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ON THE PROPERTIES OF ORTHOGONAL POLYNOMIALS OVER A REGION WITH ANALYTIC WEIGHT FUNCTION

F.G. Abdullayev[1 §], M. Küçükaslan[2]

[1]Dept. of Mathematics
Faculty of Arts & Science
Mersin University
33342 Mersin, TURKEY
e-mail: fabdul@mersin.edu.tr

[2]Dept. of Mathematics
Faculty of Arts & Science
Çukurova University
Mersin, TURKEY
e-mail: mkucukaslan@mersin.edu.tr

Abstract: Let $G \subset C$ be a finite region bounded by Jordan curve $L := \partial G$ and $h(z) > 0$ be a weight function on G; $\{K_n(z)\}_{n=0}^{\infty}$ be a orthonormal system for the pair (G, h). In this paper, we investigate some properties of the orthonormal polynomials and we are going to find an upper bound for $|K_n(z)|$, $z \in G$ in regions of the complex plane depending of geometric properties and the distance of $z \in G$ to the boundary L.

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[§]Correspondence author

1. Introduction and the Main Results

Let $G \subset \mathbb{C}$ be a finite region bounded by a Jordan curve $L = \partial G$ and $h(z) > 0$ be a weight function on G. The polynomials $\{K_n(z)\}_{n=0}^{\infty}$, $\deg K_n = n$ for $n = 0, 1, 2, ...$ satisfying

$$\iint_G h(z) K_n(z) \overline{K_m(z)} d\sigma_z = \begin{cases} 1, & n = m; \\ 0, & n \neq m, \end{cases} \tag{1}$$

where σ is a two dimensional Lebesque measure on G, are called orthonormal polynomials for the pair (G, h). They are determined uniquely if the coefficient of the term of the highest degree is positive. These polynomials are also called as Bergman polynomials in literature. Some properties of Bergman polynomials have been studied in [1, 2, 3, 9, 13].

One of the main problem in the theory of orthogonal polynomials is to investigate the rate of approximation to zero of orthonormal polynomials on compact subset of the region.

In case of $G = \{z : |z| < 1\}$ and $h(z) \equiv 1$ then

$$|K_n(z)| \leq \text{ const.} \sqrt{n} \cdot r^n \ , \quad |z| = r < 1, \tag{2}$$

that is, the sequence $\{K_n(z)\}_{n=0}^{\infty}$ tends to zero for every fixed z, $|z| = r < 1$, as a power function of r.

Suetin [13] showed that if L is an analytic curve then (2) satisfies for every r, $r_0 < r < 1$, $r_0 = r_0(G)$. At the same time (2) is true for special weight function [13]

$$h(z) = |D(z)|^2 \, , \ D \in A(\overline{G}), \ D(z) \neq 0 \, \text{for every } \ z \in \overline{G}. \tag{3}$$

If the conditions on L is diminished then the rate of $\{K_n(z)\}_{n=0}^{\infty} \to 0$, $n \to \infty$, chancing from the typically of the form r^n, $r < 1$ to typically of the form $n^{-\kappa}$, i.e.

$$|K_n(z)| \leq C(F) n^{-\kappa}, \qquad z \in F, \ F \subset G \, , \tag{4}$$

where $\kappa = \kappa(G, h) > 0$, $C(F) = C(G, \text{dist}\,(F, \partial G))$ and independent of n.

We say that $L \in C(p, \alpha)$, $p \geq 1$, $0 < \alpha \leq 1$, if L has a naturally parametrization $z = z(s)$, $0 \leq s \leq \text{ mes}\, L =: \ell$ and the function $z(s)$ is $p-$times continuously differentiable in $[0, \ell]$ such that $z^p(s) \in \text{Lip}\, \alpha$.

P.K. Suetin [13] proved that if $L \in C(p + 1, \alpha)$, $p \geq 0$, $0 < \alpha < 1$ and $D^{(p)} \in \text{ Lip}\, \alpha$ then (4) satisfies for $\kappa = p + \alpha$ and $C(F) = O(d^{-p-3}(F, \partial G))$.

We say that $G \in C_\theta$, if $L = \partial G$ has a continuous $\theta(s) := \theta(z(s))$ tangent for every point $z(s)$, and $G \in C_\theta(\lambda)$, $0 < \lambda < 2$, if $L = \partial G$ consists of the union of finite number $C_\theta-$arcs such that they have exterior angles $\lambda_j\pi$, $0<\lambda_j<2$, $\min\limits_j \lambda_j=\lambda$, at the corners (with respect to G) where two arc meet.

D. Gaier [9] proved that, if $G\in C_\theta(\lambda),\ 0<\lambda<2$, and $h(z)\equiv 1$, then the inequality

$$|K_n(z)|\le c(z)\cdot n^{-\mu}\ ,\quad z\in G, \tag{5}$$

holds for all $\mu<\min\left\{\frac{\lambda}{2-\lambda};\frac{1}{2}\right\}$, where $c(z)$ is a constant depending on z. But this dependency has not been shown explicitly in [9].

This dependency was given explicitly by F.G. Abdullayev [2].

We denote $\vartheta(\lambda;\beta)$ as

$$\vartheta(\lambda;\beta):=\begin{cases}\dfrac{1}{2-\lambda}\,, & 0<\lambda\le\dfrac{2-2\beta}{3+\beta};\\ \dfrac{1-\beta}{2\lambda(1+\beta)}\,, & \dfrac{2-2\beta}{3+\beta}<\lambda<1;\\ \dfrac{1-\beta}{2(1+\beta)}\,, & 1\le\lambda<2.\end{cases}$$

In the case of $G\in C_\theta(\lambda)$, $0<\lambda<2$, and the weight function defined by (3) and in addition $D\in \operatorname{Lip}\alpha, 0<\alpha\le 1$, then the following result was found in [2]:

$$|K_n(z_0)|\le c\delta^{-\frac{9-4\lambda}{2(2-\lambda)}}(z_0)\cdot n^{-\mu},\quad z_0\in G, \tag{6}$$

for all μ, $0<\mu<\min\left\{\frac{\lambda}{2-\lambda};\frac{1}{2}\right\}$, if $\alpha>\vartheta(\lambda;0)$ and $0<\mu<\alpha\cdot\min\{1;\lambda\}$, if $\alpha\le\vartheta(\lambda;0)$, where c is a constant independent of n and $\delta(z_0):=\operatorname{dist}(z_0,\partial G)$.

In this work, we add the case $\lambda=2$ in (6) and extend this result for more general regions.

Throughout this paper $c,c_1,c_2,...$ are positive, $\varepsilon,\varepsilon_1,\varepsilon_2,...$ are sufficiently small positive constants which depend on G in general.

Definition 1. We say that, $G\in C_\theta(\lambda;\beta)$, $0<\lambda<2$, $\beta>0$, if $L:=\partial G$ is expressed as the union of finite number $C_\theta-$arcs connecting at points $z_1,...,z_m$, L is locally smooth at z_1 and following conditions are satisfied:

i) for every points $z_j,\ j=2,...,p$ the domain G has $\lambda_j\pi,\ 0<\lambda_j<2$ exterior angles at the corners z_j, $\lambda=\min\limits_{2\le j\le p}\{\lambda_j\}$;

ii) for every points $z_j,\ j=p+1,...,m$ in (x,y) local co-ordinate system with the origin at z_j,

$$\left\{z=x+iy\ :\ c_1x^{1+\beta}\le y\le c_2x^{1+\beta},\ 0\le x\le\varepsilon_1\right\}\subset\overline{G},$$

$$\{z = x + iy \ : \ |y| \geq \varepsilon_2 x \ , \ 0 \leq x \leq \varepsilon_1\} \subset CG,$$

for some certain constants $-\infty < c_1 < c_2 < +\infty$, $\varepsilon_i > 0$, $i = 1, 2$.

It is obvious from Definition 1 that each domain $G \in C_\theta(\lambda;\beta)$ maybe have $\lambda\pi, (0 < \lambda < 2)$ exterior angles at the points z_j, $j = 2, ..., p$ and interior zero angles at the points z_j, $j = p+1, ..., m$ which the boundary arcs touching $x^{1+\beta}$ speed. If $\beta = 0$ then the domain G does not have interior zero angles and belongs to $C_\theta(\lambda)$, $0 < \lambda < 2$, i.e. $C_\theta(\lambda) \equiv C_\theta(\lambda;0)$.

If $\lambda = 1$, then the domain $G \in C_\theta(1;\beta)$, $\beta > 0$, has piecewise smooth boundary with interior zero angles.

Theorem 2. *Let $G \in C_\theta(\lambda;\beta)$ for some λ, $0 < \lambda < 2$, and β, $0 < \beta < 1$; $h(z)$ defined by (3) and in addition $D \in Lip\,\alpha, 0 < \alpha \leq 1$. Then, for each $z_0 \in G$*

$$|K_n(z_0)| \leq c\delta^{-\frac{9-4\lambda_*}{2(2-\lambda_*)}}(z_0) \cdot n^{-\mu}, \tag{7}$$

for all $\mu, 0 < \mu < \min\left\{\frac{\lambda}{2-\lambda}; \frac{1-\beta}{2(1+\beta)}\right\}$, if $\alpha > \vartheta(\lambda;\beta)$ and $0 < \mu < \alpha \min\{1;\lambda\}$, if $\alpha \leq \vartheta(\lambda;\beta)$, where $\lambda_* = \max\{1;\lambda\}$ and c is independent of z_0 and n.

Let $w = \Phi(z)$ be the conformal mapping of $\Omega = C\overline{G}$ on $\Delta := \{w : |w| \geq 1\}$ normalized by $\Phi(\infty) = \infty, \Phi'(\infty) > 0$ and $\Psi = \Phi^{-1}$.

Definition 3. We say that $G \in Q^\gamma$, $0 < \gamma \leq 1$, if

i) $L = \partial G$ is a quasiconformal curve;

ii) $\Psi \in \mathrm{Lip}\,\gamma$, $|w| \geq 1$.

Theorem 4. *Let $G \in Q^\gamma$ for some γ, $0 < \gamma \leq 1$, $h(z)$ defined by (3) and in addition $D \in Lip\,\alpha$, $0 < \alpha \leq 1$. Then, for each $z_0 \in G$*

$$|K_n(z_0)| \leq c\delta^{-\frac{5}{2}}(z_0) n^{-\eta}, \tag{8}$$

where

$$\eta := \begin{cases} \alpha\gamma, & \alpha < \frac{1}{4(2-\gamma)}; \\ \frac{\gamma}{4(2-\gamma)}, & \alpha \geq \frac{1}{4(2-\gamma)}. \end{cases}$$

Remark 5. a) If G is a convex region, then $\Psi \in \mathrm{Lip}\, 1$ [12, p. 28] and, consequently, (8) holds for each

$$\eta = \begin{cases} \alpha, & \alpha < \frac{1}{4}; \\ \frac{1}{4}, & \alpha \geq \frac{1}{4}. \end{cases} \tag{9}$$

b) If $G \in C_\theta$, then $G \in Q^\gamma$ for all $0 < \gamma < 1$. Then, (8) holds for each η as in (9).

c) If G is an $L-$shaped region, then $\Psi \in \mathrm{Lip}\, \frac{1}{2}$, hence (8) holds for each

$$\eta = \begin{cases} \frac{\alpha}{2}, & \alpha < \frac{1}{6}; \\ \frac{1}{12}, & \alpha \geq \frac{1}{6}. \end{cases}$$

d) If L is quasi-smooth, i.e. for every $z_1, z_2 \in L$ if $s(z_1, z_2)$ represents the smaller of the lengths of the arcs joining z_1 to z_2 on L, there exists a constant $c > 1$ such that $s(z_1, z_2) \leq c\,|z_1 - z_2|$, then $\Psi \in \mathrm{Lip}\,\gamma$, $\gamma = \frac{2}{(1+c)^2}$ [14]. So, η in (8) can be calculated easily.

e) If L is a $c-$quasiconformal [11] then, $\Psi \in \mathrm{Lip}\,\gamma$ for $\gamma = \frac{2(\arcsin \frac{1}{c})^2}{\pi^2 - \pi \arcsin \frac{1}{c}}$. Also, if L is an asymptotic conformal curve, then $\Psi \in \mathrm{Lip}\,\gamma$ for $\gamma < 1$ [11].

Hence, η in (8) can be calculated too.

2. Some Auxiliary Results

We shall use $a \prec b$ and $a \asymp b$ as equivalent to $a \leq cb$ and $c_1 a \leq b \leq c_2 a$ for some constants c, c_1, c_2, respectively.

Let $z_0 \in G$ and $w = \varphi(z, z_0)$ be the conformal mapping of G to the unit disc $B = \{w : |w| < 1\}$ normalized by $\varphi(z_0, z_0) = 0$, $\varphi'(z_0, z_0) > 0$. Whenever writes $w = \varphi(z)$ we assume $z_0 \in G$ is fixed point.

Let L be a Jordan arc or curve in the complex plane. L is called a K-quasiconformal ($K \geq 1$) arc or curve if there is a $K-$quasiconformal mapping f of a region D containing L such that $f(L)$ is a line segment or circle [10].

$D = C$ gives the global and $D \subset C$ gives local definition of K-quasiconformal arc or curve [1]. Throughout this work, we consider the local definitions.

For $t > 0$, let

$$L_t := \{z : |\varphi(z)| = t, \text{ if } t < 1, \quad |\Phi(z)| = t, \text{ if } t > 1\}, \quad L_1 := L$$

$$G_t := int\, L_t \quad \text{and} \quad \Omega_t := ext\, L_t.$$

For any $K-$quasiconformal curve L, the region $D \supset L$ can be chosen as $G_{R_0} \backslash G_{r_0}$ for a certain number $R_0 := R_0(\Phi, f, G)$, $1 < R_0 \leq 2$, and $r_0 := R_0^{-1}$[1, p. 28].

Let L be $K-$quasiconformal curve. Then the function

$$\alpha(\cdot) = f^{-1}\left\{\overline{[f(\cdot)]^{-1}}\right\}$$

is a K^2-quasiconformal reflection across L as shown in [5, p. 75], that is, $\alpha(\cdot)$ is a K^2-antiquasiconformal mapping leaving points on L fixed and satisfying the conditions:

$$\alpha(G_{\widetilde{R}} \setminus \overline{G}) \subset G \setminus G_{r_0}\ ,\ \alpha(G \setminus \overline{G_{\widetilde{r}}}) \subset G_{R_0} \setminus \overline{G}$$

for some $\widetilde{R}$, $1 < \widetilde{R} < R_0$ and $\widetilde{r}$, $r_0 < \widetilde{r} < 1$, where $R_0 := R_0(\Phi, f, G) > 1$ and $r_0 := r_0(\varphi, f, G) < 1$ some constants.

On the other hand there exists a $C(K)-$quasiconformal $\alpha^*(\cdot)$ (see [10, p. 98], [5, p. 76]) such that it satisfies the following:

$$|z_1 - \alpha^*(z)| \asymp |z_1 - z|\ ,\ \ z_1 \in L,\ z \in D\ . \tag{10}$$

Lemma 6. ([1]) *Let L be a $K-$quasiconformal curve; $z_1 \in L$, $z_2, z_3 \in G \cap \{z : |z - z_1| \leq c_1 d(z_1, L_{R_0})\}$; $w_j = \varphi(z_j)$*

$$(z_2, z_3 \in \Omega \cap \{z : |z - z_1| \leq c_2 d(z_1, L_{r_0})\}\,;\ w_j = \Phi(z_j)),\ j = 1, 2, 3\ .$$

Then,

i) The statements $|z_1 - z_2| \preceq |z_1 - z_3|$ and $|w_1 - w_2| \preceq |w_1 - w_3|$ are equivalent. So are $|z_1 - z_2| \approx |z_1 - z_3|$ and $|w_1 - w_2| \approx |w_1 - w_3|$.

ii) If $|z_1 - z_2| \prec |z_1 - z_3|$, then

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^{-2}} \prec \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \prec \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^2}\ . \tag{11}$$

Let $H_2'(h, G)$ $(H_2'(G) \equiv H_2'(1, G))$ be the set of analytic functions f on G satisfying

$$\|f\|^2_{H_2'(h,G)} = \iint_G h(z)\,|f(z)|^2\,d\sigma_z < \infty\ .$$

Lemma 7. *[4] Let $G \in C_\theta(\lambda; \beta)$ for some λ, $0 < \lambda < 2$, and $\beta > 0$; $z_0 \in G$. Then, for each $n \geq 1$ there exists a polynomial $Q_n(z, z_0)$ such that* *i)* $Q_n(z_0,z_0)=0,\ Q'_n(z_0,z_0)=\varphi'(z_0,z_0)$;

ii) $\|\varphi'(\cdot,z_0)-Q'_n(\cdot,z_0)\|_{H'_2(G)} \prec \delta^{-\frac{5-4\lambda_*}{2(2-\lambda_*)}}(z_0)\cdot n^{-\mu}$, *for all* $\mu,\ 0<\mu<\min\left\{\frac{\lambda}{2-\lambda};\frac{1-\beta}{2(1+\beta)}\right\}$, *where* $\lambda_*=\max\{1;\lambda\}$.

Lemma 8. *Let* $G\in C_\theta(\lambda;\beta)$ *for some* $\lambda, 0<\lambda<2$, *and* $\beta, 0<\beta<1$; $h(z)$ *defined by (3) and in addition* $D\in Lip\,\alpha, 0<\alpha\le 1$. *Then, for every* $n\ge 1$ *there exists a polynomial* $T_n(z,z_0)$ *such that*

i) $T_n(z_0,z_0)=\frac{\varphi'(z_0,z_0)}{D(z_0)}$;

ii) $\left\|\frac{\varphi'(\cdot,z_0)}{D}-T_n(\cdot,z_0)\right\|_{H'_2(G)} \prec \delta^{-\frac{5-4\lambda_*}{2(2-\lambda_*)}}(z_0)\cdot n^{-\mu}$,

for all μ, $0<\mu<\min\left\{\frac{\lambda}{2-\lambda};\frac{1-\beta}{2(1+\beta)}\right\}$, *if* $\alpha>\vartheta(\lambda;\beta)$ *and* $0<\mu<\alpha\min\{1;\lambda\}$, *if* $\alpha\le\vartheta(\lambda;\beta)$.

Proof. By the assumption of $D(z)$, $\frac{1}{D(z)}\in \mathrm{Lip}\,\alpha$, $z\in\overline{G}$. Hence, since L is quasiconformal (with some quasiconformal coefficient $C(K)>1$) by [7, Theorem3] there are polynomials $\widetilde{Q}_m(z)$ such that

$$\max_{z\in\overline{G}}\left|\frac{1}{D(z)}-\widetilde{Q}_m(z)\right| \prec d^{\alpha}(z,L_{1+\frac{1}{m}})\ . \tag{12}$$

Let $Q_m(z,z_0):=\widetilde{Q}_m(z)-\widetilde{Q}_m(z_0)+\frac{1}{D(z_0)}$. From (12) we get

$$\max_{z\in\overline{G}}\left|\frac{1}{D(z)}-Q_m(z,z_0)\right| \prec m^{\mu'} \tag{13}$$

for all $0<\mu'<\alpha\min\{1;\lambda\}$. Then using (13) and Lemma 7 we obtain

$$\begin{aligned}\left\|\frac{\varphi'(\cdot,z_0)}{D}-T_n(\cdot,z_0)\right\|_{H'_2(G)} &\prec \left\|\frac{1}{D(\cdot)}\right\|_{C(\overline{G})}\left\|\varphi'(\cdot,z_0)-P'_l(\cdot,z_0)\right\|_{H'_2(G)}\\ &+\left\|\frac{1}{D(\cdot)}-Q_m(\cdot,z_0)\right\|_{C(\overline{G})}\left\|\varphi'(\cdot,z_0)-P'_l(\cdot,z_0)\right\|_{H'_2(G)}\\ &+\left\|\varphi'(\cdot,z_0)\right\|_{H'_2(G)}\left\|\frac{1}{D(\cdot)}-Q_m(\cdot,z_0)\right\|_{C(\overline{G})}\\ &\prec \delta^{-\frac{5-4\lambda_*}{2(2-\lambda_*)}}(z_0)\ (l^{-\mu}+l^{-\mu}m^{-\mu'})+m^{-\mu'}\end{aligned}$$

for all $0 < \mu < \min\left\{\frac{\lambda}{2-\lambda}; \frac{1-\beta}{2(1+\beta)}\right\}$, $0 < \mu' < \alpha \min\{1; \lambda\}$. By defining $m := n$ if $\mu' \leq \mu$ $m := \left[n^{\frac{\mu}{\mu'}}\right] + 1$ if $\mu' > \mu$ and $T_n := P_l \cdot Q_m$ the proof is completed. □

Lemma 9. *Let $G \in Q^{\gamma}$ for some γ, $0 < \gamma \leq 1$; $z_0 \in G$. Then for all* $0 < u < R_0 - 1$

$$mes\, \varphi\left[\alpha^*(G_{1+u} \setminus G), z_0\right] \prec \delta^{-1}(z_0) \cdot \delta^{\frac{1}{2(2-\gamma)}}(\xi) \tag{14}$$

where $\xi = \varphi^{-1}(\tau, z_0)$ such that $|\tau| = \inf\{|w| : w \in \varphi[\alpha^*(L_{1+u}), z_0]\}$.

Proof. It is obvious that

$$\text{mes}\, \varphi\left[\alpha^*(G_{1+u} \setminus G), z_0\right] \prec 1 - |\tau|. \tag{15}$$

We shall give the proof under several headings. Let $D = G_{R_0} - G_{r_0}$.

(1) $z_0 \in \overline{G}_{r_0}$. Since $\Psi \in$ Lip γ, then $\varphi \in$ Lip $\frac{1}{2-\gamma}$ by [11] and

$$1 - |\tau| \prec d^{\frac{1}{2-\gamma}}(\xi, L); \tag{16}$$

(2) $z_0 \in G \cap D$ and $z_0' := \alpha^*(z_0)$, $d(z_0, L) = |z_0 - z|$, $z \in L$. Then there are two cases

(i) $z_0' \in \overline{G}_{1+u}$. In this case, we obtain

$$1 - |\tau| \prec \left|\frac{z - \xi}{z - z_0}\right| \tag{17}$$

from Lemma 6 and (10).

(ii) $z_0' \in \overline{\Omega}_{1+u} \cap D$. Let $\Gamma(z, \xi; z_0, G)$ be a family of locally rectifiable curves separating z, ξ from z_0 in G and $\Gamma' = \varphi(\Gamma)$.

We also set

$$z^* = \frac{1}{z - z_1}\ ;\ w^* = \frac{1}{w}, \tag{18}$$

where $z_1 \in G$ is some fixed point such that $d(z_1, L) \geq \varepsilon$, $|z_1 - z| > \varepsilon$. Then, the domain G transforming in some domains $G^*, \infty \in G^*$ with a quasiconformal boundary $L^* = \partial G^*; z \to z^*, \xi \to \xi^*, z_0 \to z_0^*$; $\Gamma \to \Gamma^*(z^*, \xi^*; z_0^*, G^*)$ and $\Gamma' \to \widetilde{\Gamma}'$.

According to [6, Theorem 4.2] we can write

$$m(\Gamma^*) \geq \frac{1}{2\pi} \ln c_1 \frac{|z^* - z_0^*|}{|z^* - \xi^*|}, \tag{19}$$

where c_1 is independent of z^*, ξ^*, and z_0^*.

On the other hand, since $\Psi \in \mathrm{Lip}\,\gamma$, then $\varphi \in \mathrm{Lip}\,\frac{1}{2-\gamma}$ from [8] . So, $z^* \circ \varphi \circ w^* \in \mathrm{Lip}\,\frac{1}{2-\gamma}$.

In this case we get from [6]

$$m(\widetilde{\Gamma}') \leq \frac{2-\gamma}{\pi} \ln \frac{c_2}{|\tau^*| - 1}, \tag{20}$$

where c_2 is independent of τ^*. Considering the conformal invariant of the modules (18), (19) and (20) we obtain

$$1 - |\tau| \prec \left| \frac{z-\xi}{z-z_0} \right|^{\frac{1}{2(2-\gamma)}}, \tag{21}$$

and by taking into account (16), (17) and (21) we get (15). □

Corollary 10. *For all u, $0 < u < R_0 - 1$*

$$mes\,\varphi\left[\alpha^*(G_{1+u} \setminus G), z_0\right] \prec \delta^{-1}(z_0) \cdot u^{\frac{\gamma}{2(2-\gamma)}}.$$

This follows from (10) and (14).

Lemma 11. *Let $G \in Q^\gamma$ for some γ, $0 < \gamma \leq 1$. Then, for each $n \geq 1$, there exists a polynomial $P_n(z, z_0)$ such that:*

i) $P_n(z_0, z_0) = 0,\ P_n'(z_0, z_0) = \varphi'(z_0, z_0)$;

ii) $\|\varphi'(\cdot, z_0) - P_n'(\cdot, z_0)\|_{H_2'(G)} \prec\ \delta^{-\frac{3}{2}}(z_0) n^{-\frac{\gamma}{4(2-\gamma)}}$.

Proof. By taking into account (3.18) in [2, Lemma 3.2] we obtain for $p = 2$

$$\left\|\varphi'(\cdot, z_0) - P_n'(\cdot, z_0)\right\|_{H_2'(G)} \prec\ \frac{1}{n} + \delta^{-1}(z_0) \left(\mathrm{mes}\,\varphi(\alpha^*(G_R \setminus G), z_0\right)^{\frac{1}{2}}.$$

Using Corollary 10 we get Lemma 11. □

Lemma 12. *Let $G \in Q^\gamma$ for some γ, $0 < \gamma \leq 1$; $h(z)$ be defined by (3) and in addition $D \in Lip\,\alpha, 0 < \alpha \leq 1$. Then, for every $z_0 \in G$ and each $n \geq 1$ there exists a polynomial $T_n(z, z_0)$ such that*

i) $T_n(z_0, z_0) = \frac{\varphi'(z_0,z_0)}{D(z_0)}$;

ii) $\left\|\frac{\varphi'(\cdot,z_0)}{D} - T_n(\cdot, z_0)\right\|_{H_2'(G)} \prec\ \delta^{-\frac{3}{2}}(z_0) n^{-\eta}$, *where*

$$\eta := \begin{cases} \alpha\gamma, & \alpha < \frac{1}{4(2-\gamma)}, \\ \frac{\gamma}{4(2-\gamma)}, & \alpha \geq \frac{1}{4(2-\gamma)}. \end{cases}$$

Proof. Using the same way of Lemma 8 we obtain Lemma 12. □

Lemma 13. *Let G be a Jordan domain, there exists a polynomial $T_n(z, z_0)$ degree at most n and satisfies for each $n \geq 1$,*

i) $T_n(z_0, z_0) = \frac{\varphi'(z_0,z_0)}{D(z_0)}$;

ii) $\sigma_n(z_0) := \left\| \frac{\varphi'(\cdot,z_0)}{D} - T_n(\cdot, z_0) \right\|^2_{H'_2(G)}$

for some $\{\sigma_n(z_0)\}_{n=0}^{\infty}$ *with* $\sigma_n(z_0) \to 0$ *at* $n \to \infty$ *for every* $z_0 \in G$.

Then,

$$|K_n(z_0)|^2 \leq c\delta^{-2}(z_0) \cdot \sigma_n(z_0), \tag{22}$$

where c is independent of z_0 and n.

Proof. It is well known that the function minimizing the integral

$$J(f) = \iint_G h(z) \left| f(z) \right|^2 d\sigma_z \tag{23}$$

in the class of functions f analytic in G, square integrable over G and normalised by $f_0(z_0, z_0) = \lambda_0 := \frac{\varphi'(z_0,z_0)}{D(z_0)}$, is $f_0(z, z_0) = \lambda_0 := \frac{\varphi'(z,z_0)}{D(z)}$ [13].

On the other hand, the polynomials that minimizing (23) in the class of polynomials of degree $\leq n-1$, normalized by λ_0 at $z_0 \in G$ are

$$\widetilde{Q}_{n-1}(z) := \lambda_0 \cdot \frac{\sum\limits_{i=0}^{n-1} \overline{K_i(z_0)} K_i(z)}{\sum\limits_{i=0}^{n-1} |K_i(z_0)|^2} \quad \text{and} \quad J(\widetilde{Q}_{n-1}) = \frac{|\lambda_0|^2}{\sum\limits_{i=0}^{n-1} |K_i(z_0)|^2}. \tag{24}$$

Also,

$$\pi \leq \iint\limits_G h(z) \left| \widetilde{Q}_{n-1}(z) \right|^2 d\sigma_z = \pi + \iint\limits_G h(z) \left| f_0(z, z_0) - \widetilde{Q}_{n-1}(z, z_0) \right|^2 d\sigma_z$$

$$\leq \pi + \iint\limits_G h(z) \left| f_0(z, z_0) - T_{n-1}(z, z_0) \right|^2 d\sigma_z$$

$$\leq \pi + c \iint\limits_G |f_0(z,z_0) - T_{n-1}(z,z_0)|^2 \, d\sigma_z \,, \tag{25}$$

where T_{n-1} is an arbitrary polynomial with $T_{n-1}(z_0,z_0) = \lambda_0$ and $c = \max\limits_{z\in\overline{G}} h(z)$. So, we get

$$\frac{|\lambda_0|^2}{\sum\limits_{i=0}^{n-1} |K_i(z_0)|^2} = \pi + O(\sigma_n(z_0))$$

from (24), (25) and Lemma 12. Consequently,

$$\sum_{i=0}^{n-1} |K_i(z_0)|^2 = \frac{|\lambda_0|^2}{\pi} - O\left(|\lambda_0|^2 \cdot \sigma_n(z_0)\right) \,. \tag{26}$$

Let $m > n$. From (26) we have

$$\sum_{i=n}^{m} |K_i(z_0)|^2 = O\left(|\lambda_0|^2 \cdot \sigma_n(z_0)\right) - O\left(|\lambda_0|^2 \cdot \sigma_m(z_0)\right)$$

and taking limit as $m \to \infty$ we obtain

$$\sum_{i=n}^{\infty} |K_i(z_0)|^2 = O\left(|\lambda_0|^2 \cdot \sigma_n(z_0)\right). \tag{27}$$

Since, (27) is true for every positive integer n then,

$$|K_i(z_0)|^2 \leq O(|\lambda_0|^2 \cdot \sigma_n(z_0)) \,.$$

By taking $|\lambda_0| \approx \delta^{-1}(z_0)$ [12, Corollary 1.4] the proof is completed. □

3. Proof of Theorems

By using Lemma 13 and Lemma 8 we obtain Theorem 2 and by using Lemma 13 and Lemma 12 we get Theorem 4.

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