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COMMON FIXED POINTS IN BEST APPROXIMATION THEORY

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Abstract: In this paper, some results on best approximation as a common fixed point of nonexpansive type maps in the setup of locally convex spaces and convex metric spaces are established.

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1. Introduction and Preliminaries

Using fixed point theorems, interesting and valuable results have been obtained in approximation theory, see for example [1], [2], [8], [9], [20] and [21]. Meinardus [13] was the first to employ a fixed point theorem to establish the existence of an invariant approximation in a Banach space. In this paper we extend,

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generalize and unify results on fixed points, common fixed points of best approximation in the setup of a locally convex space (not necessarily metrizable), convex metric space and metrizable topological vector space.

In the sequel, (E,τ) will be a Hausdorff locally convex topological vector space. A family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on E is said to be an associated family of seminorms for τ if the family $\{\gamma U : \gamma > 0\}$, where $U = \bigcup_{i=1}^{n} U_{\alpha_i}$ and $U_{\alpha_i} = \{x : p_{\alpha_i}(x) < 1\}$, forms a base of neighbourhoods of zero for τ. A family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on E is called an augmented associated family for τ if $\{p_\alpha : \alpha \in I\}$ is an associated family with the property that the seminorm $\max\{p_\alpha, p_\beta\} \in \{p_\alpha : \alpha \in I\}$ for any $\alpha, \beta \in I$. The associated and augmented associated families of seminorms will be denoted by $A(\tau)$ and $A^*(\tau)$, respectively. It is well known that given a locally convex space (E,τ), there always exists a family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on E such that $\{p_\alpha : \alpha \in I\} = A^*(\tau)$ (see [12], p. 203).

The following construction will be crucial. Suppose that M is a τ-bounded subset of E. For this set M we can select a number $\lambda_\alpha > 0$ for each $\alpha \in I$, such that $M \subset \lambda_\alpha U_\alpha = \{x : p_\alpha(x) \le 1\}$. Clearly $B = \bigcap_\alpha \lambda_\alpha U_\alpha$ is τ-bounded, τ-closed, absolutely convex and contains M. The linear span E_B of B in E is $\bigcup_{n=1}^{\infty} nB$. The Minkowski functional of B is a norm $\|\cdot\|_B$ on E_B. Thus $(E_B, \|\cdot\|_B)$ is a normed space with B as its closed unit ball, and $\sup_\alpha p_\alpha(x/\lambda_\alpha) = \|x\|_B$ for each $x \in E_B$ (cf. [29]).

Let I and T be selfmaps on M. The map T is called:

(i) $A^*(\tau)$-nonexpansive if for all $x, y \in M$, $p_\alpha(Tx - Ty) \le p_\alpha(x - y)$ for each $p_\alpha \in A^*(\tau)$.

(ii) $A^*(\tau)$-I-nonexpansive if for all $x, y \in M$, $p_\alpha(Tx - Ty) \le p_\alpha(Ix - Iy)$ for each $p_\alpha \in A^*(\tau)$.

For simplicity we call $A^*(\tau)$-nonexpansive ($A^*(\tau)$-I-nonexpansive) maps to be nonexpansive (I-nonexpansive). The set of all fixed points of T will be denoted by $F(T)$. Let $u \in E$. We denote by $P_M(u)$ the set of best M-approximation to u; that is, $P_M(u) = \{y \in M : p_\alpha(y - u) = d_{p_\alpha}(u, M)$, for all $p_\alpha \in A^*(\tau)\}$, where $d_{p_\alpha}(u, M) = \inf\{p_\alpha(x - u) : x \in M\}$.

A subset M of a linear space E is called q-starshaped if there exists at least one point q in M such that $tq + (1-t)x \in M$ for all $x \in M$ and $t \in [0,1]$. Let (E,d) be a metrizable locally convex space. A ball $B_r(0) = \{x \in E : d(x,0) \le$

$r\}$, $(r > 0)$ is said to be compressible if for every $s > 1$, there is $t > r$ such that $B_t(0) \subseteq sB_r(0)$. If every ball $B_r(0)$ in (E, d) is compressible (respectively convex), then we say d is compressible (respectively convex) (see [28]).

Theorem A. ([28], Theorem 2.1) *Let (E, d) be a locally convex metrizable linear space. If d is convex and compressible, then every weak sequentially compact subset M of E is proximinal.*

Let (E, τ) be a topological vector space (TVS). We assume that the topology τ of E is generated by an F-norm q which has the following properties:

(i) *$q(x) \geq 0$ and $q(x) = 0$ if and only if $x = 0$ $(x \in E)$.*

(ii) *$q(x + y) \leq q(x) + q(y)$ for all $x, y \in E$.*

(iii) *$q(\lambda x) \leq q(x)$ for all (real or complex) scalars λ with $|\lambda| \leq 1$.*

(iv) *If $q(x_n) \to 0$, then $q(\lambda x_n) \to 0$ for all scalars λ.*

(v) *If $\lambda_n \to 0$, then $q(\lambda_n x) \to 0$ for all $x \in E$.*

The relation $d(x, y) = q(x - y)$ defines a translation invariant metric on E. The space E is said to be uniformly convex if there corresponds to each pair of positive numbers (ϵ, r), a positive number δ such that if x and y lie in E with $d(x, y) \geq \epsilon$, $d(x, 0) < r + \delta$, $d(y, 0) < r + \delta$, then $d\left(\frac{x+y}{2}, 0\right) < r$.

The set of real numbers with the metric $d(x, y) = \dfrac{|x - y|}{1 + |x - y|}$ is a uniformly convex metric linear space.

The space E is said to be strictly convex if $d(x, 0) \leq r$, $d(y, 0) \leq r$ imply that $d\left(\frac{x+y}{2}, 0\right) < r$, $x \neq y$, where $x, y \in E$ and r is any positive real number.

It follows from the above definitions that a uniformly convex metric linear space is strictly convex but the converse does not hold in general.

The notion of a T-regular set has been introduced by Veeramani [31] and recently this class of nonconvex sets is extensively used by Khan and Hussain [11] to study iterative approximation of fixed points.

Definition 1.1. A subset M of the vector space E is said to be T-regular if and only if:

(i) $T : M \to M$.

(ii) $\dfrac{x + Tx}{2} \in M$ for each $x \in M$.

Example 1.2. ([11], Example 2.2) Let E be a nonzero vector space and $a, b \in E$ with $a \neq b$. Put $c = \dfrac{a+b}{2}$ and $M = \{a, b, c\}$. Define $T : M \to M$ by $T(a) = b,\ T(b) = a$ and $T(c) = c$. Then the set M is T-regular but not convex. Define $F(x) = \dfrac{x+Tx}{2}$. As $\dfrac{a+F(a)}{2} = \dfrac{a+c}{2} \neq M$ so M is not F-regular. Let $A = \{a, b, c, d, e\}$, where $d = \dfrac{a+c}{2}$, $e = \frac{b+c}{2}$. Define $T : A \to A$ by $T(a) = b$, $T(b) = a$, $T(c) = c$, $T(d) = d$ and $T(e) = e$. From $F(x) = \dfrac{x+Tx}{2}$, we obtain $F(a) = F(b) = F(c) = c$, $F(d) = d$ and $F(e) = e$. Also note that $F = TF = FT$ on the set A. So A is T-regular and F-regular (and hence TF-regular) but not a convex set in E.

Clearly for a family $\{A_\alpha : \alpha \in I\}$ of T-regular sets, the sets $\bigcup_\alpha A_\alpha$ and $\bigcap_\alpha A_\alpha$ are T-regular.

Takahashi [27] introduced convexity in a metric space. We recall briefly some facts about this concept. Let (X, d) be a metric space. A continuous mapping $W : X \times X \times [0,1] \to X$ is said to be a convex structure on X, if for all $x, y \in X, \lambda \in [0,1]$, $d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1-\lambda) d(u, y)$ for all $u \in X$.

A metric space X with convex structure is called a convex metric space. A Banach space and each of its convex subsets are simple examples of convex metric spaces with $W(x, y, \lambda) = \lambda x + (1-\lambda) y$. There are many convex metric spaces which can not be imbedded in any Banach space.

A subset M of a convex metric space X is said to be convex if $W(x, y, \lambda) \in M$ for all $x, y \in M$ and $\lambda \in [0,1]$. The set M is said to be q-starshaped if there exists $q \in M$ such that $W(x, q, \lambda) \in M$ for all $x \in M$ and $\lambda \in [0,1]$. Clearly q-starshaped subsets of X contain all convex subsets of X as a proper subclass. A convex metric space is said to satisfy property (I), if for all $x, y \in X$ and $\lambda \in [0,1]$, $d(W(x, q, \lambda),\ W(y, q, \lambda)) \leq \lambda d(x, y)$. The property (I) is always satisfied in any normed space X. For examples and other details we refer to [14, 15, 27]. A continuous function I from a closed convex subset M of a convex metric space X into itself is said to be affine if $I(W(x, y, \lambda)) = W(Ix, Iy)$ whenever $\lambda \in [0,1] \cap Q$ where Q denotes the set of rational numbers and $x, y \in M$. Two maps T and I of X will be called R-weakly commuting provided there exists a positive real number R such that $d(TIx, ITx) \leq R d(Tx, Ix)$ for each $x \in X$.

A set M in a metric space (X, d) is said to be approximately compact if for every x in X and every sequence $\{y_n\}$ in M with $\lim d(x, y_n) = d(x, M)$ there exists a subsequence $\{y_{n_k}\}$ converging to an element y in M. The Kuratowski measure of noncompactness of a nonempty bounded subset B of X is given by $\alpha(B) = \inf\{r > 0 : B$ can be covered by a finite number of sets of diameter $\leq r\}$. For any map $I : M \to E$, we follow Al-Thagafi [1] to define $C_M^I(u) = \{x \in M : Ix \in P_M(u)\}$ and $D_M^I(u) = P_M(u) \cap C_M^I(u)$. Note that $D_M^I(u) = P_M(u) = C_M^I(u)$ whenever I is the identity map on M. In the sequel, $\mathrm{cl}(M)$ and ∂M stand for the closure and boundary of M, respectively.

The following recent common fixed point theorems will be needed.

Theorem B. ([10], Corollary 2.4) *Let M be a τ-bounded, τ-sequentially closed and finite dimensional subset of a Hausdorff locally convex space (E, τ). Assume that M is q-starshaped. Suppose that $T, I : M \to M$ are nonexpansive, I is affine and leaving q fixed and $TI = IT$. Suppose that for $x, y \in M$, there exist $n = n(x, y), m = m(x, y)$ in $N_0 = \{0, 1, 2, \ldots,\}$ such that*

$$p_\alpha(Tx - Ty) \leq p_\alpha(I^m x - I^n y) \text{ for each } p_\alpha \in A^*(\tau). \tag{1}$$

Then T and I have a common fixed point.

Theorem C. ([1], Theorem 2.2) *Let M be a closed subset of a normed space X, I and T selfmaps of M with $T(M) \subset I(M)$ and $q \in F(I)$. If M is q-starshaped, $\mathrm{cl}(T(M))$ is compact, I is continuous and linear, I and T are commuting and T is I-nonexpansive, then T and I have a common fixed point.*

The following is a consequence of Theorem 1 of Pant [18].

Theorem D. (cf. [22], Theorem 5) *Let (X, d) be a complete metric space and $T, I : X \to X$ be R-weakly commuting mappings such that $T(X) \subseteq I(X)$, and $d(Tx, Ty) < d(Ix, Iy)$ whenever $Ix \neq Iy$. If either T or I is continuous, then $F(T) \cap F(I)$ is singleton.*

2. Results

We present generalization of some well-known approximation results for commuting and non-commuting maps on a normed space to the case of locally convex spaces and convex metric spaces. Our results are also new in the sense that the compactness of $P_M(u)$ is relaxed.

Theorem 2.1. *Let T and I be selfmaps of a Hausdorff locally convex space (E, τ) and let $M \subseteq E$ be such that $T(\partial M) \subseteq M$. Let $u \in F(T) \cap F(I)$ and $D = D_M^I(u)$ be nonempty τ-bounded, τ-sequentially closed and finite dimensional. Assume that D is q-starshaped with $q \in F(I)$ and T, I are commuting,* *nonexpansive and I is affine on D. If T and I satisfy (1) for all $x, y \in D \cup \{u\}$ and $I(D) \subseteq D$, then T and I have a common fixed point in $P_M(u)$.*

Proof. Let $y \in D$. Then $I^n y \in D$ for $n \in N_0$ because $I(D) \subseteq D$ and hence $I^n y \in P_M(u)$ for any $n \in N_0$. By definition of D, $y \in \partial M$ and since $T(\partial M) \subseteq M$, it follows that $Ty \in M$. By (1) for each $p_\alpha \in A^*(\tau)$,

$$p_\alpha(Ty - u) = p_\alpha(Ty - Tu) \leq p_\alpha(I^n y - I^m u) \tag{2}$$

for some $n, m \in N_0$. Now $I^m u = u$ and $I^n y \in P_M(u)$, so by (2) we have

$$p_\alpha(Ty - u) \leq p(I^n y - u) = d_{p_\alpha}(u, M).$$

Thus $Ty \in P_M(u)$. The maps T and I commute on D, so for every $p_\alpha \in A^*(\tau)$, we obtain

$$p_\alpha(ITy - u) = p_\alpha(TIy - Tu) \leq p_\alpha(I^j y, I^k u) \tag{3}$$

for some $j, k \in N_0$. As $I^j y \in P_M(u)$ and $I^k u = u$, so (3) implies that

$$p_\alpha(ITy - u) \leq p_\alpha(I^j y - u) = d_{p_\alpha}(u, M) \quad \text{for every } p_\alpha \in A^*(\tau).$$

Hence $Ty \in C_M^I(u)$. Consequently, $Ty \in D$ and so $T, I : D \to D$ satisfy all the conditions of Theorem B. Thus there exists $a \in P_M(u)$ such that $a = T(a) = I(a)$ as desired. □

Lemma 2.2. *Let M be a τ-bounded subset of a Hausdorff locally convex space (E, τ) and $T : M \to M$ an I-nonexpansive map. Then T is I-nonexpansive on M with respect to $\|\cdot\|_B$.*

Proof. Straightforward (see Lemma 2.1 [10] or Theorem 1.1 [29]). □

In [6] Habinaik extended a well-known fixed point theorem due to Dotson [5]. We obtain a common fixed point generalization of this result of Habiniak and Corollary 1.2 [29] by using Theorem C as follows:

Theorem 2.3. *Let M be a nonempty τ-bounded, τ-sequentially closed subset of a Hausdorff locally convex space (E, τ). Let T and I be selfmaps of M with $T(M) \subset I(M)$ and $q \in F(I)$. Suppose M is q-starshaped and τ-sequential-cl($T(M)$) is finite dimensional. If I is nonexpansive, linear, I and T are commuting and T is I-nonexpansive on M, then T and I have a common fixed point in M.*

Proof. Since $\|\cdot\|_B$-topology is finer than the relative τ-topology on E_B, $\|\cdot\|_B$-cl$(M) \subset \tau$-sequential-cl$(M) = M$. Therefore, M is $\|\cdot\|_B$-closed in the normed space $(E_B, \|\cdot\|_B)$. Further, $\|\cdot\|_B$-cl$(T(M))$ being a closed, bounded and finite dimensional subset of a normed space $(E_B, \|\cdot\|_B)$, is compact. By Lemma 2.2, I is $\|\cdot\|_B$-nonexpansive and T is $\|\cdot\|_B$-I-nonexpansive. We can now apply Theorem C to M as a subset of $(E_B, \|\cdot\|_B)$ to conclude that there exist $a \in M$ such that $a = T(a) = I(a)$ as desired. $\square$

We apply Theorem 2.3 to establish the following extension of the main result of Sahab, Khan and Sessa [20].

Theorem 2.4. *Let T and I be selfmaps of a Hausdorff locally convex space (E, τ) and $M \subseteq E$ be such that $T(\partial M) \subseteq M$. Let $u \in F(T) \cap F(I)$ and T is I-nonexpansive on $D = D_M^I(u) \cup \{u\}$. Suppose I is nonexpansive and linear on $D, IT = TI$ on D, $ID = D$, D is nonempty τ-bounded, τ-sequentially closed and q-starshaped with $q \in F(I)$ and τ-sequential-cl-$(T(D))$ is finite dimensional. Then I and T have a common fixed point in $P_M(u)$.*

Proof. Let $y \in D$. We can show, as in the proof of Theorem 2.1, that $Ty \in D$. Hence $T(D) \subseteq D = ID$. The result now follows from Theorem 2.3. $\square$

Theorem 2.5. *Let X be a convex metric space satisfying the property (I). Let $T, I : X \to X$ be operators, M be a subset of X such that $T : \partial M \to M$ and $u \in F(T) \cap F(I)$. Suppose that $D = D_M^I(u)$ is nonempty compact and q-starshaped with $q \in F(I)$, T is I-nonexpansive on $D \cup \{u\}$, I is nonexpansive on $P_M(u) \cup \{u\}$, I is affine on D and $I(D) = D$. If*

$$d(TIx, ITx) \leq R/\lambda d(W(Tx, q, \lambda), Ix) \tag{4}$$

for all $\lambda \in (0, 1), x \in D$ and some $R > 0$, then $P_M(u) \cap F(T) \cap F(I) \neq \phi$.

Proof. If $y \in D$, then $Iy \in D$ since $I(D) = D$, and hence $Iy \in P_M(u)$. By definition of D, $y \in \partial M$ and since $T(\partial M) \subseteq M$, it follows that $Ty \in M$. As T is I-nonexpansive so

$$d(Ty, u) = d(Ty, Tu) \leq d(Iy, Iu) = d(Iy, u). \tag{5}$$

Since $Ty \in M$ and $Iy \in P_M(u)$, so (5) implies that $Ty \in P_M(u)$. As I is nonexpansive on $P_M(u) \cup \{u\}$, we obtain

$$\begin{aligned} d(ITy, u) &= d(ITy, Iu) \leq d(Ty, u) \\ &= d(Ty, Tu) \leq d(Iy, Iu) = d(Iy, u). \end{aligned}$$

Thus $ITy \in P_M(u)$. This implies that $Ty \in C_M^I(u)$, and hence $Ty \in D$. Thus T maps D into itself.

For $k_n = 1 - \dfrac{1}{n}$, we define $T_n : D \to D$ as $T_n(x) = W(Tx, q, k_n)$ for all $x \in D$. Since I is affine, X satisfies the property (I) and T and I satisfy (4) on D, so we have

$$T_nIx = W(TIx, q, k_n)\,, \quad IT_nx = W(ITx, q, k_n)$$

and

$$\begin{aligned} d(T_nIx, IT_nx) = d(W(TIx, q, k_n), W(ITx, q, k_n)) &\le k_n d(TIx, ITx) \\ &\le k_n(R/k_n)d(W(Tx, q, k_n), Ix) = Rd(T_nx, Ix) \end{aligned}$$

for all $x \in D$. Thus T_n and I are R-weakly commuting on D for each n and $T_n(D) \subseteq D = I(D)$. Also we have

$$d(T_nx, T_ny) \le k_n d(Tx, Ty) \le k_n d(Ix, Iy) < d(Ix, Iy)$$

whenever $Ix \ne Iy$.

The map I is continuous so according to Theorem D, $F(T_n) \cap F(I) = \{x_n\}$ for each n. By the compactness of D, $\{x_n\}$ has a subsequence $\{x_{n_j}\} \to z$ (say) in D. Now

$$x_{n_j} = T_{n_j}x_{n_j} = W(Tx_{n_j}, q, k_{n_j}).$$

Since $k_{n_j} \to 1$, therefore $x_{n_j} \to Tz$. Hence $Tz = z$. The continuity of I further implies that

$$Iz = I(\lim_j x_{n_j}) = \lim_j x_{n_j} = z.$$

Thus $z \in P_M(u) \cap F(T) \cap F(I)$ as desired. □

Theorem 2.6. *Let X be a convex metric space satisfying the property (I) and T be a selfmap of X. Let M be a T-invariant subset of X and u in X be a fixed point of T. Suppose that $D = P_M(u)$ is nonempty convex and complete and*

(a) T is continuous on D and $\inf_{x \in D} \alpha[\{y \in D : d(y, Ty) \le d(x, Ty)\}] = 0$, where α is the Kuratowsksi measure of noncompactness.

(b) $d(x, y) \le d(u, M)$ implies $d(Tx, Ty) \le d(x, y)$ for all $x, y \in D \cup \{u\}$.

Then D contains a fixed point of T.

Proof. If $y \in D$, then by (b), we obtain $d(Ty, u) = d(Ty, Tu) \le d(y, u)$. So $Ty \in D$. Hence T maps D into D. Now Corollary 1 [7] implies that T has a fixed point in D. This completes the proof. □

Next, we establish results on invariant approximation for mappings defined on a class of nonconvex sets in a metrizable topological vector space. For this we require the following pair of propositions.

Proposition 2.7. ([16]) *Let E be a strictly convex metric linear space, $u \in E$ and M a subset of E. If $y_1 \neq y_2 \in P_M(u)$, then $\lambda y_1 + (1-\lambda)y_2 \notin M, 0 < \lambda < 1$.*

The proof of the following proposition is included for completeness.

Proposition 2.8. ([11], Proposition 3.12) *Let E be a strictly convex metrizable TVS, M is any subset of E and $T : M \to M$. If $P_M(u)$ is nonempty and T-regular for any $u \in E$, then each point of $P_M(u)$ is a fixed point of T.*

Proof. Suppose for some x in $P_M(u)$, we have $x \neq Tx$. Then by Proposition 2.7, $\dfrac{x+Tx}{2} \notin M$ and so it cannot be in $P_M(u)$. By hypothesis, $P_M(u)$ is T-regular and hence $x = Tx$ must hold. Thus each best M-approximation of u is a fixed point of T. □

Veeramani [31], like many other authors, has used a fixed point result to obtain an approximation result in Corollary 2.2. We achieve this goal in a different way. In fact, our method establishes this corollary without the assumptions of uniform convexity of E and nonexpansiveness of T. Moreover, our results provide several versions of a result of Brosowski [3], which itself is a generalization of a theorem of Meinardus [13].

Theorem 2.9. *Let M be a nonempty T-regular subset of a strictly convex metrizable TVS, E and u be a point in M. Suppose that $d(Tx, u) \leq d(x, u)$ for all x in M. Then each x in M which is a best approximation to u, is a fixed point of T provided one of the following conditions holds:*

(i) M is closed and T is a compact mapping.

(ii) M is approximatively compact.

(iii) M is a weak sequentially compact subset of E with convex and compressible metric d.

(iv) M is a weakly compact subset of a normed space E.

Proof. (i) Let $r = d(u, M)$. Then there is a minimizing sequence $\{y_n\}$ in M such that $\lim d(u, y_n) = r$. It is obvious that $\{y_n\}$ is a bounded sequence. As T

is compact, $\mathrm{cl}(\{Ty_n\})$ is a compact subset of M and so $\{Ty_n\}$ has a convergent subsequence $\{Ty_m\}$ with $\lim Ty_m = x$ (say) in M. Now

$$r \leq d(u,x) = \lim_m d(u, T_{y_m}) \leq \lim_m d(u, y_m) = \lim d(u, y_n) = r.$$

Hence $x \in P_M(u)$. Also, if $y \in P_M(u)$, then $Ty \in M$ and $r \leq d(Ty, u) \leq d(y,u) = r$ imply that $Ty \in P_M(u)$. Further, if $y \in P_M(u)$, then $Ty \in P_M(u)$, so $d(y,u) = r = d(Ty,u)$ and hence by the strict convexity of E, we have $r = d(u, M) \leq d((y+Ty)/2, u) < r$. Thus $(y+Ty)/2 \in P_M(u)$ implies that $P_M(u)$ is T-regular. The result follows from Proposition 2.8.

(ii) A strictly convex metric linear space is locally convex (cf. [11]). An approximately compact set in a locally convex space is proximinal (see [17]) so M is proximinal and hence $P_M(u)$ is nonempty. The result now follows as in (i).

(iii) By Theorem A, M is proximinal so $P_M(u)$ is nonempty and the conclusion follows as in (i).

(iv) $P_M(u)$ is a nonempty and T-regular subset of E (see the proof of Corollary 2.2 [31]) and so we obtain the result from Proposition 2.8. □

An analogue of Theorem 1 [22] in the setup of a metrizable topological vector space is given below.

Theorem 2.10. *Let E be a strictly convex metrizable TVS and $T: E \to E$ a mapping. If M is a nonempty T-regular subset of E, u be a point in $E \setminus M$ such that $Tu = u$ and T satisfies the following condition:*

$$q(Tx - Ty) \leq \alpha q(x-y) + \beta(q(x-Tx) + q(y-Ty)) + \gamma(q(x-Ty) + q(y-Tx)) \quad (6)$$

for all $x, y \in E$, where α, β and γ are reals with $\alpha + 2\beta + 2\gamma \leq 1$. Then each x in M which is a best approximation to u, is a fixed point of T provided one of the conditions (i) – (iv) of Theorem 2.9 holds.

Proof. If we can show that $d(Tx, u) \leq d(x, u)$ for all $x \in E$, then the result will follow from Theorem 2.9. Clearly,

$$\begin{aligned} q(Tx - Tu) &\leq \alpha q(x-u) + \beta(q(x-Tx) + q(u-Tu)) \\ &\quad + \gamma(q(x-Tu) + q(u-Tx)). \end{aligned}$$

Since $Tu = u$, we obtain

$$\begin{aligned} q(Tx-Tu) &\le \alpha q(x-u) + \beta q(x-Tx) + \gamma(q(x-u) + q(Tu-Tx)) \\ &\le \alpha q(x-u) + \beta(q(x-u) + q(Tu-Tx)) + \gamma(q(x-u) + q(Tu-Tx)) \\ &\le \frac{\alpha+\beta+\gamma}{1-\beta-\gamma} q(x-u). \end{aligned}$$

Thus $q(Tx - Tu) \le q(x - u)$ for all $x \in E$ as desired. □

Recently, Pathak and Khan [19] have introduced and studied some conditions of nonexpansive type in the setting of a normed space. We state these conditions in a metrizable topological vector space.

Definition 2.11. A selfmap T of M is said to satisfy

(a) condition 1.1 if for all $x, y \in M$, $q(Tx-Ty) \le \max\{\beta q(x-y), 1/2q(x(x-Tx) + q(y-Ty)), 1/2(q(x-Ty) + q(y-Tx))\}$;

(b) condition 1.2 if for all $x, y \in M$, $q(Tx-Ty) \le \max\{\beta q(x-y), 1/2(q(x-Tx) + q(y-Ty)), q(x-Ty), \beta q(y-Tx)\}$, where $0 \le \beta < 1$.

Note that a map satisfying the condition 1.1 also satisfies the condition 1.2 but not conversely (see [19], Example 3.5).

Recall that $T : M \to M$ is called quasi-nonexpansive if $d(Tx, u) \le d(x, u)$ for all $x \in M$ and $u \in F(T)$ (see Carbone [4]).

We now show that a mapping T satisfying condition 1.1 or condition 1.2 is a quasi-nonexpansive mapping as follows. Suppose that u is a fixed point of T.

(i) Assume that T satisfies condition 1.1

$$\begin{aligned} &q(Tx-u) = q(Tx-Tu) \\ &\le \max\{\beta q(x-u), 1/2q(x-Tx), 1/2(q(x-u) + q(u-Tx))\} \\ &\le \max\{\beta q(x-u), 1/2(q(x-u) + q(u-Tx))\} \end{aligned}$$

which implies that $q(Tx - u) \le q(x - u)$ for all $x \in M$.

(ii) Suppose that T satisfies condition 1.2.

$$\begin{aligned} &q(Tx-u) = q(Tx-Tu) \\ &\le \max\{\beta q(x-u), 1/2q(x-Tx), q(x-u), \beta q(u-Tx)\} \\ &\le \max\{\beta q(x-u), \\ &\qquad 1/2(q(x-u) + q(u-Tx)), q(x-u), \beta q(u-Tx)\}. \end{aligned}$$

Since $q(Tx-u) \le \beta q(u-Tx)$ is not possible, we have

$$q(Tx-u) \le \max\{1/2(q(x-u)+q(u-Tx)), q(x-u)\}$$

which implies that $q(Tx-u) \le q(x-u)$ for all x in M as required.

Note that the mapping T satisfying inequality (6) is usually known as generalized nonexpansive map. In case we have a map T satisfying condition 1.1 or condition 1.2 in place of generalized nonexpansive map, Theorem 2.10 still remains true. Indeed, Theorem 2.10 is valid even if T is a quasi-nonexpansive map.

The following simple example justifies the above results.

Example 2.12. ([31]) Let $M=[-2,-1]\cup[1,2]$. Define $T: M \to M$ by $Tx=-1$ for $x \in [-2,-1]$ and $Tx=1$ for $x \in [1,2]$. Then all the conditions of Theorem 2.9 are satisfied. $P_M(0)=\{-1,1\}$ and each point of $P_M(0)$ is a fixed point of T.

Remarks 2.13. (i) If $I(P_M(u)) \subseteq P_M(u)$, then $P_M(u) \subseteq C_M^I(u)$. Hence $D_M^I(u)=P_M(u)$.

(ii) If $I(C_M^I(u)) \subseteq C_M^I(u)$, then $I(D_M^I(u)) \subseteq I(C_M^I(u)) \subseteq D_M^I(u)$.

(iii) In view of (i) and (ii), Theorems 2.1, 2.4 and 2.5 hold for $D=P_M(u)$ as well as $D=C_M^I(u)$.

(iv) Theorem 4 of Jungck and Sessa [9], Theorem 2.6 of Khan, Hussain and Khan [10], Theorems 2 and 4 of Narang [16], Theorem 1 of Singh [25], the main result of Subrahmanyam [26] and the result of many other authors are special cases of our Theorems 2.1 and 2.4.

(v) Theorem 2.3 extends Corollary 2.5 [10] and Corollary 2.3 (i) [30].

(vi) Theorem 2.5 generalizes several known results due to Al-Thagafi [1], Beg et al [2], Brosowski [3], Sahab, Khan and Sessa [20] and Shahzad [22].

(vii) If $T: X \to X$ is nonexpansive and D is compact, then both the conditions (a) and (b) in Theorem 2.6 are satisfied. Thus Theorem 2.6 extends the results of Beg et al [2], Brosowski [3], Singh [24] and Subrahmanyam [26].

(viii) The maps T_n and I can not be R-weakly commuting on $P_C(\hat{x})$ for each n, in the proof of Theorem 6 due to Shahzad [22] unless I and T satisfy the following normed space analogue of inequality (4)

$$\|ITx - TIx\| \le \left(\frac{R}{K}\right) \|(kTx+(1-k)q)-Ix\|$$
$$\text{for all } x \in P_C(\hat{x}), \quad k \in (0,1) \text{ and some } R>0.$$

So this theorem of Shahzad, as it stands, is incorrect (see [23] for correct version).

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