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OPTIMIZATION PROBLEMS FOR SOME FUNCTIONALS RELATED TO SOLUTIONS OF PDE's

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Abstract: We investigate maxima and minima of some functionals associated with solutions to Dirichlet problems for elliptic equations. In particular, we shall discuss the characterization and the geometry of the corresponding optimal configurations.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$ and let $D \subset \Omega$ be Lebesgue measurable. Consider the Dirichlet problem

$$-\Delta u(x) = \chi_D(x) \quad \text{in} \quad \Omega, \quad u(x) = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\chi_D(x) = 1$ if $x \in D$ and $\chi_D(x) = 0$ if $x \in \Omega \setminus D$.

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Since $\chi_D(x)$ is not continuous, equation (1.1) is understood in the weak sense. By standard results on elliptic equations, problem (1.1) has a unique (positive) solution $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$ [11]. The "energy integral" associated with the solution $u_D = u_D(x)$ of problem (1.1) is the integral

$$I_D = \int_\Omega |\nabla u_D|^2 dx = \int_\Omega \chi_D u_D \, dx.$$

Let $0 < \beta < |\Omega|$. The problem

$$\max_{|D|=\beta} I_D$$

has been investigated in [1,4,5,6,7,8], and the similar problem

$$\min_{|D|=\beta} I_D$$

has been discussed in [5,6].

In the present paper, we shall investigate the problems

$$\max_{|G|=\alpha, |D|=\beta} \int_\Omega \chi_G u_D dx, \tag{1.2}$$

$$\min_{|G|=\alpha, |D|=\beta} \int_\Omega \chi_G u_D dx, \tag{1.3}$$

where $0 < \alpha \le |\Omega|$ and u_D is the solution to problem (1.1). We shall find results of existence as well as results concerning the location of the couple (G, D) corresponding to optimal configurations. We also investigate the symmetry or the symmetry breaking of G and D whenever Ω is symmetric.

A physical model leading to these problems is the following. Let $\Omega \subset \mathbb{R}^3$ be a region surrounded by a thermostat regulated at a zero temperature, and let D be a region inside Ω, where heat is produced at a constant unity density. If we are in the stationary situation, the integral $\int_\Omega \chi_G u_D dx$ represents, up to a normalization constant, the amount of heat presented in the region occupied by G. Therefore, it might be useful to find the location of G and D which maximize or minimize such amount of heat.

2. Existence Results

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain and let $0 < \beta < |\Omega|$. For $D \subset \Omega$ with $|D| = \beta$, let $u = u_D$ be a (weak) solution to the Dirichlet problem

$$u \in H_0^1(\Omega): \quad \int_\Omega \nabla u \cdot \nabla \phi \, dx = \int_\Omega \chi_D \phi \, dx \quad \forall \phi \in H_0^1(\Omega). \tag{2.1}$$

We start to discuss problems (1.2) and (1.3) in the simplest case $\alpha = |\Omega|$, that is $G = \Omega$. Let us introduce the solution w of the Saint-Venant problem

$$w \in H_0^1(\Omega): \qquad \int_\Omega \nabla w \cdot \nabla \phi \, dx = \int_\Omega \phi \, dx \qquad \forall \phi \in H_0^1(\Omega). \tag{2.2}$$

Using the solution w of problem (2.2), define the domains:

$$D_M = \{x \in \Omega : w(x) > t\}, \; |D_M| = \beta;$$

$$D_m = \{x \in \Omega : w(x) < \tau\}, \; |D_m| = \beta.$$

We have

Theorem 2.1. *Let u_{D_M} and u_{D_m} be the solutions to problem* (2.1) *corresponding to D_M and D_m, respectively. Then we have*

$$\int_\Omega u_{D_M} \, dx = \max_{|D|=\beta} \int_\Omega u_D \, dx,$$

and

$$\int_\Omega u_{D_m} \, dx = \min_{|D|=\beta} \int_\Omega u_D \, dx.$$

These domains D_M and D_m realize the unique optimal configurations up to sets of zero measure.

Proof. Putting $\phi = w$ in (2.1) and $\phi = u_D$ in (2.2) we find

$$\int_\Omega u_D dx = \int_\Omega \chi_D w \, dx. \tag{2.3}$$

Now let D_i, with $|D_i| = \beta$, be a sequence so that

$$I_M = \sup_{|D|=\beta} \int_\Omega \chi_D w \, dx = \lim_{i \to \infty} \int_\Omega \chi_{D_i} w \, dx.$$

Since the sequence $\chi_{D_i}(x)$ is bounded in $L^2(\Omega)$, a subsequence (again denoted with χ_{D_i}) converges weakly to a function $f \in L^2(\Omega)$ with $0 \le f(x) \le 1$ a.e. and with $\int_\Omega f \, dx = \beta$. Hence, by using the bathtub principle ([13], Theorem 1.14) we have

$$I_M = \int_\Omega f \, w \, dx \le \int_\Omega \chi_{D'} w \, dx \le I_M,$$

with

$$\{x \in \Omega : w(x) > t\} \subset D' \subset \{x \in \Omega : w(x) \ge t\}, \quad |D'| = \beta.$$

Since the function w is analytic, the set $\{x \in \Omega : w(x) = t\}$ has zero measure for any $t \geq 0$ and we must have $D' = D_M$ (up to a set of zero measure). The uniqueness of D_M is obvious: if an optimal domain $\tilde{D}$ did not a level set of w then one could increase the integral $\int_\Omega \chi_{\tilde{D}} w\, dx$ by shifting a subset of $\tilde{D}$, where $w(x) < t$ to a subset of $\Omega \setminus \tilde{D}$ where $w(x) > t$.

The argument for the minimum is similar and is omitted. The theorem is proved. □

As a consequence of Theorem 2.1, we have uniqueness for problems (1.2) and (1.3) whenever $\alpha = |\Omega|$. Now we investigate problems (1.2) and (1.3) when $0 < \alpha < |\Omega|$. We have

Theorem 2.2. *Let $0 < \alpha < |\Omega|$ and let u_D be the solution to problem* (2.1) *corresponding to D. We have:*

i) There exist $G_M \subset \Omega$ with $|G_M| = \alpha$ and $D_M \subset \Omega$ with $|D_M| = \beta$ such that

$$\max_{|G|=\alpha, |D|=\beta} \int_\Omega \chi_G u_D dx = \int_\Omega \chi_{G_M} u_{D_M} dx.$$

Moreover, if $\alpha = \beta$ then $G_M = D_M$.

ii) There exist $G_m \subset \Omega$ with $|G_m| = \alpha$ and $D_m \subset \Omega$ with $|D_m| = \beta$ such that

$$\min_{|G|=\alpha, |D|=\beta} \int_\Omega \chi_G u_D dx = \int_\Omega \chi_{G_m} u_{D_m} dx.$$

Proof. i) Let C_α denote the family of all functions χ_G with $G \subset \Omega$, $|G| = \alpha$, and let K_α denote the family of all weak limits (in $L^2(\Omega)$) of sequences from C_α. Similarly, we define C_β and K_β. It is well known [1,4,5] that K_α is convex and weakly compact. In particular, if $g \in K_\alpha$, then $0 \leq g(x) \leq 1$ a.e. in Ω and $\int_\Omega g(x)\, dx = \alpha$. Similar remarks hold for K_β.

For $f \in L^2(\Omega)$ define the linear operator $u = u_f$ as

$$u \in H_0^1(\Omega) : \qquad \int_\Omega \nabla u \cdot \nabla \phi\, dx = \int_\Omega f \phi\, dx \quad \forall \phi \in H_0^1(\Omega). \tag{2.4}$$

Putting $\phi = u$ and using Poincaré Theorem we find

$$\|\nabla u\|_2 \leq C \|f\|_2, \tag{2.5}$$

with C independent of f. If $f = \chi_D$ then we have $u_{\chi_D} = u_D$. We use both these notations. Let

$$I = \sup_{|G|=\alpha, |D|=\beta} \int_\Omega \chi_G u_D dx.$$

Assume $\{G_i, D_i\}$ is a maximizing sequence for I. Since the sequence $\{\chi_{G_i}\}$ is bounded in $L^2(\Omega)$, a subsequence (again denoted by $\{\chi_{G_i}\}$) converges weakly in $L^2(\Omega)$ to a function $g \in K_\alpha$. Similarly, a subsequence of $\{\chi_{D_i}\}$ (again denoted by $\{\chi_{D_i}\}$) converges weakly in $L^2(\Omega)$ to a function $f \in K_\beta$. By using (2.5) and Rellich's Theorem one finds that $\{u_{D_i}\}$ converges strongly to u_f. Hence,

$$I = \lim_{i\to\infty} \int_\Omega \chi_{G_i} u_{D_i} dx = \int_\Omega g\, u_f dx.$$

Take G_M such that

$$\{x \in \Omega :\, u_f(x) > t\} \subset G_M \subset \{x \in \Omega :\, u_f(x) \geq t\}, \quad |G_M| = \alpha.$$

Then we have

$$I = \int_\Omega g\, u_f dx \leq \int_\Omega \chi_{G_M}\, u_f dx = \int_\Omega f\, u_{G_M} dx.$$

The last identity follows easily from the equations for u_{G_M} and u_f. Now take D_M such that

$$\{x \in \Omega :\, u_{G_M}(x) > \tau\} \subset D_M \subset \{x \in \Omega :\, u_{G_M}(x) \geq \tau\}, \quad |D_M| = \beta.$$

We have

$$I \leq \int_\Omega f\, u_{G_M} dx \leq \int_\Omega \chi_{D_M} u_{G_M} = \int_\Omega \chi_{G_M} u_{D_M} \leq I.$$

The existence of the maximum follows.

Now let $\alpha = \beta$. Observe that

$$0 \leq \int_\Omega |\nabla(u_G - u_D)|^2 dx = \int_\Omega (\chi_G - \chi_D)(u_G - u_D) dx \tag{2.6}$$

with equality if and only if $G = D$. Since $|G| = |D| = \beta$, we have $(\chi_G + \chi_D)/2 \in K_\beta$. Let G and D be an optimal pair for the maximum of I. Using (2.6) and the identity

$$\int_\Omega \chi_G u_D dx = \int_\Omega \chi_D u_G dx$$

we find

$$\begin{aligned} I = \int_\Omega \chi_G u_D dx &\leq \int_\Omega \frac{\chi_G + \chi_D}{2} \frac{u_G + u_D}{2} dx \\ &= \int_\Omega \frac{\chi_G + \chi_D}{2} u_{\frac{\chi_G + \chi_D}{2}} dx \leq I \end{aligned} \tag{2.7}$$

Since equality must hold in (2.7), we must have equality also in (2.6), hence $G = D$.

ii) The proof for the minimum is very similar to the previous one. One finds an optimal pair G_m, D_m such that

$$\{x \in \Omega : u_{D_m}(x) < t'\} \subset G_m \subset \{x \in \Omega : u_{D_m}(x) \le t'\}, \quad |G_m| = \alpha,$$

$$\{x \in \Omega : u_{G_m}(x) < \tau'\} \subset D_m \subset \{x \in \Omega : u_{G_m}(x) \le \tau'\}, \quad |D_m| = \beta.$$

The theorem is proved. □

Remark 1. If $\alpha = \beta$ in case ii) of Theorem 2.2 we cannot conclude, in general, that $G_m = D_m$. Let us exhibit a counterexample for $N = 1$. If $\Omega = (-L, L)$, the minimum of the functional

$$\min_{|D|=\beta} \int_{-L}^{L} \chi_D u_D dx$$

is attained for $D = (-L, -L + \beta/2) \cup (L - \beta/2, L)$ ([6], Theorem 3.1). This minimum value can be easily computed and equals to $\beta^3/12$. Now let $G = (-L, -L + \beta)$ and $D = (L - \beta, L)$. Take $\beta < 2L/3$. By computation one finds

$$\int_{-L}^{L} \chi_G u_D dx = \beta^4/(8L) < \beta^3/12.$$

Hence, a pair (D, G) with $D = G$ cannot be a minimizer.

3. Symmetry of the Optimal Configurations

Consider first problems (1.2) and (1.3) with $G = \Omega$. Since the optimal configurations D_M and D_m can be expressed in terms of level sets of the solution w to the Saint-Venant problem (2.2), we can say that if Ω is convex, then also D_M and $\Omega \setminus D_m$ are convex [12]. Moreover, if Ω is Steiner symmetric, then also D_M and $\Omega \setminus D_m$ have the same property [12].

Now take problem (1.2) with $\alpha = \beta$. It is known that if Ω is Steiner symmetric with respect to some hyperplane, then also D_M is Steiner symmetric with respect the same hyperplane [7]. Observe that it is not known if the convexity of Ω is inherited by the optimal configuration D_M. It is known [6,7] that $u = u_{D_M}$ satisfies the equation

$$\Delta u + \phi(u) = 0 \;\text{ in } \Omega, \quad u = 0 \;\text{ on }\; \partial\Omega,$$

where $\phi(t)$ is not smooth. Results about convexity of level sets of u whenever Ω is convex are known for $\phi(t)$ smooth and satisfying special restrictions [12].

Now consider problem (1.2) for general α and β. We have

Theorem 3.1. *Assume Ω is Steiner symmetric with respect to a hyperplane Π. If (G, D) is a solution to problem (1.2) then $(G^\#, D^\#)$ is also a solution, where $G^\#$ and $D^\#$ denote Steiner symmetrization of G and D respectively, with respect to Π.*

Proof. Denote a point in Ω by (x, y) with $x = (x_1, \cdots, x_{N-1})$ and $y = x_N$. We may assume that Ω is Steiner symmetric with respect to the hyperplane $y = 0$. Let $u_D = u_D(x, y)$ satisfy

$$-\Delta u_D = \chi_D \text{ in } \Omega, \quad u_D = 0 \text{ on } \partial\Omega,$$

and let $u_{D^\#} = u_{D^\#}(x, y)$ be the solution to the problem

$$-\Delta u_{D^\#} = \chi_{D^\#} \text{ in } \Omega, \quad u_{D^\#} = 0 \text{ on } \partial\Omega.$$

Recall that $\Omega^\# = \Omega$ by assumption. Denoting by $u_D^\#$ the Steiner symmetrization of u_D with respect to $y = 0$, we shall prove the inequality

$$\int_{G^\#} u_D^\# dxdy \le \int_{G^\#} u_{D^\#}\, dxdy. \tag{3.1}$$

We have

$$\int_{G^\#} u_D^\#(x, y) dxdy = \int_{G^\# \cap \{y=0\}} \Big(\int_{G_x^\#} u_D^\#(x, y) dy \Big) dx$$

with $G_x^\# = \{(x', y) \in G^\# : x' = x\}$. Similarly, we have

$$\int_{G^\#} u_{D^\#}(x, y) dxdy = \int_{G^\# \cap \{y=0\}} \Big(\int_{G_x^\#} u_{D^\#}(x, y) dy \Big) dx.$$

Therefore, to prove (3.1) it is sufficient to show that

$$\int_{G_x^\#} u_D^\#(x, y) dy \le \int_{G_x^\#} u_{D^\#}(x, y) dy \tag{3.2}$$

for almost all $x \in G^\# \cap \{y = 0\}$. Now we use a result proved in [2] (Theorem 11 and its Remarks) for parabolic operators, but also true in the stationary case. Denoting by $u_D^*(x, s)$ the decreasing rearrangement of the function $y \mapsto u_D(x, y)$, we have, for $r > 0$

$$\int_0^r u_D^*(x, s) ds \le \int_0^r u_{D^\#}^*(x, s) ds$$

for almost all $x \in G^{\#} \cap \{y=0\}$. Hence, recalling that $(u_D^{\#})^* = u_D^*$ and denoting $\Omega_x = \{(x', y) \in \Omega : x' = x\}$ we have

$$\int_{G_x^{\#}} u_D^{\#}(x,y)dy = \int_0^{|G_x^{\#}|} (u_D^{\#})^*(x,s)ds = \int_0^{|G_x^{\#}|} u_D^*(x,s)ds$$

$$\leq \int_0^{|G_x^{\#}|} u_{D^{\#}}^*(x,s)ds = \sup_{\substack{E \subset \Omega_x \\ |E| = |G_x^{\#}|}} \int_E u_{D^{\#}}(x,y)dy = \int_{G_x^{\#}} u_{D^{\#}}(x,y)dy.$$

In the last step we have used the Steiner symmetry of $u_{D^{\#}}$ with respect to $y=0$. Inequality (3.2) follows for almost all $x \in G^{\#} \cap \{y=0\}$ and inequality (3.1) is proved.

Now, by Hardy-Littlewood inequality and (3.1) we find

$$\int_\Omega \chi_G u_D dxdy \leq \int_{G^{\#}} u_D^{\#} dxdy \leq \int_{G^{\#}} u_{D^{\#}}\, dxdy = \int_\Omega \chi_{G^{\#}} u_{D^{\#}} dxdy.$$

Hence the pair $(G^{\#}, D^{\#})$ is a solution to problem (1.2) and the theorem is proved. □

Corollary. *If Ω is a disc then a solution to problem* (1.2) *is the pair* (G, D), *where G and D are discs concentric with Ω.*

Remark 2. It is not possible to prove the analogous of Theorem 3.1 for a solution to problem (1.3). Let us discuss a counterexample for $N=1$. The interval $\Omega = (-L, L)$ is, of course, Steiner symmetric with respect to $x=0$. Observe first that the minimum of the functional

$$\min_{|G|=\alpha, |D|=\beta} \int_{-L}^{L} \chi_G u_D dx,$$

for G and D symmetric with respect to $x=0$, is attained for $G_{\#} = (-L, -L + \alpha/2) \cup (L - \alpha/2, L)$ and $D_{\#} = (-L, -L+\beta/2) \cup (L - \beta/2, L)$. Indeed, since $-u'' = \chi_D \geq 0$, $u(-L) = u(L) = 0$, the function $u = u_D$ is concave and symmetric with respect to $x=0$. Therefore, for such a D, the integral

$$\int_{-L}^{L} \chi_G u_D dx$$

attains its minimum value whenever $G = G_{\#}$. Moreover, since

$$\int_{-L}^{L} \chi_G u_D dx = \int_{-L}^{L} \chi_D u_G dx,$$

we find $D = D_{\#}$.

This minimum value, for $\alpha \leq \beta$, equals to $\alpha^2(3\beta - \alpha)/24$. Now let $G = (-L, -L + \alpha)$ and $D = (L - \beta, L)$. Take $\beta < L$ and $\alpha < 3\beta(L - \beta)/L$. One finds

$$\int_{-L}^{L} \chi_G u_D dx = \frac{\alpha^2 \beta^2}{8L} < \frac{\alpha^2}{24}(3\beta - \alpha).$$

This shows that the symmetric configuration cannot be a minimizer.

If Ω is a disc and if $\alpha = \beta$, the uniqueness for problem (1.2) has been observed in [6,7]. Let us prove now the uniqueness for general α.

Theorem 3.2. *If Ω is a ball and (G, D) is an optimal pair for problem* (1.2) *then G and D are balls concentric with Ω.*

Proof. Assume the ball Ω is centered at the origin. For $E \subset \Omega$ we denote by E^{Δ} the ball concentric with Ω and $|E^{\Delta}| = |E|$. Recall a result by G. Talenti [14]. Let u_D and $u_{D^{\Delta}}$ satisfy

$$-\Delta u_D = \chi_D \quad \text{in} \quad \Omega, \qquad u_D = 0 \quad \text{on} \quad \partial\Omega, \tag{3.3}$$

and

$$-\Delta u_{D^{\Delta}} = \chi_{D^{\Delta}} \quad \text{in} \quad \Omega, \qquad u_{D^{\Delta}} = 0 \quad \text{on} \quad \partial\Omega. \tag{3.4}$$

Following [14], by (3.3) we find

$$1 \leq \frac{1}{N^2 C_N^{2/N}} (-\mu'(t))(\mu(t))^{-2+2/N} \int_0^{\mu(t)} \chi_{(0,\beta)}(s)\, ds := g(t)\,, \tag{3.5}$$

where $\mu(t) = |\{x \in \Omega : u_D(x) > t\}|$ and C_N is the measure of the unit ball in R^N.

By (3.4) we get

$$u_{D^{\Delta}}(x) = \frac{1}{N^2 C_N^{2/N}} \int_{C_N|x|^N}^{|\Omega|} r^{-2+2/N} \int_0^r \chi_{(0,\beta)}(s)\, ds dr. \tag{3.6}$$

Denote by $u_D^{\Delta}(x)$ the spherically decreasing rearrangement of the function $u_D(x)$. Integrating (3.5) from 0 to $u_D^{\Delta}(x)$ and using (3.6) we find

$$u_D^{\Delta}(x) \leq \int_0^{u_D^{\Delta}(x)} g(t)\, dt = u_{D^{\Delta}}(x). \tag{3.7}$$

Using the Hardy-Littlewood inequality, (3.7) and recalling that (G, D) is maximal we find

$$\int_{\Omega} \chi_G u_D dx \leq \int_{\Omega} \chi_{G^{\Delta}} u_D^{\Delta} dx \leq \int_{\Omega} \chi_{G^{\Delta}} u_{D^{\Delta}} dx \leq \int_{\Omega} \chi_G u_D dx.$$

Hence,

$$\int_\Omega \chi_{G^\Delta} u_D^\Delta dx = \int \chi_{G^\Delta} u_{D^\Delta} dx.$$

This equality and inequality (3.7) imply that $u_D^\Delta(x) = u_{D^\Delta}(x)$ in G^Δ (recall that $u_D^\Delta(x)$ and $u_{D^\Delta}(x)$ are continuous functions). In particular, $u_D^\Delta(0) = u_{D^\Delta}(0)$. The latter equality, (3.5) and (3.7) imply that $g(t) = 1$ a.e. in $(0, u_{D^\Delta}(0))$. As a consequence, equality must hold in (3.5) and in (3.7), thus

$$u_D^\Delta(x) = u_{D^\Delta}(x) \quad \text{in } \Omega. \tag{3.8}$$

By using standard inequalities about rearrangements and (3.8) we find

$$\int_\Omega |\nabla u_D|^2 dx = \int_\Omega (2\chi_D u_D - |\nabla u_D|^2)\, dx \le \int_\Omega (2\chi_{D^\Delta} u_D^\Delta - |\nabla u_D^\Delta|^2)\, dx$$
$$= \int_\Omega (2\chi_{D^\Delta} u_{D^\Delta} - |\nabla u_{D^\Delta}|^2)\, dx = \int_\Omega |\nabla u_{D^\Delta}|^2 dx.$$

Using again a standard inequality on rearrangements, the latter inequality and (3.8), we find

$$\int_\Omega |\nabla u_D^\Delta|^2 dx \quad \le \quad \int_\Omega |\nabla u_D|^2 dx \quad \le \quad \int_\Omega |\nabla u_{D^\Delta}|^2 dx \quad = \quad \int_\Omega |\nabla u_D^\Delta|^2 dx.$$

Thus $\int_\Omega |\nabla u_D^\Delta|^2 dx = \int_\Omega |\nabla u_D|^2 dx$. Hence, recalling a well known result of Brothers and Ziemer [3] we have $u_D^\Delta(x) = u_D(x)$ provided that $\{x \in \Omega : \nabla u_D^\Delta(x) = 0\}$ has zero measure. But this is obvious because $\nabla u_{D^\Delta}(x) = 0$ for $x = 0$ only. Then $u_D(x) = u_D^\Delta(x)$ in Ω. Since

$$G = \{x : u_D(x) > t\} \quad \text{for some } \ t > 0\,,$$

we have $G = G^\Delta$. Finally, since

$$\int_\Omega \chi_G u_D dx = \int_\Omega \chi_D u_G dx\,,$$

one proves similarly that $D = D^\Delta$. The theorem is proved. □

By using the example of Remark 2, we may observe that we do not have uniqueness, in general, for problem (1.3).

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