ON THE SOLVABILITY OF THE CAUCHY PROBLEM
FOR SOME CLASSES OF EVOLUTION EQUATIONS
WITH NON-DENSELY DEFINED OPERATOR
COEFFICIENTS

M.K. Balaev

Department of Mathematics
Faculty of Science and Arts
Cumhuriyet University
58140, Sivas, TURKEY
and
Institute of Mathematics and Mechanics
Azerbaijan Academy of Science
Baku, AZERBAIJAN

Abstract: In this paper we describe some classes of arbitrary order evolution equations in a Banach space with linear unbounded non-densely defined operator coefficients, for which the Cauchy problem is well-posed solvable.

AMS Subject Classification: 34K30, 35J25, 47D06
Key Words: differential-operator equations, Cauchy problem, analytic semigroups, well-posed solvability

In this paper we describe some classes of differential operator equations of an arbitrary order with linear unbounded non-densely defined operator coefficients in a Banach space, for which the Cauchy problem is well-posed solvable. Also, we establish the well-posed solvability of non-regular initial-boundary value problems with functional boundary conditions for some classes of well-posed in the sense of Petrovsky equations.

Received: November 12, 2002 © 2003, Academic Publications Ltd.
§Correspondence address: Department of Mathematics, Faculty of Science and Arts, Cumhuriyet University, 58140, Sivas, TURKEY
Notations. Let $E$ be a Banach space with the norm $\|\cdot\|$, and let $A$ be a closed operator in $E$ with the domain $D(A)$. Denote

$$E(A) := \{ u : u \in D(A), \|u\|_{E(A)} = \|Au\| + \|u\| \}, \quad u^{(k)}(t) := \frac{d^k}{dt^k}u(t),$$

$$C^n([0, T]; E(A_n), \ldots, E(A_1), E) := \{ f(\cdot) : A_n f, \ldots, A_1 f^{(n-1)}, f^{(n)} \in C([0, T]; E) \},$$

$$F_{p, \mu}^\alpha((0, T); E) := \{ f(\cdot) : f \in C((0, T]; E), \|f\|_{F_{p, \mu}^\alpha((0, T); E)} \},$$

$$= \sup_{t \in [0, T]} \|t^\mu f(t)\| + \sup_{0 < \tau < t \leq T} \int_\tau^t \frac{\|f(t) - f(s)\|^p}{|t - s|^{p\alpha}} s^{p\mu} \, ds,$$

$$\alpha \in (0, 1], \quad \mu \geq 0, \quad p \geq 1.$$

1. Differential Equations with Constant Operator Coefficients

We consider the Cauchy problem for the differential operator equation

$$u^{(n)}(t) + \sum_{k=1}^{n} \left( A_k + B_k \right) u^{(n-k)}(t) = f(t), \quad (1)$$

$$u^{(k)}(0) = u_k, \quad k = 0, 1, \ldots, n - 1, \quad (2)$$

in a Banach space $E$. Here $A_k$ and $B_k$ ($k = 1, \ldots, n$) are linear unbounded operators with the domains $D(A_k)$ and $D(B_k)$, respectively, $f(t)$ is a function with values in $E$, and $u_0, \ldots, u_{n-1}$ is elements from $E$.

Boundary value problems for differential operator equations for $n = 1, 2$ have been studied in various aspects by many authors (see, for example, [1-5] and the references therein). Higher-order evolution equations were investigated in [6-9,10,11] and other works.

It is usually assumed that the domains $D(A_k)$ of the operators $A_k$ ($k = 1, \ldots, n$) are dense in $E$, and these operators generate strongly continuous and analytic semigroups. However, in some problems we deal with non-densely defined operators and semigroups with a singularity at zero. In such cases P.E.

A. Favini and H. Tanabe [8] studied the existence, the uniqueness and the regularity of the solution of the problem (1)-(2) for some classes of arbitrary order evolution equations in the case when the operator $A_k$ is basic, and other operators are subordinate to it. We note that in this paper it is not assumed that the domains of $A_k$ are dense.

S.Ya. Yakubov [10] investigated higher-order equations in the case when $A_1$ or $A_2$ is basic and other operators are subordinate to them. Without these suggestions a class of equations of the form (1) for $n = 2$ was studied fairly completely in [12-13].

In this paper the problem (1)-(2) is studied without conditions of the subordination to the basic operator in the case when the domains $D(A_k)$ and $D(B_k)$ of the operators $A_k$ and $B_k$, $k = 1, \ldots, n$, generally speaking, are not dense in $E$. Moreover, $f(t)$, $t > 0$ is assumed to be weak continuous.

**Theorem 1.** Let the following conditions be fulfilled:

1) The linear operators $A_k$, $k = 1, \ldots, n - 1$, have the bounded inverse operators $A_k^{-1}$, $k = 1, \ldots, n - 1$ in $E$.

2) The closed operators $A_k A_k^{-1}$, $k = 1, \ldots, n$, satisfy the condition

$$\|R(\lambda; -A_k A_k^{-1})\| \leq C|\lambda|^{-\sigma_k}, \quad |\arg \lambda| \leq \frac{\pi}{2} + \eta, \quad |\lambda| \to \infty,$$

and

$$\delta_{pj} := \sum_{k=p}^{j} \sigma_k \in [j - p, n], \quad 1 \leq p \leq n.$$

3) For each $\varepsilon > 0$ and $u \in D(A_k A_k^{-1}) \subset D(A_{k+1} A_k^{-1})$, $k = 1, \ldots, n - 1$, the following inequality is valid

$$\|A_{k+1} A_k^{-1} u\| \leq \varepsilon \|A_k A_k^{-1} u\| + \Phi_{k, \varepsilon}(u),$$

where $\Phi_{k, \varepsilon}(u)$ is a continuous convex functional of $u$.

4) For each $\varepsilon > 0$ and $u \in D(A_{k-1}) \cap D(A_{k-1}) \subset D(B_k)$, $k = 1, \ldots, n$, the following inequality is valid

$$\|B_k u\| \leq \varepsilon (\|A_{k-1} u\| + \|A_k u\|) + \Phi_{k, \varepsilon}(u).$$

5) $f \in F^\alpha_{p, \mu}((0, T); E)$, $\alpha \in (n - \delta_1 n, 1]$, $\mu \in [0, \delta_1 n - n + 1)$, $p \in [1/\beta, \infty)$. 


6) If \( \overline{D(A_k)} \neq E \), \( k = 0, 1, \ldots, n-1 \), then \( u_k \in D(A_{n-k-1}) \cap D(A_{n-k}) \), and if \( \overline{D(A_k)} = E \), \( k = 0, 1, \ldots, n-1 \), then \( u_k \in D(A_{n-k-1}) \).

Then the problem (1)-(2) has the unique solution

\[
\begin{align*}
  u \in C^{n-1}((0, T]; \mathbb{E}(A_{n-1}), \ldots, \mathbb{E}(A_1), E) \cap C^n(0, T]; \\
  E(A_{n-1}) \cap E(A_n), \ldots, E(A_1), E).
\end{align*}
\]

Moreover, the following estimate is valid

\[
\begin{align*}
  \|u^{(n)}(t)\| + \|A_1u^{(n-1)}(t)\| + \ldots + \|A_nu(t)\| \\
  \leq Ct^{-1}(\|A_1u_{n-1}\| + \|A_1u_{n-2}\| + \ldots + \|A_{n-1}u_0\| + \|A_nu_0\| + u_{n-1}\| \\
  + \|f\|_F^{\alpha_p,\mu}((0, T); E), \quad t \in (0, T].
\end{align*}
\]

2. Differential Equations with Variable Operator Coefficients

We consider the following Cauchy problem for differential-operator equations with variable coefficients

\[
\begin{align*}
  u^{(n)}(t) + \sum_{k=1}^{n} \left( A_k + B_k(t) \right) u^{(n-k)}(t) = f(t), \\
  u^{(k)}(0) = u_k, \quad k = 0, 1, \ldots, n-1.
\end{align*}
\]

**Theorem 2.** Let the following conditions be fulfilled:

1) The operators \( A_k \), \( k = 1, \ldots, n-1 \), have the bounded inverse operators \( A_k^{-1} \), \( k = 1, \ldots, n-1 \) in \( E \).

2) For a certain real \( \alpha \), the closed operators \( A_kA_{k-1}^{-1} \), \( k = 1, \ldots, n \) (\( A_0 := I \)), satisfy the condition

\[
\|R(\lambda; A_kA_{k-1}^{-1})\| \leq C|\lambda|^{-1}, \quad k = 1, \ldots, n, \quad \text{Re } \lambda \geq \alpha.
\]

3) For each \( \varepsilon > 0 \) and \( u \in D(A_2^2) \subset D(A_2) \),

\[
\|A_2u\| \leq \varepsilon(\|A_1u\| + \|A_2^2u\|) + \Phi_{1,\varepsilon}(u),
\]

and for each \( \varepsilon > 0 \) and \( u \in D(A_kA_{k-1}^{-1}) \subset D(A_{k+1}A_k^{-1}) \),

\[
\|A_{k+1}A_k^{-1} u\| \leq \varepsilon\|A_kA_{k-1}^{-1} u\| + \Phi_{k,\varepsilon}(u), \quad k = 2, \ldots, n-1,
\]

where \( \Phi_{k,\varepsilon}(u) \) is a continuous convex functional of \( u \).
4) For each $\varepsilon > 0$ and $u \in D(A_{k-1}) \cap D(A_{k-1}) \subset D(B_k)$, $k = 1, \ldots, n$, the following inequality is valid
\[
\|B_k(t)u\| \leq \varepsilon (\|A_{k-1}u\| + \|A_ku\|) + \Phi_{k,\varepsilon}(u).
\]

5) For a certain $\varepsilon > 0$ and each $t, \tau \in [0, T]$,
\[
\|B_k(t) - B_k(\tau)\| \leq C(t - \tau)^{\beta} (\|A_{k-1}u\| + \|A_ku\| + \|u\|),
\]
\[
f \in F^{\alpha}_{\beta, \mu}((0, T); E), \quad \beta \in (0, 1), \quad \mu \in (0, 1), \quad p \in [1, \infty).
\]

6) If $\overline{D(A_k)} \neq E$, $k = 0, 1, \ldots, n - 1$, then $u_k \in D(A_{n-k}) \cap D(A_{n-k})$, and if $\overline{D(A_k)} = E$, $k = 0, 1, \ldots, n - 1$, then $u_k \in D(A_{n-k})$.

Then the problem (3)-(4) has the unique solution
\[
u \in C^n([0, T]; \{E(A_n), \ldots, E(A_1), E\}).
\]

Moreover, the following estimate is valid
\[
\|u^{(n)}(t)\| + \sum_{k=1}^n \|A_ku^{(n-k)}(t)\| \leq C\left(\sum_{k=1}^n \|A_{k-1}u_{n-k}\| \right. \\
+ \|f\|_{F^{\alpha}_{\beta, \mu}((0, T); E)} \bigg), \quad t \in [0, T].
\]

We investigate the problems (1)-(2) and (3)-(4) by the method of the semigroup theory as in [9].

3. Initial-boundary Value Problems for Partial Differential Equations

In the rectangle $\Omega = [0, T] \times [0, 1]$ we consider the equation
\[
Lu := \frac{\partial^n u(t, x)}{\partial t^n} + (-1)^{p_1} \frac{\partial^{2p_1+n-1} u(t, x)}{\partial t^{p_1} \partial x^{2p_1}} + \cdots + (-1)^{p_n} \frac{\partial^{2p_n} u(t, x)}{\partial x^{2p_n}} \\
+ \sum_{\alpha=0}^{2p_1-1} b_{1\alpha}(x) \frac{\partial^{\alpha+n-1} u(t, x)}{\partial t^{p_1} \partial x^{\alpha}} + \cdots + \sum_{\alpha=0}^{2p_n-1} b_{n\alpha}(x) \frac{\partial^{\alpha} u(t, x)}{\partial x^{\alpha}} = f(x, t),
\]
with the boundary conditions
\[
\begin{align*}
L_{\nu} u &= \alpha_{\nu} u^{(k_{\nu})(t, 0)} + \beta_{\nu} u^{(k_{\nu})(t, 1)} + T_0 u(t, \cdot) = 0, \\
\nu &= 1, 2, \ldots, 2m - r, \\
L_{\nu} u &= \int_0^1 \varphi_{\nu}(x) u(t, x) \, dx = 0, \\
\nu &= 2m - r + 1, \ldots, 2m \quad (0 \leq r \leq 2m),
\end{align*}
\]
and the initial conditions
\[ u_t^{(k)}(0,x) = u_k(x), \quad k = 0, 1, \ldots, n - 1. \] (7)

Here \( \alpha_\nu \) and \( \beta_\nu \) are complex numbers, \( |\alpha_\nu| + |\beta_\nu| \neq 0 \), \( 0 \leq k_1 \leq \ldots \leq k_\nu \leq k_{\nu+1} \leq \ldots \leq 2m - 1 \), \( T_\nu \) are linear functionals in \( W_p^{k_\nu-1}(0,1) \) for a certain \( p \in [1, \infty) \), \( m = \max_{1 \leq k \leq n} p_k \), \( 2p_k > p_{k+1} - p_{k-1}, k = 1, \ldots, n, p_0 = 0 \), and \( \{\varphi_k(x)\} \) are continuous and linear independent on \([0,1]\). Boundary conditions (6) is not assumed to be regular (see [2], p. 17). However, we suppose that boundary conditions (6) satisfy the condition by Y.T. Silchenko [14].

Reducing this problem to the Cauchy problem for the evolution equation (1) and applying Theorem 1 and the estimate from [14], one can prove the well-posed solvability of the problem (5)-(7) in \( L_p(0,1) \) for some \( p_k \) and \( p \).

Particular cases of this problem with regular boundary conditions were investigated in [3, 12]. For \( p_2 > p_1 \), equation (5) is quasi-elliptic or quasi-parabolic. For \( p_2 < 2p_1 \), it is well-posed by Petrovsky, and mixed problem (5)-(7) has not been studied, yet. Here we point out sufficiently wide class of the boundary conditions (6) for which the problem (5)-(7) is well-posed for \( p_2 < 2p_1 \), and equation (5) is parabolic in the sense that the problem (5)-(7) is reduced to the Cauchy problem for a parabolic differential-operator equation.

References


