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WEIERSTRASS POINTS AND WEIERSTRASS PAIRS ON ALGEBRAIC CURVES

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Abstract: Here we prove (in characteristic 0) the following result. Fix integers k, g such that $g \geq 13$ and $k \geq 4$. Then a general k-gonal curve of genus g has only ordinary Weierstrass points, only finitely many Weierstrass pairs of weight at least two and no Weierstrass pair of weight at least 3.

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1. Introduction

Let X be a smooth complex projective curve of genus $g \geq 2$. Recall (see [2], p. 42) that a Weierstrass point P of X is said to be normal if its gap sequence is given by the integers t such that $1 \leq t \leq g-1$ and the integer $g+1$ or, equivalently, if $h^0(X, \mathcal{O}_X((g-1)P)) = 1$, $h^0(X, \mathcal{O}_X(gP)) = 2$ and $h^1(X, \mathcal{O}_X((g+1)P)) = 0$ or, equivalently, if it has weight 1. A Weierstrass point P of X is said to be ordinary if $h^1(X, \mathcal{O}_X((g+1)P)) = 0$. For all integers g, n, a_i, $1 \leq i \leq n$, x, t such that $g \geq 2$, $n > 0$, $\sum_{i=1}^n a_i = x \leq 2g-2$ and every smooth projective curve X of genus g, set $W(X; n, x, t; a_1, \dots, a_n) := \{(P_1, \dots, P_n) \in X \times \cdots \times X : h^0(X, \mathcal{O}_X(\sum_{i=1}^n a_i P_i)) = t\}$. Fix integers g, n, a_i, $1 \leq i \leq n$, x, t such that $g \geq 2$, $n > 0$, $t > 0$, $\sum_{i=1}^n a_i \geq g + t - 1$ and

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n distinct points $P_1, \dots, P_n$ of X. The ordered n-ple $(P_1, \dots, P_n)$ is called a Weierstrass n-ple of type $\geq (a_1, \dots, a_n; t)$ if $h^0(X, \mathcal{O}_X(\sum_{i=1}^n a_i P_i)) \geq t$. It is usual to say type $\geq (a_1, \dots, a_n)$ instead of type $\geq (a_1, \dots, a_n; 1)$. This is the notion studied in [6]. For $n = 2$ this is the notion of Weierstrass pair given in [13] and related but different from the notion of Weierstrass pair given in [2], p. 365. For any fixed n-ple $(a_1, \dots, a_n)$ the set of all Weierstrass n-ples of type $\geq (a_1, \dots, a_n)$ is a locally closed algebraic subset of X^n. In [6], Definition 0.1, there is a definition of weight for Weierstrass n-ples on smooth curves. We work over an algebraically closed field $\mathbf{K}$ but we stress that the definition of weight given in [6] is the good one only if $\mathrm{char}(\mathbf{K}) = 0$, as examples contained in [17], [14] and [15] show even when $n = 1$. The first aim of this paper is to prove the following result.

Theorem 1. *Assume* $\mathrm{char}(\mathbf{K}) = 0$. *Fix integers* k, g *such that* $g \geq 13$ *and* $k \geq 4$. *Then a general* k*-gonal curve of genus* g *has only ordinary Weierstrass points, only finitely many Weierstrass pairs of weight at least two and no Weierstrass pair of weight at least* 3.

For curves with general moduli much more is true. Indeed, the smoothness part of the following result is essentially a translation of [16], Theorem 1.1 (see [3] or the last part of Section 3).

Theorem 2. *Fix positive integers* g, n, a_i, $1 \leq i \leq n$, x, t *such that* $g \geq 2$, $\sum_{i=1}^n a_i = x \leq 2g - 2$. *Assume either* $\mathrm{char}(\mathbf{K}) = 0$ *or* $\mathrm{char}(\mathbf{K}) > g$. *Let* X *be a general curve of genus* g. *The scheme* $W(X; n, x, t; a_1, \dots, a_n)$ *is a non-empty smooth algebraic set of dimension* $g+t+1-x-n$ *if* $x+n < g+t+2$. *If* $W(X; n, x, t; a_1, \dots, a_n) \neq \emptyset$, *then* $W(X; n, x, t; a_1, \dots, a_n)$ *is connected. If* $W(X; n, x, t; a_1, \dots, a_n) \neq \emptyset$, *then* $W(X; n, x, t+1; a_1, \dots, a_n)$ *and* $W(X; n, x-1, t; b_1, \dots, b_n)$, $a_i - 1 \leq b_i \leq a_i$, $\sum_{i=1}^n b_i = x - 1 = -1 + \sum_{i=1}^n a_i$, *are in the closure of* $W(X; n, x, t; a_1, \dots, a_n)$.

The proof of Theorem 1 uses a method employed in [4] to study Weierstrass points and Weierstrass n-ples of plane curves (see Lemmas 1 and 2). We believe that this method has an independent interest. In Section 3 we apply this method to curves contained in a smooth quadric surface and obtain some results on such curves and, as a byproduct, a proof of Theorem 1. Indeed, the normalization of any integral curve $C \subset \mathbf{P}^1 \times \mathbf{P}^1$ with C of type (a, b) has two pencils, one of degree a and the other of degree b, and, in particular, it is k-gonal for some $k \leq \min\{a, b\}$. In the last section we use these constructions to show the existence of smooth curves in which a pair of points has high weight but neither of these points is a Weierstrass point. Hence multiple Weierstrass points may be used to obtain finer invariants of smooth curves.

2. The Main Lemmas

Fix $P \in \mathbf{P}^1 \times \mathbf{P}^1$ and a positive integer z. Set $M(P, z) := \{Z : Z$ is a curvilinear lenght z subscheme of $\mathbf{P}^1 \times \mathbf{P}^1$ with $Z_{red} = \{P\}\}$. By the theory of the local Hilbert scheme (see [11] or [12] or [7]) M(P,z) is smooth and irreducible of dimension $z - 1$. If a, b, z, w are non-negative integers such that $z \leq (a+1)(b+1)$, set $M(P, z; a, b, w) := \{Z : Z$ is a curvilinear length z subscheme of $\mathbf{P}^1 \times \mathbf{P}^1$ with $Z_{red} = \{P\}$ and $h^1(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{I}_Z(a, b)) \geq w\}$. We use the same notation also for $z > (a+1)(b+1)$, but, of course, if $w < (a+1)(b+1) - z$, then $M(P, z; a, b, w) = M(P, z; a, b, w+1)$. If $z < (a+1)(b+1)$, set $M(P, z; a, b)(*) := \{Z \in M(P, z)$: every curve of type (a, b) of $\mathbf{P}^1 \times \mathbf{P}^1$ containing Z is singular at $P\}$. More generally, fix strictly positive integers $y, z_1, \dots, z_y, a, b, w$ such that $\sum_{i=1}^{y} z_i \leq (a+1)(b+1)$ and y distinct points P_i, $1 \leq i \leq y$, of $\mathbf{P}^1 \times \mathbf{P}^1$. Set $M(P_1, \dots, P_y; z_1, \dots, z_y) := \{Z : Z$ is a curvilinear subscheme of $\mathbf{P}^1 \times \mathbf{P}^1$ with y connected components $Z_1, \dots, Z_y$ such that length$(Z_i) = z_i$ and $(Z_i)_{red} = \{P_i\}\}$. By the theory of the local Hilbert scheme ([11] or [12] or [7]) $M(P_1, \dots, P_y; z_1, \dots, z_y)$ is smooth and irreducible of dimension $\sum_{i=1}^{y} z_i - y$. Set $M(P_1, \dots, P_y; z_1, \dots, z_y; a, b, w) := \{Z \in M(P_1, \dots, P_y; z_1, \dots z_y) : h^1(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{I}_Z(a, b)) \geq w\}$. If $z_1 + \cdots + z_y < (a+1)(b+1)$, set $M(P_1, \dots, P_y; z_1, \dots, z_y; a, b)(*) := \{Z \in M(P_1, \dots, P_y; z_1, \dots, z_y)$: every curve of type (a, b) of $\mathbf{P}^1 \times \mathbf{P}^1$ containing Z is singular at least one of the points $P_1, \dots, P_y\}$.

Lemma 1. *Assume char*$(\mathbf{K}) = 0$. *The following results hold.*

(a) *For all non-negative integers z, a and b such that $z \leq (a+1)(b+1)$ the scheme $M(P, z; a, b, 1)$ has codimension at least one in $M(P, z)$.*

(b) *For all non-negative integers z, a and b such that $z \leq (a+1)(b+1)$ the scheme $M(P, z; a, b, 2)$ has codimension at least two in $M(P, z)$.*

(c) *For all non-negative integers z, a and b such that $z \leq (a+1)(b+1)$ a general $Z \in M(P, z)$ is not in $M(P, z; a, b)(*)$.*

(d) *For all non-negative integers z, a and b such that $z \leq (a+1)(b+1)$ the scheme $M(P, z; a, b, 1) \cap M(P, z; a, b)(*)$ has codimension at least two in*

$M(P, z)$.

Proof. Part (a) is trivial because $M(P, z)$ is irreducible.

Now we prove part (c). Since $z < (a+1)(b+1)$, for every $Z \in M(P, z)$ we have $h^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{I}_Z(a, b)) > 0$. There exists a smooth curve D of type (a, b) with $P \in D$. Let $E \subset D$ be the effective Cartier divisor of degree z of D with $E_{red} = \{P\}$. Since $E \subset D$ and D is smooth, we have $E \notin M(P, z; a, b)(*)$. Hence, we obtain part (c) by the openness of smoothness.

Now we consider part (b). We will simultaneously prove part (d). Just to fix the notation we assume $b \geq a$, the other case being similar. If $a + b \leq 2$ and $b \geq a$, the triple (a, b, z) is one of the following ones: $(0, 0, 0)$, $(0, 1, 0)$, $(0, 1, 1)$, $(0, 2, 0)$, $(0, 2, 1)$, $(0, 2, 2)$, $(1, 1, 0)$, $(1, 1, 1)$, $(1, 1, 2)$ and $(1, 1, 3)$. Assume $(a, b, z) = (0, 2, 2)$. A length two subscheme Z with $Z_{red} = \{P\}$ has $h^1(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{I}_Z(a, b)) = 0$, unless Z is contained in the line D of type $(0, 1)$ passing through P and $h^1(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{I}_Z(a, b)) = 1$ if and only if Z is the Cartier divisor $2P$ on D. Thus we checked the case $(0, 2, 2)$ and the cases $(0, 0, 0)$, $(0, 1, 0)$, $(0, 1, 1)$, $(0, 2, 0)$ and $(0, 2, 1)$ are easier. Now assume $(a, b, z) = (1, 1, 3)$. Take a curvilinear scheme Z such that $Z_{red} = \{P\}$ and $\text{length}(Z) = 3$. Since $h^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(1, 1)) = 4$, Z is contained in a curve of type $(1, 1)$. Since a singular curve of type $(1, 1)$ is the union of two lines and Z is curvilinear, Z is contained in a singular curve of type $(1, 1)$ if and only if it is contained in a line through P. There are exactly two such collinear length 3 schemes Z. Since $\dim(M(P, 3)) = 2$, we obtain part (b) when $(a, b, z) = (1, 1, 3)$. Hence, from now on, we may assume $a + b \geq 3$ and use induction on the integer $a + b$. Fix a triple (a, b, z) with $b \geq a \geq 0$ and $a + b \geq 3$. First assume $z < b(a + 1)$. If part (b) is true for a triple (a_1, b_1, z_1), then it is true for all triples (a_2, b_2, z_1) with $a_2 \geq a_1$ and $b_2 \geq b_1$. Since $z < ((b-1)+1)(a+1)$, we may apply the inductive assumption to the triple $(a, b-1, z)$ and hence obtain part (b) for the triple (a, b, z). Now assume $z \geq b(a+1)$. By induction on z (for the fixed pair (a, b)) we may assume the result for all integers $z' < z$. By the theory of the local Hilbert scheme (see [11] or [12] or [8]), for every $W \in M(P, z-1)$ the algebraic set $q(W) := \{B \in M(P, z) : W \subset B\}$ is irreducible and one-dimensional; $q(W)$ will be called the prolongation set of W and every elemnent of $q(W)$ will be called a length z prolongation (or just a prolongation) of W. Every $Z \in M(P, z)$ is the prolongation of a unique $W \in M(P, z-1)$. Since $M(P, z-1; a, b, 2)$ has codimension at least two in $M(P, z-1)$ by the inductive assumption, the set of all prolongations of elements of $M(P, z-1; a, b, 2)$ has codimension at least two in $M(P, z)$. Hence, to prove parts (b) and (d) of the lemma it is sufficient to prove the following three assertions:

(i) for every $W \in M(P, z-1)\backslash(M(P, z-1; a, b, 2) \cup M(P, z-1; a, b)(*))$ a general $Z \in q(W)$ is not an element of $M(P, z; a, b, 2)$;

(ii) for a general $W \in M(P, z-1)$ no $Z \in q(W)$ is an element of $M(P, z; a, b, 2)$;

(iii) for a general $W \in M(P, z-1)$ no $Z \in q(W)$ is an element of $M(P, z; a, b, 1) \cap M(P, z; a, b)(*)$).

Proofs of (ii) and (iii): Fix a general $W \in M(P, z-1)$. Since $z-1 < (a+1)(b+1)$, we have $h^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{I}_W(a, b)) > 0$. By part (c) we may assume $W \notin M(P, z; a, b)(*)$. Hence, there is a curve D of type (a, b) with $P \in D_{reg}$ and $W \subset D$. By the generality of W and part (a) we have $h^1(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{I}_W(a, b)) = 0$. Hence, for every prolongation Z of W we have $h^1(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{I}_Z(a, b)) \leq \text{lenght}(Z) - \text{length}(W) = 1$. If Z is contained in D, then $Z \notin M(P, z; a, b)(*)$. If Z is not contained in D, then $h^1(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{I}_Z(a, b)) = 0$.

Proof of (i): Fix $W \in M(P, z-1)\backslash(M(P, z-1; a, b, 2) \cup M(P, z-1; a, b)(*))$. Since $z-1 < (a+1)(b+1)$, we have $h^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{I}_W(a, b)) > 0$. Since $W \notin M(P, z; a, b)(*)$, there is a curve D of type (a, b) with $W \subset D$ and $P \in D_{reg}$. There is a unique prolongation Z' of W which is contained in D. For any prolongation Z of W with $Z \neq Z'$ the scheme Z is not contained in D. Thus $h^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{I}_Z(a, b)) < h^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{I}_W(a, b))$. Hence, $Z \notin M(P, z; a, b, 2)$. Since $q(W)$ is one-dimensional, there is such prolongation Z, proving (i) and hence concluding the proof of the lemma. □

The same proof gives the following result.

Lemma 2. *Assume char$(\mathbf{K}) = 0$. Fix positive integers z_1, z_2, a, b such that $a + b \geq 5$ and $z_1 + z_2 \leq (a+1)(b+1)$. Fix distinct points P_1 and P_2 of $\mathbf{P}^1 \times \mathbf{P}^1$. Then $M(P_1, P_2; z_1, z_2; a, b, 1)$ has codimension at least one in $M(P_1, P_2; z_1, z_2)$, while $M(P_1, P_2; z_1, z_2; a, b, 2)$ and $M(P_1, P_2; z_1, z_2; a, b, 1) \cap M(P_1, P_2; z_1, z_2; a, b)(*)$ have codimension at least two in $M(P_1, P_2; z_1, z_2)$.*

3. Proofs of Theorems 1 and 2

In this section we apply Lemmas 1 and 2 to prove Theorem 1 and four more refined results on the normalization of a sufficiently general integral nodal curve contained in a smooth quadric surface (see Propositions 1, 2, 3 and 4). At the end of the section we will prove Theorem 2.

Proposition 1. *Assume char($\mathbf{K}$) = 0. Fix integers a, b, x such that $a \ge 4$, $b \ge 4$ and $0 \le 3x < 2a + 2b$. Fix a general subset S of $\mathbf{P}^1 \times \mathbf{P}^1$ with card(S) = x. For each $Q \in S$ assign a label "Q is an ordinary node" or a label "Q is an ordinary cusp". Let $C \subset \mathbf{P}^1 \times \mathbf{P}^1$ be a general curve of type (a, b) such that $S \subseteq Sing(C)$. Then C is integral, $S = Sing(C)$ and C has at each point of S an ordinary node or an ordinary cusp according to the chosen label. Let X be the normalization of C. Then X has only normal Weierstrass points.*

Proof. We divide the proof into two steps. In the second step we will pass from the statement "only ordinary Weierstrass points" to the statement "only normal Weierstrass points".

Step 1. Set $g := ab - a - b + 1 - x$. Thus $0 \le g \le (a-1)(b-1)$. For any general $S \subset \mathbf{P}^1 \times \mathbf{P}^1$ with card$(S) = x$ set $A(S, a, b) :=$ {integral curves of type (a, b) with S as singular locus}. By assumption, we have $0 \le 3x \le 2a + 2b$. Thus for a general S the set $A(S, a, b)$ is a non-empty open subset of a projective space of dimension $ab+a+b-3x$. For any integer $z > 0$ set $C(S, z) := \{(P, Z) : P \in (\mathbf{P}^1 \times \mathbf{P}^1 \backslash S)$ and Z is a curvilinear length z subscheme of $\mathbf{P}^1 \times \mathbf{P}^1$ with $Z_{red} = \{P\}\}$. Notice that for any $(P, Z) \in C(S, z)$ we have $Z_{red} = \{P\}$ and hence P is uniquely determined by Z. Since $\dim(\mathbf{P}^1 \times \mathbf{P}^1 \backslash S) = 2$, the theory of the local Hilbert scheme ([11] or [12] or [7]) gives that $C(S, z)$ is smooth and irreducible of dimension $z + 1$. Let $\Gamma(a, b, S, z) := \{(C, P, Z) : C \in A(S, a, b), (P, Z) \in C(S, z), Z \subset C\}$ be the incidence correspondence. Let $\pi_1(z) : \Gamma(a, b, S, z) \to A(S, a, b)$ and $\pi_2(z) : \Gamma(a, b, S, z) \to C(S, z)$ be the projections. We will use $C(S, z)$ for $z = g - 1, g$, and $g + 1$. Set $C(S, z, =) := \{(P, Z) \in C(S, z) : h^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{I}_Z(a-2, b-2)) \neq 0\}$. Since $h^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(a-2, b-2)) = g + x$, for every $(P, Z) \in C(S, g-1)$ we have $h^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{I}_Z(a-2, b-2)) \neq 0$.

First Claim: For every $z \ge g+1$ the algebraic set $C(S, z, =)$ has codimension at least two in $C(S, z)$.

Proof of the First Claim: First we assume $x = 0$, i.e. $S = \emptyset$. By part (a) of Lemma 1, $C(\emptyset, g, =)$ is a proper subset of $C(\emptyset, g)$. We will use the notion of prolongation introduced in the proof of Lemma 1. For every $(P, W) \in (C(\emptyset, g) \backslash C(\emptyset, g, =))$ and every prolongation Z of W we have $(P, Z) \in (C(\emptyset, g+1) \backslash C(\emptyset, g+1, =))$. Hence, to check the claim for $C(\emptyset, g+1, =)$ it is sufficient to prove the existence of an algebraic subset Γ of $C(\emptyset, g)$ with codimension at least two in $C(\emptyset, g)$ and such that for every $(P, W) \in (C(\emptyset, g) \backslash \Gamma)$ a general prolongation W' of W satisfies $(P, W') \notin C(\emptyset, g+1, =)$. Set $\Gamma := \{(W_{red}, W) : W \in M(P, g; a-2, b-2)(*) \cup M(P, g; a-2, b-2, 2)$. By part (c) of Lemma

**2.1* we have $\dim(M(P, g; a-2, b-2)(*)) \le g-2$. By part (b) of Lemma 1 we have $\dim(M(P, g; a-2, b-2, 2)) \le g-2$. Hence, using assertion (i) in the proof of Lemma 1 we obtain $\dim(\Gamma) \le g-1$, as wanted. Now assume $z > g+1$ and that $C(\emptyset, z-1, =)$ has codimension at least two in $C(\emptyset, z-1)$. For every $(P, W) \in (C(\emptyset, z-1) \backslash C(\emptyset, z-1, =))$ and every prolongation Z of W we have $(P, Z) \in (C(\emptyset, z) \backslash C(\emptyset, z, =))$. Hence, every element of $C(\emptyset, z, =)$ if of the form (P, Z) with Z prolongation of some W such that $(P, W) \in C(\emptyset, z-1, =)$. Thus every irreducible component of $C(\emptyset, z, =)$ has dimension at most $\dim(C(\emptyset, z-1, =)) + 1$, and hence $C(\emptyset, z, =)$ has codimension at least two in $C(\emptyset, z)$ by the First Claim for the integer $z-1$. Now we will prove the First Claim when $x > 0$ by induction on x. Assume $x > 0$ and that the First Claim is true for the integer $x-1$. Take a general $S' \subset \mathbf{P}^1 \times \mathbf{P}^1$ with $\text{card}(S') = x-1$. Fix one general point Q_i, $1 \le i \le \alpha$, of every irreducible component of $C(S', z, =)$. Notice that for every zero-dimensional scheme W we have $h^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{I}_{W \cup S' \cup \{Q\}}(a-2, b-2)) = \max\{0, h^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{I}_{W \cup S'}(a-2, b-2)) - 1\}$ for a general $Q \in \mathbf{P}^1 \times \mathbf{P}^1$. Apply this trivial observation to every Q_i, $1 \le i \le \alpha$, and set $S := S' \cup \{Q\}$ with Q general. Since $C(S'z)$ and $C(S, z)$ are open subschemes of $C(\emptyset, z)$, and passing from $x-1$ to x we drop by 1 the geometry genus, we obtain the First Claim for all pairs (x, z) such that $z \ge g+2$. Now assume $z = g+1$. The proof just given works if we know that the set $C(S', g+1, =)' := \{(P, Z) \in C(S', g+1) : h^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{I}_Z(a-2.b-2)) \ne 0$ is a proper subset of $C(S', g+1)$. This is easily shown by induction, but it is a triviality, just meaning that the general point of the normalization, X', of a general curve of type (a, b) with $x-1$ double points is not a Weierstrass point of X'; indeed, in characteristic zero every smooth curve of genus at least two has only finitely many Weierstrass points.

Second Claim: $A(S, a, b)$ is a non-empty open subset of a projective space of dimension $ab + a + b - 3x$.

Proof of the Second Claim: First, we will check the existence of an integral curve $T \subset \mathbf{P}^1 \times \mathbf{P}^1$ of type (a, b) such that $S \subseteq \text{Sing}(T)$. We assume that this assertion is true for all triples (a', b', x') such that $0 \le 3x' \le 2a' + 2b'$ and $a' + b' < a + b$; essentially, we use induction on the integer $a+b$. In particular, we may assume $x > 0$ and that the result is true for the triple $(a, b-1, x-1)$. Let $A \subset \mathbf{P}^1 \times \mathbf{P}^1$ be a general set with $\text{card}(A) = x-1$. By the inductive assumption there is an integral curve B of type $(a, b-1)$ such that $A \subseteq \text{Sing}(B)$. Since $3(x-1) \le (a+1)b - 2$ and to be integral is an open condition in a flat family of reduced curves, for a general $P \in \mathbf{P}^1 \times \mathbf{P}^1$ there is an integral curve E of type $(a, b-1)$ such that $A \subseteq \text{Sing}(E)$ and $P \in E$. Set $S := A \cup \{P\}$. Since A and P are general, S is a general subset of $\mathbf{P}^1 \times \mathbf{P}^1$ with $\text{card}(S) = x$. Let

$A \subset \mathbf{P}^1 \times \mathbf{P}^1$ be the line of type $(0,1)$ passing through P. By construction $D \cup E$ is a reduced curve of type (a,b) with exactly two irreducible components and $S \subseteq \mathrm{Sing}(D \cup E)$. The linear system $\Gamma := \mathbf{P}(H^0(\mathbf{P}^1 \times \mathbf{P}^1, (\mathcal{I}_S)^2(a,b))^*)$ parametrizes all curves of type (a,b) containing S in their singular locus. We need to prove that its general element is irreducible. By semicontinuity, we just proved that the general element of Γ is either irreducible or with two irreducible components, one of type $(a,b-1)$ and one of type $(0,1)$. Applying in the same way the inductive assumption to the triple $(a-1,b,x-1)$ we see that the general element of Γ is either irreducible or with exactly two irreducible components, one of type $(a-1,b)$ and one of type $(1,0)$. Hence, the general element of Γ is irreducible, i.e. there is an integral curve T of type (a,b) such that $S \subseteq \mathrm{Sing}(T)$. Since $3x < (a+1)(b+1)$, $x \leq ab-a-b+1$ and $\mathbf{P}^1 \times \mathbf{P}^1$ has ample anticanonical divisor, the Second Claim follows from [1], Proposition 3.7 and [8], Lemma 3.

Since

$$3x < 2a + 2b = h^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(a,b)) - h^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(a-2,-2b))$$

and a curve has dimension 1, using the First Claim and its proof we will check that a general $C \in A(S,d)$ contains no scheme Z with Z_{red} a point of $C \backslash S$, lenght$(Z) \geq g+1$ and Z contained in an adjoint curve of type $(a-2,b-2)$ to C. Fix S, an integer $z \geq g+1$ and $P \in (\mathbf{P}^1 \times \mathbf{P}^1 \backslash S)$. Set $U(P,S,z) := \{C \in A(S,a,b) : P \in C$ and the length z subscheme of C supported by P is contained in a curve of type $(a-2,b-2)$ containing $S\}$; since $P \notin S$, every curve $C \in A(S,a,b)$ containing P is smooth at P; hence for any such curve C there is a unique scheme $Z \subset C$ such that $Z_{red} = \{P\}$ and $\mathrm{length}(Z) = z$: the effective Cartier divisor zP of C; we have $C \in U(P,S,z)$ if and only if $h^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{I}_{Z \cup S}(a-2.b-2)) \neq 0$. Now we will check that for a general $S \subset \mathbf{P}^1 \times \mathbf{P}^1$ with $\mathrm{card}(S) = z$ the linear system $\Phi(S)$ associated to $H^0(\mathbf{P}^1 \times \mathbf{P}^1, (\mathcal{I}_S)^2(a,b))$ has no base points outside S. Fix any $A \subset \mathbf{P}^1 \times \mathbf{P}^1$ such that $\mathrm{card}(S) = z$ and no line of $\mathbf{P}^1 \times \mathbf{P}^1$ contains two or more points of A. Since $3x < 2a + 2b$, $\Phi(A)$ contains reducible curves union of a lines of type $(1,0)$ and b lines of type $(0,1)$. Using these reducible curves we see that the base locus of $\Phi(A)$ is contained in lines containing at least one point of A. However, for each $Q \in A$ the linear system $\Phi(A)$ contains a reducible curve E (respectively F) containing the line of type $(1,0)$ (respectively $(0,1)$) through Q but not the line of type $(0,1)$ (respectively $(1,0)$) through Q. By the first part of the proof of the Second Claim applied to the triple $(a-1,b,x-1)$ (respectively $(a,b-1,x-1)$) and the set $S \backslash \{Q\}$ we may even assume that E (respectively F) has only two irreducible components. Varying such curves E and F we see that $\Phi(A)$ has exactly S as base locus. Thus for general S the linear system $\Phi(S)$ has exactly S as base locus, i.e. $h^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{I}_{S\cup\{P\}}(a,b)) = h^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{I}_S(a,b)) - 1$ for every $P \notin S$. There is at most a one-dimensional subset Ω of $\mathbf{P}^1 \times \mathbf{P}^1 \backslash S$ such that for every $P \notin (S \cup \Omega)$ the algebraic set $U(P,S,z)$ has codimension two in $A(S,a,b)$: here we use that in the proof of the First Claim we may take as point Q any point outside a suitable one-dimensional subset of $\mathbf{P}^1 \times \mathbf{P}^1$. For every $P \in (\mathbf{P}^1 \times \mathbf{P}^1 \backslash S)$ the algebraic set $U(P,S,z)$ is a proper subset of $A(S,a,b)$ because $h^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{I}_{S\cup\{P\}}(a,b)) = h^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{I}_S(a,b)) - 1$. Varying P in $\mathbf{P}^1 \times \mathbf{P}^1 \backslash S$ and looking at the dimensions of the fibers of the projection $\pi_2(z)$, we conclude the checking.

Let X be the normalization of a general $C \in A(S,a,b)$. Just to fix the notation we will consider only the case in which all labels are "ordinary node". Now we will check that for every $P \in S$ there is $C \in A(S,a,b)$ such that the two local branches of C at P have as intersection with the infinitesimaal neighborhood of order $g-1$ of P in $\mathbf{P}^1 \times \mathbf{P}^1$ a length g scheme $Z \in M(P,g;a-2,b-2,0)'$. Since this is an open condition and the symetric product of x copies of $\mathbf{P}^1 \times \mathbf{P}^1$ is irreducible, it is sufficient to show that this is true for at least one $P \in S$. By Lemma 1 we know that $M(P,g;a-2,b-2,1)$ is a proper closed subscheme of the irreducible variety $M(P,g)$. Take a general $Z \in M(P,g)$. Since $S\backslash\{P\}$ is general, we have $h^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{I}_{Z\cup(S\backslash\{P\})}(a-2,b-2)) = 0$. Since $3(x-1)+1+g \leq ab+a+b$, there is $U \in A(S,a,b)$ such that $Z \subset U$, concluding the checking. Hence, for a general $C \in A(S,a,b)$ the counterimages of the nodes of C are not Weierstrass points of X. Alternatively, one could look again at the proof of Lemma 1 and use the proof of the assertion (ii) made in that proof. Hence, X has only ordinary Weierstrass points.

Step 2. Let X be the normalization of a general $C \in A(S,a,b)$. Here we will check that X has only normal Weierstrass points. We checked at the end of Step 1 that the points of X going to the nodes of C are not Weierstrass points of X. Take a point Q of X whose image, P, in C is a smooth point of C. It is sufficient to check that $h^0(X, \mathcal{O}_X((g-1)Q)) = 1$ and $h^1(X, \mathcal{O}_X((g+1)Q)) = 0$ because if these equalities are satisfied either $h^0(X, \mathcal{O}_X(gQ)) = 2$ (i.e. Q is a normal Weierstrass point) or $h^0(X, \mathcal{O}_X(gQ)) = 1$ (i.e. Q is not a Weierstrass point). These equalities are true for a general $C \in A(S,a,b)$ by the First Claim. $\square$

Now we will give an extension of Proposition 1 needed for the constructions of Section 4. Its proof is left to the reader.

Proposition 2. *Assume char*$(\mathbf{K}) = 0$. *Fix integers a, b, x such that $a \geq 4$, $b \geq 4$ and $0 \leq 3x < 2a+2b$. We fix a general subset S of $\mathbf{P}^1 \times \mathbf{P}^1$ with* *card*$(S) = x$ *and a line* $D \subset \mathbf{P}^1 \times \mathbf{P}^1$ *of type* $(0,1)$ *such that* $D \cap S = \emptyset$. *Fix* $P, P' \in D$. *Let* Z, Z' *be the zero-dimensional subschemes of* D *with* $Z_{red} = \{P\}$, $Z'_{red} = \{P'\}$, *length*$(Z) = [a/2]$ *and length*$(Z') = [(a+1)/2]$. *For each* $Q \in S$ *assign a label "*Q *is an ordinary node" or a label "*Q *is an ordinary cusp". Let* $C \subset \mathbf{P}^1 \times \mathbf{P}^1$ *be a general curve of type* (a,b) *such that* $S \subseteq$ *Sing*(C) *and* $Z \cup Z' \subset C$. *Then* C *is integral,* $S =$ *Sing*(C) *and* C *has at each point of* S *an ordinary node or an ordinary cusp according to the chosen label. Let* X *be the normalization of* C. *Then* X *has only normal Weierstrass points.*

Using Lemma 2 instead of Lemma 1 and the definition of weight, the proofs of Propositions 1 and 2 give the following results.

Proposition 3. *Assume char*$(\mathbf{K}) = 0$. *Fix integers* a, b, x *such that* $a \geq 4$, $b \geq 4$ *and* $0 \leq 3x < 2a + 2b$. *Fix a general subset* S *of* $\mathbf{P}^1 \times \mathbf{P}^1$ *with card*$(S) = x$. *For each* $Q \in S$ *assign a label "*Q *is an ordinary node" or a label "*Q *is an ordinary cusp". Let* $C \subset \mathbf{P}^1 \times \mathbf{P}^1$ *be a general curve of type* (a,b) *such that* $S \subseteq$ *Sing*(C). *Then* C *is integral,* $S =$ *Sing*(C) *and* C *has at each point of* S *an ordinary node or an ordinary cusp according to the chosen label. Let* X *be the normalization of* C. *Then* X *has at most finitely many Weierstrass pairs of weight at least two and no Weierstrass pair of weight at least three.*

Proposition 4. *Assume char*$(\mathbf{K}) = 0$. *Fix integers* a, b, x *such that* $a \geq 4$, $b \geq 4$ *and* $0 \leq 3x < 2a + 2b$. *We fix a general subset* S *of* $\mathbf{P}^1 \times \mathbf{P}^1$ *with card*$(S) = x$ *and a line* $D \subset \mathbf{P}^1 \times \mathbf{P}^1$ *of type* $(0,1)$ *such that* $D \cap S = \emptyset$. *Fix* $P, P' \in D$. *Let* Z, Z' *be the zero-dimensional subschemes of* D *with* $Z_{red} = \{P\}$, $Z'_{red} = \{P'\}$, *length*$(Z) = [a/2]$ *and length*$(Z') = [(a+1)/2]$. *For each* $Q \in S$ *assign a label "*Q *is an ordinary node" or a label "*Q *is an ordinary cusp". Let* $C \subset \mathbf{P}^1 \times \mathbf{P}^1$ *be a general curve of type* (a,b) *such that* $S \subseteq$ *Sing*(C) *and* $Z \cup Z' \subset C$. *Then* C *is integral,* $S =$ *Sing*(C) *and* C *has at each point of* S *an ordinary node or an ordinary cusp according to the chosen label. Let* $\pi : X \to C$ *be the normalization map. Then* X *has at most finitely many Weierstrass pairs of weight at least two and no Weierstrass pair of weight at least three not containing any of the points* $\pi^{-1}(P)$ *and* $\pi^{-1}(P')$.

Proof of Theorem 1.1. Set $a := 4$. Let b be the unique integer such that $3b + 1 \leq g \leq 3b+3$. Hence, $b \geq 4$. Set $x := 3b+3-g$. Hence, $0 \leq 3x \leq 2a+2b$. Apply Propositions 1 and 2 with respect to these data. By semicontinuity we obtain that a general 4-gonal curve of genus g has only ordinary Weierstrass points, only finitely many Weierstrass pairs of weight at least two and no Weierstrass pair of weight at least three. Since for every integer $k > 4$ the closure of the set of all k-gonal curves of genus g contains the set of all 4-gonal curves of genus g (e.g. by the theory of admissible coverings due to J. Harris and D. Mumford) we obtain the thesis even if $k > 4$.

Proof of Theorem 1.2. The non-emptyness and connectedness of the scheme $W(X; n, x, t; a_1, \dots, a_n)$ follow from the proof of [9], §2; more precisely, these results follow from [9], Proposition 1.10 and Lemma 2.2; if $\mathrm{char}(\mathbf{K}) > 0$, use also [9], Remark 2.2. The other results are just a translation of the hyperplane versality of the canonical embedding of C proved in [16], Theorem 1.1.

4. Weierstrass Pairs with High Weight

In this section we apply Propositions 2 and 4 to show the existence of smooth curves in which a pair of points has high weight but neither of these points is a Weierstrass point.

Example 1. We fix integers a, b and t with $b > a \geq 4$, $0 \leq t \leq ab-a-b+1$ and $3t < (a+1)(b+1)$. We fix a general subset S of $\mathbf{P}^1 \times \mathbf{P}^1$ with $\mathrm{card}(S) = t$ and we assume the existence of an irreducible curve $Y \subset \mathbf{P}^1 \times \mathbf{P}^1$ of type (a, b) with $S = \mathrm{Sing}(Y)$ and with only ordinary nodes as singularities. For the existence of such a curve Y for almost all triples (a, b, t) when $\mathrm{char}(\mathbf{K}) = 0$, see [1], 3.7, or [8], Lemma 3. If $\mathrm{char}(\mathbf{K}) > 0$ one has to use a more refined version of Bertini's theorem; the existence of such a curve Y follows when t is low from [10]. Furthermore, we assume the existence of a line $D \subset \mathbf{P}^1 \times \mathbf{P}^1$ of type $(0, 1)$ such that $\mathrm{card}((Y \cap D)_{red}) = 2$, say $(Y \cap D)_{red} = \{P_1, P_2\}$, P_1 and P_2 smooth points of Y, and such that Y and D have intersection multiplicity $[a/2]$ at P_1 and $[(a+1)/2]$ at P_2. The existence of such a curve Y for t low is easy using Bertini's theorem; if $\mathrm{char}(\mathbf{K}) = 0$ one follows the proofs of [1], 3.7, or [8], Lemma 3, with an added base locus: the union Z of two zero-dimensional subschemes of length $[a/2]$ and $[(a+1)/2]$ with P_1 and P_2, respectively, as support. If $\mathrm{char}(\mathbf{K}) > 0$ and t is low, use [10]. Set $g := ab - a - b + 1 - t$. Let $\pi : X \to Y$ be the normalization. Set $Q_i = \pi^{-1}(P_i)$. By [5] Q_1 and Q_2 are not Weierstrass points of X. Since $h^0(X, \mathcal{O}_X(z[a/2]Q_1 + z[(a+1)/2]Q_2)) \geq z + 1$ for every integer $z \geq 0$ and $b > a$, it is obvious that the weight $w(Q_1, Q_2)$ of the pair (Q_1, Q_2) is quite large.

In the next example we will always use the notations just introduced.

Example 2. Here we assume $t = 0$ (i.e. Y smooth) and that Y is a general curve of type (a, b) containing Z. By Example 1 Q_1 and Q_2 are not Weierstrass points. By Proposition 2, for every $P \in Y \backslash \{Q_1, Q_2\}$ we have $h^0(Y, \mathcal{O}_Y((g-1)P)) = 1$ and $h^1(Y, \mathcal{O}_Y((g+1)P)) = 0$. Hence, Y has only ordinary Weierstrass points. By Proposition 4, for all pairs $A, A' \in Y$, $A \neq A'$, such that $\{A, A'\} \cap \{Q_1, Q_2\} = \emptyset$ we have $w(A, A') \leq 2$. As in the proof of Proposition 4 we easily see that for general Y containing Z we have $w(P, Q_1) \leq 2$ and $w(P, Q_2) \leq 2$ for every $P \in Y \backslash \{Q_1, Q_2\}$. Hence, (Q_1, Q_2) is the only pair of Y with weight at least three.

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