A NOTE ON THE TAFT’S PROBLEM

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Abstract: Let $F$ be a field of characteristic zero. In this paper we work out the linearly recursive relation on Lie multiplication $[f, g]$ in Witt algebras $(W^{(i)})^o$ (resp. $(W^{(i)})^o$). This is an open problem proposed by Earl J. Taft. We show that if the characteristic polynomial $p(x)$ (resp. $q(x)$) of $f$ (resp. $g$) $\in (W^{(i)})^o$ or $(W^{(i)})^o$ satisfy $p(x)|(x^i - a^i)$ and $q(x)|(x^i - a^i)$ for $a$ in the algebraically closure of $F$, then $[f, g]$ satisfies $\text{LCM}(p(x), q(x))$, the least common multiple of $p(x)$ and $q(x)$. Some examples illustrate the results.

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1. Introduction

Let $F$ be a field of characteristic zero. In this paper we describe some special recursive relations on Lie multiplications in Lie duals of the (one-sided) Witt algebra in one variable $W_1 = \text{Der } F[x]$ and of the 2-sided Witt algebra (or Vi-
the Lie bialgebra structures on closed of characteristic zero. We assume
Ng and Tafft show that Tafft's Lie bialgebra structure satisfies the classical Yang-Baxter equation (CYBE) for $f$, $g$ have only singular root, and for any root $a$ of characteristic polynomial of $f$, $b$ of characteristic polynomial of $g$ satisfy $a^i = b^i$. The Lie dual structure of $W$ and $W_1$ have been studied by Michaelias [3, 4, 6], Nichols [7, 8] and Tafft [13, 14, 15, 16].

Recall that a Lie algebra $L$ over $F$ has a skew-symmetric multiplication $[,]$ satisfying the Jacobi identity. Reversing the arrows, a Lie coalgebra $M$ over $F$ has a comultiplication $\delta$ from $M$ to $M \wedge M$, the skew-symmetric tensors in $M \otimes M$, which satisfies the co-Jacobi identity $(1 + \sigma + \sigma^2)(1 \otimes \delta)\delta = 0$, where $\sigma$ is the permutation (123) in $S_3$ acting in the usual way on $M \otimes M \otimes M$. A Lie algebra $L$, which is simultaneously a Lie coalgebra is called a Lie bialgebra if $\delta \in Z^1(L, L \wedge L)$. If $\delta = \delta_r \in B^1(L, L \wedge L)$ for some $r \in L \wedge L$, $L$ is called a coboundary Lie bialgebra. The condition is that $\delta_r(x) = [r, x]$ for all $x \in L$.

Every Lie algebra $L$ over $F$ has a dual Lie coalgebra $L^\circ$, which is the sum of the good subspaces of $L^\ast$. A subspace $V$ of $L^\ast$ is good if the map $L^\ast \to (L \otimes L)^\ast$ dual to the Lie multiplication of $L$ takes $V$ to $V \otimes V$ (see [3, 15] for more details).

Let Witt algebras $W_1 = \text{Der } F[x]$, the Lie algebra of derivations of the polynomial algebra $F[x]$. $W_1$ has a basis $\{e_i\}$ for $i \geq -1$, where $e_i = x^{i+1}d/dx$ and $[e_i, e_j] = (j - i)e_{i+j}$. We identify $W_1^\ast$ with sequences $(f_i)_{i \geq -1}$, where $f \leftrightarrow (f_i)$ means $f_1 = f(e_1)$, then $W_1^\circ$ has been identified as the space of linearly recursive sequences (see [7] for details). Similarly, for (full) Witt algebras $W = \text{Der } F[x, x^{-1}]$, $W$ has a basis $\{e_i\}$ for $i \in \mathbb{Z}$, where $e_i = x^{i+1}d/dx$ and $[e_i, e_j] = (j - i)e_{i+j}$. We identity $W^\circ$ as the space of back-solving linearly recursive sequences. The sequence $(f_i)_{i \in \mathbb{Z}}$ is back-solving linearly recursive sequences if $f_1$ satisfying the recursive relation over $F$.

The various Lie coalgebra structure $W_1^{(i)}$ for $i \geq -1$, which are naturally non-isomorphic, are construct in [15] as follows. Let $r_i = e_0 \wedge e_i$. $r_i$ satisfy the classical Yang-Baxter equation (CYBE) for $W_1$, i.e., $r_i \in W_1 \otimes W_1$ is a solution of the triple tensor product condition

$$\text{(CYBE)} \quad [r^{12}_i, r^{13}_i] + [r^{12}_i, r^{23}_i] + [r^{13}_i, r^{23}_i] = 0,$$

in $W_1 \otimes W_1 \otimes W_1$. The notation is that if $r_i = \sum a_j \otimes b_j$, then $r^{12}_i = \sum a_j \otimes b_j \otimes 1$, $r^{13}_i = \sum a_j \otimes 1 \otimes b_j$ and $r^{23}_i = \sum 1 \otimes a_j \otimes b_j$. Thus, $W_1^{(i)} = (W_1, \delta_{r_i})$ is a triangular coboundary Lie bialgebra (see [1] and [15] Proposition 1). In [9, 10], Ng and Tafft show that Tafft's Lie bialgebra structure $W_1^{(i)}$ on $W_1$ are all of the Lie bialgebra structures on $W_1$ up to isomorphism when $F$ is algebraically closed of characteristic zero. We assume $i \in \mathbb{Z}^+$, as $\delta_0 = 0$ gives $(W_1^{(0)})^\circ$ the
structure of an abelian Lie algebra. Let \( \delta_i = \delta_{r_i} \), i.e., \( \delta_i(x) = [e_0 \wedge e_i, x] \). Then \( \delta_i(e_n) = n(e_n \wedge e_i) + (n - i)(e_0 \wedge e_{n+i}) \) be the Lie cobracket in \( W^{(i)} \). The Lie multiplication in \( (W^{(i)})^o \) is described by

\[
[e^*_0, e^*_n] = (n - 2i)e^*_{n-i}, \quad \text{for } n \neq 0,
\]

\[
[e^*_n, e^*_i] = ne^*_n, \quad \text{for } n \neq 0, i,
\]

with all other Lie multiplication of the \( e^*_n \) being zero. Thus let \( f = \sum a_n e^*_n, g = \sum b_m e^*_m \in (W^{(i)})^o, \)

\[
[f, g] = \sum c_p e^*_p,
\]

where

\[
c_p = p(a_0 b_{p+i} - b_0 a_{p+i} + b_i a_p - a_i b_p) + i(a_{p+i} b_0 - a_0 b_{p+i})
\]

The formulas (1) is very important for obtain the algorithm on recursive relations of \([f, g] \in (W^{(i)})^o\). For \( i \in \mathbb{Z} \), a similar discussion is possible for the 2-sided Witt algebra \( W \). Let \( W^{(i)} \) is the Lie bialgebra \((W, \delta_i)\) with \( \delta_i(x) = [e_0 \wedge e_i, x] \) for \( i \in \mathbb{Z} \). The formulas (1) is also held for \( (W^{(i)})^o \). Note that \( e^*_n \not\in (W^{(i)})^o \), we will use the basis \((a^i p^o)_{i \geq -1}^n\) for \( a \in F^x \) and \( n \in \mathbb{N} \).

In [13, 14, 15], Taft proposes an open problem on finding an algorithm for multiplying two given linearly recursive sequences under the above Lie multiplication. Let \( F \) be an algebraically closed field of characteristic zero, we have consider the Taft’s problem in [2]. We show that for \( f, g \in (W^{(i)})^o \) with \( i \neq 0 \), \([f, g]\) satisfies

\[
x^{\max(r_0, s_0)+1}(x - a_1)^{\max(r_1, s_1)+1} \ldots (x - a_k)^{\max(r_k, s_k)+1}(x - c_1)^{r_{k+1}+1} \cdot (x - c_2)^{r_{k+1}+1} \cdot \ldots \cdot (x - c_n)^{r_{k+1}+1} \cdot (x - d_1)^{s_{k+1}+1} \cdot \ldots \cdot (x - d_m)^{s_{k+1}+1}
\]

\[
(r \text{resp. } (x - a_1)^{\max(r_1, s_1)+1} \ldots (x - a_k)^{\max(r_k, s_k)+1}(x - c_1)^{r_{k+1}+1} \cdot (x - c_2)^{r_{k+1}+1} \cdot \ldots \cdot (x - c_n)^{r_{k+1}+1} \cdot (x - d_1)^{s_{k+1}+1} \cdot \ldots \cdot (x - d_m)^{s_{k+1}+1})
\]

in general, where the characteristic polynomial of \( f \) is \( p(x) = x^{r_0}(x - a_1)^{r_1} \ldots (x - a_k)^{r_k}(x - c_{k+1})^{r_{k+1}} \ldots (x - c_{k+n})^{r_{k+n}} \) (resp. \( p(x) = (x - a_1)^{r_1} \ldots (x - a_k)^{r_k}(x - c_{k+1})^{r_{k+1}} \ldots (x - c_{k+n})^{r_{k+n}} \)) and the characteristic polynomial of \( g \) is \( q(x) = x^{s_0}(x - a_1)^{s_1} \ldots (x - a_k)^{s_k}(x - d_{k+1})^{s_{k+1}} \ldots (x - d_{k+m})^{s_{k+m}} \) (resp. \( q(x) = (x - a_1)^{s_1} \ldots (x - a_k)^{s_k}(x - d_{k+1})^{s_{k+1}} \ldots (x - d_{k+m})^{s_{k+m}} \)), and \( \deg(p(x)) > 0, \deg(q(x)) > 0 \).

Where \( a_1, \ldots, a_k, c_{k+1}, \ldots, c_{k+n}, d_{k+1}, \ldots, d_{k+m} \) are distinct in \( F^x \), \( r_0, s_0 \in \mathbb{N} \) and \( r_1, \ldots, r_{k+n}, s_1, \ldots, s_{k+m} \in \mathbb{Z}^+ \). But for the case, which the root \( a_1, \ldots, a_k \) (resp. \( b_1, \ldots, b_k \)) of the characteristic polynomial of \( f \) (resp. \( g \)) \( (W^{(i)})^o \) (or \( (W^{(i)})^o \) satisfy
\[ a^i = b^j \] it is more complex. In this paper, we will give a more explicit algorithm for the recursive relations on \([f, g] \in (W_1^{(i)})^o, (W^{(i)})^o\) in this case.

Throughout the set of non-zero elements of \(F\) is denoted \(F^\times\). We use \(\mathbb{Z}\) denote integers, \(\mathbb{N}\) for the non-negative integers, \(\mathbb{N}_\leq\) for the integers greater than \(-1\), and \(\mathbb{Z}^+\) for the positive integers. In Section 2, 3, 4 and 5 (except for the last section, Section 6), we assume that \(F\) is an algebraically closed field. See [11, 16] for a development of the Hopf algebraic structure of linearly recursive sequences and see [12] for Hopf algebra and coalgebra background.

**Lemma 1.** Let \(F\) be an algebraic closure filed. If \(f = \{f_j\} \in (W_1^{(i)})^o\) is a linearly recursive sequence with the characteristic polynomial \(p(x) = (x - a_1)\cdots(x - a_n)\) and \(a_1, \cdots, a_n\) are distinct in \(F^\times\) then

\[ f_j = t_1 a_i^j + \cdots + t_n a_n^j, \]

for \(j \in \mathbb{Z}\).

**Lemma 2.** Let \(\{a^j\}, \{b^k\} \in (W^{(i)})^o\) for \(j, k \in \mathbb{Z}\), where \(a, b \in F^\times\) such that \(a^i = b^j\). Then \(\{a^j\}, \{b^k\}\) = 0 for \(a = b\) and \(\{a^j\}, \{b^k\}\) satisfies \((x - a)(x - b)\) for \(a \neq b\).

**Proof.** If \(a = b\) then \(\{a^j\} = \{b^k\}\). Thus \(\{a^j\}, \{b^k\}\) = 0.

Now let \(a \neq b\). By the formulas \(c_p = \{a^j\}, \{b^k\}\) of (1)

\[ c_p = p(b^{p+i} - a^{p+i} + b^ia^p - a^i b^p) + i(a^{p+i} - b^{p+i}). \]

Since \(a^i = b^j\),

\[ c_p = p((b^ia^p - a^{p+i} + b^ia^p - a^i b^p) + i(a^{p+i} - b^{p+i}) \]

\[ = i(a^{p+i} - b^{p+i}). \]

So \(\{a^j\}, \{b^k\}\) = \(c_p\) satisfies \((x - a)(x - b)\). This completes the proof of Lemma.

**Lemma 3.** Let \(f, g\) be the linearly recursive sequences. Let the characteristic polynomial of \(f\) be \(p(x)\) and the characteristic polynomial of \(g\) be \(q(x)\) with \(\deg(p(x)) > 0, \deg(q(x)) > 0\). Then \(f + g\) satisfies \(\text{LCM}(p(x), q(x))\), the least common multiple of \(p(x)\) and \(q(x)\).

Now we can proof our main result.

**Theorem 4.** Let \(F\) be an algebraically closed field. Let \(f, g \in (W^{(i)})^o\) and \(f\) with the characteristic polynomial of \(p(x) = (x - a_1)\cdots(x - a_k)(x - c_1)\cdots(x - c_n)\), \(g\) with the characteristic polynomial of \(q(x) = (x - a_1)\cdots(x -
If $a_1, \ldots, a_k, d_1, \ldots, d_m$ are distinct in $F^*$ and satisfy $a_\alpha^i = b_\beta = c_\gamma$ for $1 \leq \alpha \leq k, 1 \leq \beta \leq n, 1 \leq \gamma \leq m$, then $[f, g]$ satisfies

$$(x - a_1) \cdots (x - a_k)(x - c_1) \cdots (x - c_n)(x - d_1) \cdots (x - d_m).$$

**Proof.** By Lemma 1,

$$f = \{f_i\} = \{t_1a_1^j + \cdots + t_ka_k^j + t_{k+1}c_1^j + \cdots + t_{k+m}c_n^j\},$$

and

$$g = \{g_j\} = \{s_1a_1^{j_1} + \cdots + s_ka_k^{j_1} + s_{k+1}d_1^{j_1} + \cdots + s_{k+m}d_m^{j_1}\}.$$

Thus,

$$[f, g] = \{(t_1a_1^j + \cdots + t_ka_k^j + t_{k+1}c_1^j + \cdots + t_{k+n}c_n^j), \{s_1a_1^{j_1} + \cdots + s_ka_k^{j_1} + s_{k+1}d_1^{j_1} + \cdots + s_{k+m}d_m^{j_1}\}\} = \sum_{u=1}^k \sum_{v=1}^n s_us_k^{j_1}\{a_u^j\}, \{a_v^{j_1}\}\}
+ \sum_{u=1}^k \sum_{v=1}^n t_us_k^{j_1}\{a_u^j\}, \{d_v^{j_1}\}\}
+ \sum_{u=1}^n \sum_{v=1}^m t_{k+u}s_k^{j_1}\{c_{k+v}^j\}, \{a_v^{j_1}\}\}
+ \sum_{u=1}^n \sum_{v=1}^m t_{k+u}s_k^{j_1}\{c_{k+v}^j\}, \{d_v^{j_1}\}\}
= \text{sum (1)} + \text{sum (2)} + \text{sum (3)} + \text{sum (4)}.$$

By Lemma 2, the sum (1) satisfies

$$(x - a_1) \cdots (x - a_k),$$

the sum (2) satisfies

$$(x - a_1) \cdots (x - a_k)(x - d_1) \cdots (x - d_m),$$

the sum (3) satisfies

$$(x - a_1) \cdots (x - a_k)(x - c_1) \cdots (x - c_n),$$

the sum (4) satisfies

$$(x - c_1) \cdots (x - c_n)(x - d_1) \cdots (x - d_m).$$

So $[f, g]$ satisfies

$$(x - a_1) \cdots (x - a_k)(x - c_1) \cdots (x - c_n)(x - d_1) \cdots (x - d_m).$$
by Lemma 3. This completes the proof of theorem. \qed

As for the case that $F$ may be not algebraically closed field, we have the following corollary, which followed from [7] Lemma 2.

**Corollary 5.** Let $f, g \in (W^{(i)})^o$ and $f$ with the characteristic polynomial of $p(x)$, $g$ with the characteristic polynomial of $q(x)$. If $p(x) | (x^{i} - a^{i})$ and $q(x) | (x^{i} - a^{i})$, where $a$ is in the algebraically closure of $F$, then $[f, g]$ satisfies

$$\text{LCM}(p(x), q(x)).$$

The following lemma is the Lemma 5 in [2]. We omit the proof.

**Lemma 6.** Let \{a^i\}, \{b^k\} \in (W^{(i)})^o for $j, k \in \mathbb{Z}$, where $a, b \in F^\times$ such that $a^i \neq b^j$. Then \{\{a^i\}, \{b^j\}\} satisfies $(x - a)^2(x - b)^2$.

As a consequence of Lemma 6, we have the following corollary. The proof is similar to Theorem 4.

**Corollary 7.** Let $F$ be an algebraically closed field. Let $f, g \in (W^{(i)})^o$ and $f$ with the characteristic polynomial of $p(x) = (x - a_1) \cdots (x - c_1)(x - d_1) \cdots (x - b_1^{(1)}) \cdots (x - b_s^{(1)})$, $g$ with the characteristic polynomial of $q(x) = (x - a_1) \cdots (x - a_k)(x - d_1) \cdots (x - d_m)(x - b_1^{(2)}) \cdots (x - b_t^{(2)})$. If $a_1, \ldots, a_k, c_1, \ldots, c_n, d_1, \ldots, d_m, b_1^{(1)}, \ldots, b_s^{(1)}, b_1^{(2)}, \ldots, b_t^{(2)}$ are distinct in $F^\times$, they satisfy $a_\alpha^i = c_\beta^i = d_\gamma^i$ for $1 \leq \alpha \leq k, 1 \leq \beta \leq n, 1 \leq \gamma \leq m$ and $a_\alpha^i \neq (b_\eta^{(1)})^i, b_\eta^{(1)} \neq (b_\lambda^{(2)})^i$ for $1 \leq \alpha \leq k, 1 \leq \eta \leq s, 1 \leq \lambda \leq t$, then $[f, g]$ satisfies

1. $(x - a_1) \cdots (x - a_k)(x - c_1) \cdots (x - c_n)(x - d_1) \cdots (x - d_m)$ for $s = t = 0$,
2. $(x - a_1)^2 \cdots (x - a_k)^2(x - c_1)^2 \cdots (x - c_n)^2(x - d_1)^2 \cdots (x - d_m)^2(x - b_1^{(1)})^2 \cdots (x - b_t^{(2)})^2$ for $s \neq 0, t = 0$ (resp. $s = 0, t \neq 0$),
3. $(x - a_1)^2 \cdots (x - a_k)^2(x - c_1)^2 \cdots (x - c_n)^2(x - b_1^{(1)})^2 \cdots (x - b_t^{(2)})^2(x - d_1)^2 \cdots (x - d_m)^2(x - b_1^{(2)})^2(x - b_t^{(2)})^2$ for $s \neq 0, t \neq 0$.

We close this paper with the following examples.

**Example 8.** Denote the linearly recursive sequences $f = \{1\}$ be the sequence $(\cdots, 1, 1, 1, 1, \cdots)$, $g = \{-1\}$ be the sequence $(\cdots, -1, 1, -1, 1, \cdots)$, where the 0-th term is 1 and $h = \{2\}$ be the sequence $(\cdots, \frac{1}{2}, 1, 2, 4, 8, \cdots)$, where the 0-th term is 1. Note that the characteric polynomial of $f$ (resp. $g$ and $f + h$) is $x - 1$ (resp. $x + 1$ and $(x - 1)(x - 2)$).

For $i = 2$, $[f, g] \in (W^{(2)})^o$ is $(\cdots, 0, 4, 0, 4, 0, 4, \cdots)$. It satisfies $x^2 - 1$. Obviously, $x^2 - 1$ is also the characteristic polynomial of $[f, g] \in (W^{(2)})^o$. $[f +
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$h, g \in (W^{(2)})^o$ is $[f, g]_p = (-3p - 2)(-1)^p + (-3p + 8)2^p + 2$. It satisfies
$x^5 - 3x^4 - x^3 + 7x^2 - 4 = (x-1)(x+1)^2(x-2)^2$. Obviously, $x^5 - 3x^4 - x^3 + 7x^2 - 4$
is also the characteristic polynomial of $[f, g] \in (W^{(2)})^o$.

However, for $i = -1$, $[f, g] \in (W^{(-1)})^o$ is $[f, g]_p = (-2p - 1)((-1)^p + 1)$. Thus, $x^4 - 2x^2 + 1 = (x^2 - 1)^2$ is the characteristic polynomial of $[f, g] \in (W^{(-1)})^o$.

$[f + h, g] \in (W^{(-1)})^o$ is $[f, g]_p = (-2p - 1) + (-7p - 2)(-1)^p + (-\frac{3}{2}p - \frac{1}{2})2^p$. Thus, $x^6 - 4x^5 + 2x^4 + 8x^3 - 7x^2 - 4x + 4 = (x^2 - 1)^2(x - 2)^2$ is the characteristic polynomial of $[f + h, g] \in (W^{(-1)})^o$.

Example 9. Let the linearly recursive sequence $f = (f_i)_{i \in \mathbb{Z}}$ be $f_i = -2f_{i-1} - 4f_{i-2}$, and $f_0 = 1, g_1 = 1$. The characteristic polynomial of $f$ is $x^2 + 2x + 4$. Denote the linearly recursive sequences $g = \{2\}$ be the sequence $(\cdots, \frac{1}{2}, 1, 2, 4, 8, \cdots)$, where the $0$-th term is 1 and $h = \{1\}$ be the sequence $(\cdots, 1, 1, 1, 1, \cdots)$. The characteristic polynomial of $g$ (resp. $f + h$ and $g+h$) is $x - 2$ (resp. $x^3 + x^2 + 2x - 4 = (x^2 + 2x + 4)(x - 1)$ and $(x - 1)(x - 2)$). Note that $(x^2 + 2x + 4)(x - 3)$. So $f_{p+3} = 8f_p$ for $p \in \mathbb{Z}$.

For $i = 3$, $[f, g] \in (W^{(3)})^o$ is $[f, g]_p = 24f_p - 24 \cdot 2^p$. Thus, the characteristic polynomial of $[f, g] \in (W^{(3)})^o$ is $x^3 - 8$. $[f + h, g] \in (W^{(3)})^o$ is $[f, g]_p = (7p + 3) + (7p - 32x - 32 = (x - 1)^2(x - 2)(x^2 + 2x + 4)^2$. Since $[f + h, g + h] = [f + h, g] + [f, g + h]$, the characteristic polynomial of $[f + h, g + h] \in (W^{(3)})^o$ is $x^8 - 2x^7 + x^6 - 16x^5 + 32x^4 - 16x^3 + 64x^2 - 128x + 64 = (x - 1)^2(x^3 - 8)^2$.

However, for $i = 2$, $[f, g] \in (W^{(2)})^o$ is $[f, g]_p = (10p - 8)2^p + (-p + 2)4f_p + 4p + 2). Thus, the characteristic polynomial of $[f, g] \in (W^{(2)})^o$ is $x^6 - 16x^3 + 64 = (x^2 - 8)^2$. $[f + h, g] \in (W^{(2)})^o$ is $[f, g]_p = (5p + 2) + (-3p - 16)2p + 4f_p + (p + 2)f_{p+2}$. $[f + h, g + h] \in (W^{(2)})^o$ is $[f, g]_p = (7p - 2) + (10p - 8)2p + 5pf_p + (-2p + 4)f_{p+2}$. $[f + h, g + h] = [f, g + h] + [f, g + h] \in (W^{(2)})^o$. Thus, the characteristic polynomial of $[f + h, g]$, $[f, g + h]$ and $[f + h, g + h] \in (W^{(-1)})^o$ are the same. It is $x^8 - 2x^7 + x^6 - 16x^5 + 32x^4 - 16x^3 + 64x^2 - 128x + 64 = (x - 1)^2(x^3 - 8)^2$.

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