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CANTOR ORDER AND COMPLETENESS

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Abstract: Nine necessary and sufficient conditions for a class of metric spaces with Cantor order to be ordered complete are given.

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1. Introduction

Park [2] and Turinici [3] introduced the notions of Cantor order and order complete metric space, respectively. Conserva and Rizzo [1] characterized a class of order complete metric spaces as those ones in which every mapping of a suitable family has at least one fixed point. The purpose of this note is to establish necessary and sufficient conditions for a class of metric spaces with Cantor order to be ordered complete.

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2. Notations and Definitions

Throughout this note, N denotes the set of all positive integers and $\omega = N\bigcup\{0\}$. Let (X,d) be a metric space and $\leq$ be an order on X (i.e. a reflexive, antisymmetric and transitive relation on X). A sequence $\{x_n\}_{n\in N}$ in X is said to be *nondecreasing* if $x_n \leq x_{n+1}$ for all $n \in N$, and the sequence $\{x_n\}_{n\in N}$ is called *strictly increasing* if it is nondecreasing and $x_n \neq x_{n+1}$ for all $n \in N$. A metric space (X,d) is said to be $\leq$-*complete* if every nondecreasing Cauchy sequence in X is convergent. Let 2^X denote the family of all nonempty subsets of X.

For every $A \subset X$ and $x \in X$, let $A(x)$ denote the set of all $y \in X$ with $x \leq y$, i.e.

$$A(x) = \{y \in X : x \leq y\},$$

$\overline{A}$ and $\delta(A)$ be the closure and diameter of A, respectively. By a Cantor sequence we mean a sequence $\{F_n\}_{n\in N}$ of nonempty closed subsets of (X,d) such that $F_1 \supset F_2 \supset \cdots \supset F_n \supset F_{n+1} \supset \cdots$ and $\delta(F_n) \to 0$ as $n \to \infty$. Park [2] introduced an order $\leq$ on X corresponding to a Cantor sequence $\{F_n\}_{n\in N}$ by $x \leq y$ in X if and only if $x = y$ or there exists an $n \in N$ such that $x \notin F_n$ and $y \in F_n$.

The order $\leq$ of X corresponding to a Cantor sequence will be called a *Cantor order*.

3. Characterizations of Completeness

We are now in position to prove the main result of this note.

Theorem. *Let (X,d) be a metric space, $\{F_n\}_{n\in N}$ be a Cantor sequence in X and $\leq$ be the Cantor order corresponding to $\{F_n\}_{n\in N}$. Then the following statements are equivalent:*

1. *(X,d) is $\leq$-complete.*
2. *$\bigcap_{n\in N} F_n \neq \emptyset$.*
3. *There exists $p \in X$ such that $\bigcap_{n\in N} F_n(x_n) = \{p\}$ for each nondecreasing sequence $\{x_n\}_{n\in N}$ in X.*
4. *There exists $p \in X$ such that $\bigcap_{n\in N} X(x_n) = \{p\}$ for each strictly increasing sequence $\{x_n\}_{n\in N}$ in X.*
5. *There exists $p \in X$ such that $\bigcap_{n\in N} F_n(x) = \{p\}$ for each $x \in X - \{p\}$.*

6. *If $T : X \to 2^X$ is a multi-valued mapping such that, for any $x \in X - Tx$, there exists $y \in X - \{x\}$ satisfying*

$$\bigcap_{n\in N} F_n(y) \neq \{x\},$$

then T has a fixed point.

7. *If $f : X \to X$ is a mapping such that, for any $x \in X - \{fx\}$, there exists $y \in X - \{x\}$ satisfying*

$$\bigcap_{n\in N} F_n(y) \neq \{x\},$$

then f has a fixed point.

8. *If $T : X \to 2^X - \{\emptyset\}$ is a multi-valued mapping satisfying*

$$\bigcap_{n\in N} F_n(y) \neq \{x\}$$

for any $x \in X$ and $y \in Tx - \{x\}$, then T has a stationary point w, i.e. $Tw = \{w\}$.

9. *If $\mathcal{F}$ is a nonempty family of mappings $f : X \to X$ satisfying*

$$\bigcap_{n\in N} F_n(fx) \neq \{x\}$$

for any $x \in X$ with $x \neq fx$, then $\mathcal{F}$ has a common fixed point.

10. *If $f : X \to X$ is a mapping satisfying*

$$\bigcap_{n\in N} F_n(fx) \neq \{x\}$$

for any $x \in X$ with $x \neq fx$, then f has a fixed point.

Proof. (3) $\Longrightarrow$ (5) is clear. Take $\mathcal{F} = \{f\}$. Then (10) is follows immediately from (9).

(1) $\Longrightarrow$ (2) Choose a sequence $\{x_n\}_{n\in N}$ in X satisfying $x_n \in F_n - F_{n+1}$ for all $n \in N$. It is clear that $\{x_n\}_{n\in N}$ is strictly increasing. Note that $\delta(\{x_k : n \leq k\}) \leq \delta(F_n) \to 0$ as $n \to \infty$. Consequently, $\{x_n\}_{n\in N}$ is a Cauchy sequence. By (1), $\{x_n\}_{n\in N}$ converges to some point $p \in X$. This means that p is in $\overline{\{x_k : n \leq k\}} \subset \overline{F_n} = F_n$ for all $n \in N$, i.e. $p \in \bigcap_{n\in N} F_n \neq \emptyset$.

(2) $\Longrightarrow$ (3) Note that $\delta(\bigcap_{n\in N} F_n) \le \delta(F_n) \to 0$ as $n \to \infty$. It follows from (2) that $\bigcap_{n\in N} F_n$ is a singleton, i.e. $\bigcap_{n\in N} F_n = \{p\}$ for some $p \in X$. Let $\{x_n\}_{n\in N}$ be a nondecreasing sequence in X. For each $n \in N$, if $x_n \notin F_m$ for some $m \in N$, then $x_n \le p$ in view of $p \in F_m$, and, if $x_n \in F_m$ for all $m \in N$, then $x_n \in \bigcap_{m\in N} F_m = \{p\}$. Consequently,

$$p \in \bigcap_{n\in N} F_n(x_n) \subset \bigcap_{n\in N} F_n = \{p\},$$

i.e. $\bigcap_{n\in N} F_n(x_n) = \{p\}$.

(3) $\Longrightarrow$ (4) Let $\{x_n\}_{n\in N} \subset X$ be a strictly increasing sequence. Then, for each $n \in N$, we can find $k(n) \in N$ satisfying $x_n \in F_{k(n)}$ and $x_n \notin F_{k(n)+1}$. Since $x_n \le x_{n+1}$ and $x_n \ne x_{n+1}$, it follows that $k(n+1) \ge k(n) + 1 \to \infty$ as $n \to \infty$. It is easy to check that

$$X(x_n) = \{x_n\} \bigcup F_{k(n)+1} \subset F_{k(n)}$$

for all $n \in N$. This means that

$$\delta(\bigcap_{n\in N} X(x_n)) \le \delta(X(x_n)) \le \delta(F_{k(n)}) \to 0$$

as $n \to \infty$. By (3), we have

$$\{p\} = \bigcap_{n\in N} F_n(x_n) \subset \bigcap_{n\in N} X(x_n).$$

Consequently, $\bigcap_{n\in N} X(x_n) = \{p\}$.

(4) $\Longrightarrow$ (1) Let $\{x_n\}_{n\in N} X$ be a nondecreasing Cauchy sequence in X. We consider two cases:

Case 1. There exists $m \in N$ such that $x_n = x_m$ for each $n > m$. Then $x_n \to x_m$ as $n \to \infty$.

Case 2. For each $m \in N$, there exists $n \in N$ such that $x_n \ne x_m$ and $m < n$. It is easy to see that there exists a strictly increasing sequence $\{x_{n(m)}\}_{m\in N}$ in $\{x_n\}_{n\in N}$. (4) implies that

$$\bigcap_{m\in N} X(x_{n(m)}) = \{p\}.$$

As in the proof of (3) $\Longrightarrow$ (4), we can find a sequence $\{k(n(m))\}_{n\in N} \subset N$ such that

$$x_{n(m)} \in F_{k(n(m))}, \quad x_{n(m)} \notin F_{k(n(m))+1}$$

and

$$X(x_{n(m)}) = \{x_{n(m)}\} \bigcup F_{k(n(m))+1}.$$

Thus, we have

$$d(x_{n(m)}, p) \leq \delta(X(x_{n(m)}) \leq \delta(F_{k(n(m))}) \to 0$$

as $m \to \infty$, i.e. $x_{n(m)} \to p$ as $m \to \infty$. Since $\{x_n\}_{n \in N}$ is nondecreasing, it follows that $x_n \to p$ as $n \to \infty$.

(5) $\Longrightarrow$ (6) Suppose that $p \notin Tp$. Then $p \in X - Tp$ and therefore there exists $y \in X - \{p\}$ with $\bigcap_{n \in N} F_n(y) \neq \{p\}$. But, by (5), we have $\bigcap_{n \in N} F_n(y) = \{p\}$, which is a contradiction.

(6) $\Longrightarrow$ (7) Define a multi-valued mapping $T : X \to 2^X$ by $Tx = \{fx\}$ for any $x \in X$. By (6), T has a fixed point w, i.e. $fw = w$.

(7) $\Longrightarrow$ (8) Suppose that T has no stationary point. Then $Tx - \{x\} \neq \emptyset$ for each $x \in X$. Choose a choice function f on $\{Tx - \{x\} : x \in X\}$. Then f is a self-mapping on X and f has not a fixed point. It is easy to verify that, for each $x \in X - \{fx\} \subset X$, there exists $fx \in Tx - \{x\}$ with $\bigcap_{n \in N} F_n(fx) \neq \{x\}$. By (7), f has a fixed point, which is impossible.

(8) $\Longrightarrow$ (9) Define a multi-valued mapping $T : X \to 2^X - \{\emptyset\}$ by $Tx = \{fx : f \in \mathcal{F}\}$ for any $x \in X$. If $x \in X$ and $y \in Tx - \{x\}$, then there exists $f \in \mathcal{F}$ such that $y = fx \neq x$. This means that

$$\bigcap_{n \in N} F_n(y) = \bigcap_{n \in N} F_n(fx) \neq \{x\}.$$

By (8), T has a stationary point w, that is, $fw = w$ for all $f \in \mathcal{F}$.

(10) $\Longrightarrow$ (2) Suppose that $\bigcap_{n \in N} F_n = \emptyset$. Put $F_0 = X$. Then we have

$$X = \bigcup_{n \in N} (X - F_n) = \bigcup_{n \in \omega} (F_n - F_{n+1}).$$

Choose a sequence $\{x_n\}_{n \in N}$ in X such that $x_n \in F_n - F_{n+1}$ for $n \in N$. Define a self-mapping f on X by $fx = x_{n+2}$ if $x \in F_n - F_{n+1}$ and $n \in \omega$. Then $x \leq fx$ for all $x \in X$ and f has not a fixed point. For each $x \in X$, there exists $n \in \omega$ such that $x \in F_n - F_{n+1}$. Consequently, $x \notin F_{n+2}(fx)$ in view of $fx \in F_{n+2} - F_{n+3}$. Hence $x \notin \bigcap_{n \in N} F_n(fx)$, i.e. $\{x_n\} \neq \bigcap_{n \in N} F_n(fx)$. But, by (10), f has a fixed point, which is a contradiction.

This completes the proof. □

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