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ON THE SOLVABILITY OF THE CAUCHY PROBLEM FOR SOME CLASSES OF EVOLUTION EQUATIONS WITH NON-DENSELY DEFINED OPERATOR COEFFICIENTS

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Abstract: In this paper we describe some classes of arbitrary order evolution equations in a Banach space with linear unbounded non-densely defined operator coefficients, for which the Cauchy problem is well-posed solvable.

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In this paper we describe some classes of differential operator equations of an arbitrary order with linear unbounded non-densely defined operator coefficients in a Banach space, for which the Cauchy problem is well-posed solvable. Also, we establish the well-posed solvability of non-regular initial-boundary value problems with functional boundary conditions for some classes of well-posed in the sense of Petrovsky equations.

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Notations. Let E be a Banach space with the norm $\|.\|$, and let A be a closed operator in E with the domain $D(A)$. Denote

$$E(A) := \{u :\ u \in D(A),\ \|u\|_{E(A)} = \|Au\| + \|u\|\},\ u^{(k)}(t) := \frac{d^k}{dt^k}u(t)\,,$$

$$C^n([0,T];\, E(A_n), \dots, E(A_1), E) := \{f(\cdot) :\ A_n f, \dots, A_1 f^{(n-1)}, \\ f^{(n)} \in C([0,T]; E)\}\,,$$

$$F^{\alpha}_{p,\mu}((0,T); E) := \{f(\cdot) :\ f \in C((0,T]; E),\ \|f\|_{F^{\alpha}_{p,\mu}((0,T);E)}$$

$$= \sup_{t\in(0,T]} \|t^{\mu} f(t)\| + \sup_{0<\tau<t\le T} \int_{\tau}^{t} \frac{\|f(t)-f(s)\|^p}{|t-s|^{p\alpha}} s^{p\mu}\, ds\},$$

$$\alpha \in (0,1], \quad \mu \ge 0, \quad p \ge 1.$$

1. Differential Equations with Constant Operator Coefficients

We consider the Cauchy problem for the differential operator equation

$$u^{(n)}(t) + \sum_{k=1}^{n} \Big(A_k + B_k\Big) u^{(n-k)}(t) = f(t), \tag{1}$$

$$u^{(k)}(0) = u_k, \quad k = 0, 1, \dots, n-1, \tag{2}$$

in a Banach space E. Here A_k and B_k $(k = 1, \dots, n)$ are linear unbounded operators with the domains $D(A_k)$ and $D(B_k)$, respectively, $f(t)$ is a function with values in E, and $u_0, \dots, u_{n-1}$ is elements from E.

Boundary value problems for differential operator equations for $n = 1, 2$ have been studied in various aspects by many authors (see, for example, [1-5] and the references therein). Higher-order evolution equations were investigated in [6-9,10,11] and other works.

It is usually assumed that the domains $D(A_k)$ of the operators $A_k\,(k = 1, \dots, n)$ are dense in E, and these operators generate strongly continuous and analytic semigroups. However, in some problems we deal with non-densely defined operators and semigroups with a singularity at zero. In such cases P.E.

Sobolevskii and Y.T. Silchenko [5] worked out a method for the solution of the Cauchy problem for the first-order evolution equations in a Banach space.

A. Favini and H. Tanabe [8] studed the existence, the uniqueness and the regularity of the solution of the problem (1)-(2) for some classes of arbitrary order evolution equations in the case when the operator A_k is basic, and other operators are subordinate to it. We note that in this paper it is not assumed that the domains of A_k are dense.

S.Ya. Yakubov [10] investigated higher-order equations in the case when A_1 or A_2 is basic and other operators are subordinate to them. Without these suggestions a class of equations of the form (1) for $n = 2$ was studied fairly completely in [12-13].

In this paper the problem (1)-(2) is studied without conditions of the subordination to the basic operator in the case when the domains $D(A_k)$ and $D(B_k)$ of the operators A_k and B_k, $k = 1, \dots, n$, generally speaking, are not dense in E. Moreover, $f(t)$, $t > 0$ is assumed to be weak continuous.

Theorem 1. *Let the following conditions be fulfilled:*

1) The linear operators A_k, $k = 1, \dots, n-1$, have the bounded inverse operators A_k^{-1}, $k = 1, \dots, n-1$ in E.

2) The closed operators $A_k A_{k-1}^{-1}$, $k = 1, \dots, n$ $(A_0 := I)$, for some $\eta > 0$, $\sigma_k \in (0, 1]$, $k = 1, \dots, n$, satisfy the condition

$$\|R(\lambda; -A_k A_{k-1}^{-1})\| \le C|\lambda|^{-\sigma_k}, \quad |\arg \lambda| \le \frac{\pi}{2} + \eta, \quad |\lambda| \to \infty,$$

and

$$\delta_{pj} := \sum_{k=p}^{j} \sigma_k \in [j - p, n], \quad 1 \le p \le n.$$

3) For each $\varepsilon > 0$ and $u \in D(A_k A_{k-1}^{-1}) \subset D(A_{k+1} A_k^{-1})$, $k = 1, \dots, n-1$, the following inequality is valid

$$\|A_{k+1} A_k^{-1} u\| \le \varepsilon \|A_k A_{k-1}^{-1} u\| + \Phi_{k,\varepsilon}(u),$$

where $\Phi_{k,\varepsilon}(u)$ is a continuous convex functional of u.

4) For each $\varepsilon > 0$ and $u \in D(A_{k-1}) \cap D(A_{k-1}) \subset D(B_k)$, $k = 1, \dots, n$, the following inequality is valid

$$\|B_k u\| \le \varepsilon(\|A_{k-1} u\| + \|A_k u\|) + \Phi_{k,\varepsilon}(u).$$

5) $f \in F_{p,\mu}^{\alpha}((0, T); E)$, $\alpha \in (n - \delta_{1n}, 1]$, $\mu \in [0, \delta_{1n} - n + 1)$, $p \in [1/\beta, \infty)$.

6) If $\overline{D(A_k)} \neq E,\ k = 0, 1, \ldots, n-1$, then $u_k \in D(A_{n-k-1}) \cap D(A_{n-k})$, and if $\overline{D(A_k)} = E,\ k = 0, 1, \ldots, n-1$, then $u_k \in D(A_{n-k-1})$.

Then the problem (1)-(2) has the unique solution

$$u \in C^{n-1}([0,T], E(A_{n-1}), \ldots, E(A_1), E) \cap C^n((0,T]; \\ E(A_{n-1}) \cap E(A_n), \ldots, E(A_1), E).$$

Moreover, the following estimate is valid

$$\|u^{(n)}(t)\| + \|A_1 u^{(n-1)}(t)\| + \ldots + \|A_n u(t)\| \\ \leq Ct^{-1}(\|A_1 u_{n-1}\| + \|A_1 u_{n-2}\| + \ldots + \|A_{n-1} u_0\| + \|A_n u_0\| + \|u_{n-1}\| \\ + \|f\|_{F^{\alpha}_{p,\mu}((0,T);E)}), \qquad t \in (0,T].$$

2. Differential Equations with Variable Operator Coefficients

We consider the following Cauchy problem for differential-operator equations with variable coefficents

$$u^{(n)}(t) + \sum_{k=1}^{n} \Big(A_k + B_k(t)\Big) u^{(n-k)}(t) = f(t), \tag{3}$$

$$u^{(k)}(0) = u_k, \quad k = 0, 1, \ldots, n-1. \tag{4}$$

Theorem 2. *Let the following conditions be fulfilled:*

1) The operators A_k, $k = 1, \ldots, n-1$, have the bounded inverse operators A_k^{-1}, $k = 1, \ldots, n-1$ in E.

2) For a certain real α, the closed operators $A_k A_{k-1}^{-1}$, $k = 1, \ldots, n$ $(A_0 := I)$, satisfy the condition

$$\|R(\lambda; A_k A_{k-1}^{-1})\| \leq C|\lambda|^{-1}, \quad k = 1, \ldots, n, \quad \operatorname{Re}\lambda \geq \alpha.$$

3) For each $\varepsilon > 0$ and $u \in D(A_1^2) \subset D(A_2)$,

$$\|A_2 u\| \leq \varepsilon(\|A_1 u\| + \|A_1^2 u\|) + \Phi_{1,\varepsilon}(u),$$

and for each $\varepsilon > 0$ and $u \in D(A_k A_{k-1}^{-1}) \subset D(A_{k+1} A_k^{-1})$,

$$\|A_{k+1} A_k^{-1} u\| \leq \varepsilon \|A_k A_{k-1}^{-1} u\| + \Phi_{k,\varepsilon}(u), \quad k = 2, \ldots, n-1,$$

where $\Phi_{k,\varepsilon}(u)$ is a continuous convex functional of u.

4) For each $\varepsilon > 0$ *and* $u \in D(A_{k-1}) \cap D(A_{k-1}) \subset D(B_k)$, $k = 1, \dots, n$, *the following inequality is valid*

$$\|B_k(t)u\| \leq \varepsilon(\|A_{k-1}u\| + \|A_k u\|) + \Phi_{k,\varepsilon}(u).$$

5) For a certain $\varepsilon > 0$ *and each* $t, \tau \in [0, T]$,

$$\|[B_k(t) - B_k(\tau)]\| \leq C(t-\tau)^{\beta}(\|A_{k-1}u\| + \|A_k u\| + \|u\|),$$

$$f \in F^{\alpha}_{p,\mu}((0,T); E),\ \beta \in (0,1],\ \mu \in (0,1],\ p \in [1,\infty).$$

6) If $\overline{D(A_k)} \neq E,\ k = 0, 1, \dots, n-1$, *then* $u_k \in D(A_{n-k-1}) \cap D(A_{n-k})$, *and if* $\overline{D(A_k)} = E,\ k = 0, 1, \dots, n-1$, *then* $u_k \in D(A_{n-k-1})$.

Then the problem (3)-(4) has the unique solution

$$u \in C^n([0,T];\ E(A_n), \dots, E(A_1), E).$$

Moreover, the following estimate is valid

$$\|u^{(n)}(t)\| + \sum_{k=1}^{n} \|A_k u^{(n-k)}(t)\| \leq C(\sum_{k=1}^{n} \|A_{k-1}u_{n-k}\| + \|f\|_{F^{\beta}_{p,\mu}((0,T);E)}), \quad t \in [0,T].$$

We investigate the problems (1)-(2) and (3)-(4) by the method of the semigroup theory as in [9].

3. Initial-boundary Value Problems for Partial Differential Equations

In the rectangle $\Omega = [0,T] \times [0,1]$ we consider the equation

$$Lu := \frac{\partial^n u(t,x)}{\partial t^n} + (-1)^{p_1} \frac{\partial^{2p_1+n-1} u(t,x)}{\partial t^{n-1} \partial x^{2p_1}} + \dots + (-1)^{p_n} \frac{\partial^{2p_n} u(t,x)}{\partial x^{2p_n}}$$
$$+ \sum_{\alpha=0}^{2p_1-1} b_{1\alpha}(x) \frac{\partial^{\alpha+n-1} u(t,x)}{\partial t^{n-1} \partial x^{\alpha}} + \dots + \sum_{\alpha=0}^{2p_n-1} b_{n\alpha}(x) \frac{\partial^{\alpha} u(t,x)}{\partial x^{\alpha}} = f(x,t)\,, \tag{5}$$

with the boundary conditions

$$\left.\begin{aligned} &L_\nu u = \alpha_\nu u_x^{(k_\nu)}(t,0) + \beta_\nu u_x^{(k_\nu)}(t,1) + T_0 u(t,\cdot) = 0, \\ &\qquad \nu = 1, 2, \dots, 2m - r, \\ &L_\nu u = \textstyle\int_0^1 \varphi_\nu(x) u(t,x)\, dx = 0, \\ &\qquad \nu = 2m - r + 1, \dots, 2m \quad (0 \leq r \leq 2m)\,, \end{aligned}\right\} \tag{6}$$

and the initial conditions

$$u_t^{(k)}(0,x) = u_k(x), \quad k = 0, 1, \ldots, n-1. \tag{7}$$

Here α_ν and β_ν are complex numbers, $|\alpha_\nu| + |\beta_\nu| \neq 0$, $0 \leq k_1 \leq \ldots \leq k_\nu \leq k_{\nu+1} \leq \ldots \leq 2m-1$, T_ν are linear functionals in $W_p^{k_\nu - 1}(0,1)$ for a certain $p \in [1,\infty)$, $m = \max\limits_{1\leq k\leq n} p_k$, $2p_k > p_{k+1} - p_{k-1}$, $k = 1,\ldots,n$, $p_0 = 0$, and $\{\varphi_k(x)\}$ are continuous and linear independent on $[0,1]$. Boundary conditions (6) is not assumed to be regular (see [2], p. 17). However, we suppose that boundary conditions (6) satisfy the condition by Y.T. Silchenko [14].

Reducing this problem to the Cauchy problem for the evolution equation (1) and applying Theorem 1 and the estimate from [14], one can prove the well-posed solvability of the problem (5)-(7) in $L_p(0,1)$ for some p_k and p.

Particular cases of this problem with regular boundary conditions were investigated in [3, 12]. For $p_2 > p_1$, equation (5) is quasi-elliptic or quasi-parabolic. For $p_2 < 2p_1$, it is well-posed by Petrovsky, and mixed problem (5)-(7) has not been studied, yet. Here we point out sufficiently wide class of the boundary conditions (6) for which the problem (5)-(7) is well-posed for $p_2 < 2p_1$, and equation (5) is parabolic in the sense that the problem (5)-(7) is reduced to the Cauchy problem for a parabolic diffrential-operator equation.

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