states that $L(\alpha) = \sup \mathcal{L}(\alpha)$. Now let

$$\mathcal{N}(\alpha) := \{ L \in \mathbb{R}_0^+ : \text{ infinitely many pairs of integers } (p, q) \\ \text{with } q \neq 0 \text{ satisfy } |\alpha - (p/q)| \leq 1/Lq^2 \} \supset \mathcal{L}(\alpha).$$

It is known [4] (see also the interesting paper [5]) that if $L > 2$ and $L \in \mathcal{N}(\alpha)$, then $L \in \mathcal{L}(\alpha)$, and that, for each $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\sqrt{5} \in \mathcal{L}(\alpha)$. Thus,

$$L(\alpha) = \sup \mathcal{L}(\alpha) = \sup \mathcal{N}(\alpha)\,.$$

Proposition 1. *$L(\alpha)$ is finite if and only if the sequence $(a_i)_{i \in \mathbb{N}}$ of partial quotients of α is bounded.*

Proof. We have

$$\begin{aligned} \mu_i &= q_i^{-2} |\alpha - \frac{p_i}{q_i}|^{-1} = q_i^{-2} |(-1)^i q_i (\alpha_{i+1} q_i + q_{i-1})| \\ &= \left| \alpha_{i+1} + \frac{q_{i-1}}{q_i} \right| = \left| [a_{i+1}, a_{i+2}, \ldots] + \frac{1}{[a_i, a_{i-1}, \ldots, a_1]} \right| \\ &= |[a_{i+1}, a_{i+2}, \ldots] + [0, a_i, a_{i-1}, \ldots, a_1]| \\ &= |[a_{i+1}] + \theta_i + \eta_i|, \end{aligned}$$

with $0 < \theta_i, \eta_i < 1$ for all positive integers i. Thus, if $(a_i)_{i\in\mathbb{N}}$ is unbounded, one has

$$\limsup_{i\to\infty} \mu_i \geq \limsup_{i\to\infty}([a_{i+1}] - 2) = +\infty,$$

and $L(\alpha) = \infty$. If $(a_i)_{i\in\mathbb{N}}$ is bounded, then

$$L(\alpha) = \limsup_{i\to\infty} \mu_i \leq \limsup_{i\to\infty}([a_{i+1}] + 2) < \infty. \qquad \square$$

2. Arithmetical Point of View

Theorem 1. *Let α and β be two irrational numbers such that*

$$\beta = \frac{a\alpha + b}{c\alpha + d}$$

with $|ad - bc| \neq 0$, and $a, b, c, d \in \mathbb{Z}$. Then

$$\frac{1}{|ad - bc|} L(\alpha) \leq L(\frac{a\alpha + b}{c\alpha + d}) \leq |ad - bc| L(\alpha).$$

Proof. Let $L \in \mathcal{L}(\alpha)$. Then there exist infinitely many pairs (p_i, q_i) with $q_i \neq 0$, such that

$$\left|\alpha - \frac{p_i}{q_i}\right| \leq \frac{1}{Lq_i^2}.$$

Now

$$\left|\beta - \frac{ap_i + bq_i}{cp_i + dq_i}\right| = |ad - bc| \frac{|\alpha - (p_i/q_i)|}{|c(p_i/q_i) + d||c\alpha + d|}$$

Let $\epsilon > 0$. Then there exists i_ϵ such that

$$\frac{1}{|c\alpha + d|} \leq \frac{1 + \epsilon}{|c(p_i/q_i) + d|} \quad \text{for all} \quad i \geq i_\epsilon\,,$$

and

$$\left|\beta - \frac{ap_i + bq_i}{cp_i + dq_i}\right| \leq \frac{(1 + \epsilon)|ad - bc|}{L(c(p_i/q_i) + d)^2 q_i^2} = \frac{(1 + \epsilon)|ad - bc|}{L} \frac{1}{(cp_i + dq_i)^2}$$

for all $i \geq i_\epsilon$. Therefore,

$$\frac{L}{(1 + \epsilon)|ad - bc|} \in \mathcal{N}(\beta) \quad \text{for all} \quad \epsilon > 0\,.$$

Now, if $\epsilon \to 0$, we get

$$\frac{L}{|ad-bc|} \leq M(\beta)\,,$$

and then

$$\frac{L(\alpha)}{|ad-bc|} \leq M(\beta).$$

Rewrite $\alpha = \dfrac{-d\beta + b}{c\beta - a}$. Then the second inequality follows immediately, so that the proof is complete. □

3. Another Proof Using Geometrical Interpretation of Best Approximations

Before that, we recall the algebraic sense. We say that a fraction p/q $(q > 0)$ is a *best approximation* to α if, for all fractions p'/q' with $0 < q' \leq q$, $|q\alpha - p| < |q'\alpha - p'|$ unless $q = q',\ p = p'$.

Now in the geometrical sense, following Stark [6] and Irwin [2], we use the same notations $C = (a, b)$ for points in the plane and for vectors. The *lattice* generated by two vectors C and D is the set of all points $mC + nD$ where m and n are integers. *The distance from a point A to a line l in the direction C* (not parralel to l) is the lenght of the line segment joining A to the unique point $A + aC$ on l. Now let l be the line $y = \alpha x$. The geometrical interpretation of p/q being the best approximation to α is that, amongst all points (q', p') of the integer lattice $\mathbb{Z}^2$ (the lattice generated by $(1, 0)$ and $(0, 1)$) with x-coordinate satisfying $0 < q' \leq q$, (q, p) is uniquely the nearest to l. Here we use the fact that $|q'\alpha - p'|$ is the vertical distance from (q', p') to l (i.e. in the direction $(0, 1)$). We call (q, p) a *best approximation* for l.

Proof. Let $a, b, c, d \in \mathbb{Z}$, $|ad - bc| \neq 0$ and consider the following function: $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$f(x, y) := (x', y') := (dx + cy, bx + ay),$$

then it is clear that f is one-to-one from $\mathbb{R}^2$ to $f(\mathbb{R}^2)$, takes the line $y = \alpha x$ onto the line $y = \beta x$, where $\beta := \dfrac{a\alpha + b}{c\alpha + d}$ and f^{-1} is defined by

$$f^{-1}(x', y') = (x, y) := \left(\frac{ax' - cy'}{ad - bc}, \frac{bx' - dy'}{bc - ad}\right).$$

Let $(q,p) \in \mathbb{Z}^2$ be a best approximation for the line $y = \beta x$ and $(Q,P) := f^{-1}(q,p)$. Then we have

$$\begin{aligned} -\alpha Q + P &= -\alpha \frac{aq - cp}{ad - bc} + \frac{bq - dp}{bc - ad} \\ &= \frac{q(\alpha a + b) - p(\alpha c + d)}{bc - ad} \\ &= \frac{\alpha c + d}{bc - ad}(q\beta - p)\,, \end{aligned}$$

and

$$\begin{aligned} Q\,|Q\alpha - P| &\leq \left| \frac{aq - cp}{ad - bc} . \frac{\alpha c + d}{bc - ad} . (q\beta - p) \right| \\ &= \left| \frac{1}{ad - bc} . \frac{aq - cp}{\beta c - a}(q\beta - p) \right| \\ &= \frac{1}{|ad - bc|} \left| \frac{a - c\frac{p}{q}}{a - c\beta} \right| q|q\beta - p|. \end{aligned}$$

As p/q is the best approximation to β and by the definitions of $L(\beta)$ and $\mathcal{L}(\alpha)$

$$|ad - bc|L(\beta) \in \mathcal{L}(\alpha)$$

i.e.

$$L(\alpha) \leq |ad - bc|L(\beta).$$

Rewrite $\alpha = \dfrac{-d\beta + b}{c\beta - a}$. Then the second inequality follows immediatly. □

Remark 1. 1) For $|ad - bc| = 1$, α and $\beta := \dfrac{a\alpha + b}{c\alpha + d}$ are said equivalent and it is already known as a classical result that

$$L(\alpha) = L(\beta)\,.$$

2) An interesting question is to find, for a given integer $n \geq 1$ and irrational α, the set of integers a, b, c, d, $|ad - bc| = n$, i.e. the set of irrational numbers $\beta = \dfrac{a\alpha + b}{c\alpha + d}$ such that $L(\alpha) = nL(\beta)$ [1].

As consequence of Proposition 1 and Theorem 1 we have

$$L(\alpha) < \infty \Longleftrightarrow M(\beta) < \infty\,,$$

i.e. the sequence of partial quotients of α is bounded if and only if the sequence of partial quotients of β is bounded.

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