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A NASH-MOSER THEOREM WITH NEAR-MINIMAL HYPOTHESIS

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Abstract: A proof of a Nash-Moser type inverse function theorem is given under substantially weaker hypothesis than previously known. Our method is associated with continuous Newton's method rather than the more conventional Newton's method.

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1. Introduction

Inverse function theorems form an important part of the subject of nonlinear partial differential equations. Following the work of Nash [7], Schwartz [10] gave a more general implicit function theorem. This was followed by celebrated work of Moser [6], which gives an inverse function theorem using a version of (discrete) Newton's method. Later works of Hamilton [4] and Hörmander [5] give further

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refinements in this line of development. In [2], Castro and the present writer give an inverse function theorem, which uses continuous Newton's method (in distinction to disrecte Newton's method used in [6], for example). This allows for a much simpler argument in that the 'loss of derivatives' problem so central to [6] is entirely avoided. No need is made for a 'scale of spaces' or a family of smoothing operators ([5], [10], for example). The present work reduces the hypothesis of [2] in a substantial way to be pointed out after some notation is established. In [2] an application is given concerning the range of two maximal monotone operators. General theory from [1] seems not sufficient to prove this result. The present work allows for a considerably simplified argument for this application.

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2. An Inverse Function Theorem

Suppose that each of H, K, J is a Banach space with H compactly embedded in J in the sense that H forms a linear subspace of J and every bounded sequence $\{x_k\}_{k=1}^{\infty}$ in H has a subsequence convergent in J to a member of $x \in H$ so that

$$\|x\|_H \leq \limsup_{k\to\infty} \|x_k\|_H.$$

If $t > 0$, B_t denotes the closed ball in H with radius t and center 0.

For Theorems 1 and 2, suppose that $M, r > 0$, F: $B_r \to K$ is continuous as a function on $J \cap D(F)$, $F(0) = 0$ and $g \in K$.

Theorem 1. *Suppose that for each $\epsilon > 0$ and x in the interior of B_r there is a sequence $\{x_n\}_{n=1}^{\infty}$ in H and a sequence $\{t_n\}_{n=1}^{\infty}$ of numbers decreasing to 0 such that*

$$\|\frac{1}{t_n}(F(x_n) - F(x)) - g\|_K \leq \epsilon, \text{ and } \|x_n - x\|_H \leq Mt_n,\ n = 1, 2, \ldots. \tag{1}$$

If $\lambda \in [0, r/M)$ there is $x \in B_{\lambda M}$ such that

$$F(x) = \lambda g. \tag{2}$$

Proof. Denote by λ a member of $[0, r/M)$ and suppose that $\epsilon > 0$. Define

$$S_{\epsilon,\lambda} = \{t \in [0, \lambda] : \text{ there is } x \in B_{Mt} \text{ such that } \|F(x) - tg\|_K \leq t\epsilon\}.$$

We will show that

$$\lambda \in S_{\epsilon,\lambda}. \tag{3}$$

To this end suppose $\lambda_0 = \sup S_{\epsilon,\lambda} < \lambda$. Then there are sequences $\{s_k\}_{k=1}^{\infty}$, $\{y_k\}_{k=1}^{\infty}$ in $[0, \lambda/M]$ and H respectively so that $\lambda_0 = \lim_{k\to\infty} s_k$, $y_k \in B_{s_k M}$ and

$$\|F(y_k) - s_k g\|_K \;\le\; s_k\epsilon, \; k = 1, 2, \dots . \tag{4}$$

Due to the nature of the compact embedding of H in J, there is an increasing sequence $\{k_j\}_{j=1}^{\infty}$ of positive integers so that $\{y_{k_j}\}_{j=1}^{\infty}$ converges to $x \in H$ (in the norm of J) and $x \in B_{\lambda_0 M}$. Since F is continuous on J it follows that

$$\|F(x) - \lambda_0 g\|_K \;\le\; \lambda_0\epsilon.$$

Denote by $\{x_k\}_{k=1}^{\infty}$ and $\{s_k\}_{k=1}^{\infty}$ two sequences so that (1) is satisfied for this choice of x. Choose k so that

$$\|x_k - x\|_H \;\le\; Mt_k \le r - \|x\|_H. \tag{5}$$

Then

$$x_k \in B_{M(t_k+\lambda_0)}\,, \tag{6}$$

and

$$\begin{aligned} &\|F(x_k) - (t_k + \lambda_0)g\|_K \\ &\qquad \le \|F(x_k) - F(x) - t_k g\|_K \;+\; \|F(x) - \lambda_0 g\|_K \;\le\; (t_k + \lambda_0)\epsilon, \end{aligned} \tag{7}$$

a contradiction since (7), (6) place $(t+\lambda_0)$ in $S_{\epsilon,\lambda}$. Thus, $\sup S_{\epsilon,\lambda} = \lambda$. That (3) holds follows from the existence of a nonincreasing sequence $\{s_k\}_{k=1}^{\infty}$ convergent to λ and $\{y_k\}_{k=1}^{\infty}$, $y_k \in B_{s_k M}$, $k = 1, 2, \dots$, such that

$$\|F(y_k) - s_k g\|_K \;\le\; s_k\epsilon, \; k = 1, 2, \dots$$

(take a subsequence again).

Suppose now that $\lambda \in [0, r/M)$ and for each $\epsilon > 0$ denote by x^ϵ a member of $B_{\lambda M}$ so that

$$\|F(x^\epsilon) - \lambda g\|_K \le \lambda\epsilon. \tag{8}$$

Since $\{x^\epsilon\}_{\epsilon>0}$ is bounded in H, there is a decreasing sequence $\{\epsilon_k\}_{k=1}^{\infty}$ so that $\{x^{\epsilon_k}\}_{k=1}^{\infty}$ converges in J to an element $x \in B_{\lambda M}$. By continuity, this element x satisfies (2). ◻

Theorem 1 indicates roughly, that g is in a kind of approximate tangent set to the image of F at each point $F(x)$, $x \in B_r$. The following theorem is under the stronger hypothesis that there is $M > 0$ such that at each $x \in B_r$, F has directional derivative g in some direction h so that $\|h\|_H \leq M$. Theorem 2 follows immediately from Theorem 1.

Theorem 2. *Suppose that for each x in the interior of B_r there is*

$$h \in B_M ,$$

so that

$$\lim_{t \to 0+} \frac{1}{t}[F(x + th) - F(x)] = g,$$

this limit being taken in the norm of K. If $\lambda \in [0, r/M)$ there is $x \in B_{\lambda M}$ so that

$$F(x) = \lambda g.$$

Thus, Theorem 2, is a substantial improvement over the result of [2] since in that reference it was required that there be a locally Lipschitizian function h with domain B_r and range in B_M so that

$$\lim_{t \to 0+} \frac{1}{t}[F(x + th(x)) - F(x)] = g, \quad x \in B_r.$$

To be more specific, in [2], one has given g, the problem of finding u so that

$$-\Delta u + f(u) = g ,$$

under zero Dirichlet boundary conditions on a smooth bounded region $\Omega \subset R^n$ with $n > 2$. The function f is chosen to have supercritical growth. In the application of that reference one is concerned with solutions h to

$$-\Delta h + f'(u)h = g , \tag{9}$$

for appropriate u. For the present setting it is only required to have uniform boundedness of the various h over all u in some appropriate region. To apply results from [2], these solutions h must fit together to form a locally Lipschitzian function as indicated above. Consequently, the present Theorem 2 may be applied to the setting of [2] with considerably less verification of hypothesis, but the main point of the simpler hypothesis is its potential to yield stronger results. See also [10] concerning this point.

3. A Comparison with Section 5 of Moser's Paper

Section 5 of [6] gives an inverse function theorem and its application to coercive systems. Our Theorem 3 gives an inverse function result for coercive systems under much weaker hypothesis. Its proof is a corollary of Theorem 2 and itself follows a development for linear systems in Section 3 of [6].

Following a proof of Theorem 3 there is a technical discussion of details on the hypotheses of Moser's theorem and Theorem 3. There is also a discussion of how Theorems 1 and 2 might be applied to systems, which are not coercive. We first give some needed notation and definitions.

Denote by

$$\{V_s : s \geq 0\} \tag{10}$$

a family of Hilbert spaces so that V_t is compactly embedded, in the sense given above, in V_s if $t > s$. It is supposed that

$$\|v\|_\rho \;\leq\; \|v\|_0^{1-\rho/r}\|v\|_r^{\rho/r}, \quad v \in V_r,\ 0 < \rho < r. \tag{11}$$

Suppose in addition that $0 < l < \rho < r$, F is a function with domain

$$\Omega = \overline{\{u \in V_r : \|u\|_0 < 1\}}^{\rho},$$

$F(0) = 0$ and $R(F) \subset V_0$, where the above superscript ρ indicates closure in V_ρ. Suppose also that the following conditions (12), (13), (14) hold:

$$F \text{ is continuous as a function on } D(F) \cap V_l. \tag{12}$$

If $u \in \Omega, v \in V_r,$ then

$$F'(u)v = \lim_{t\to 0+} \frac{1}{t}(F(u+tv) - F(u))$$

$$\text{exists and is in } V_0, \text{ this limit being taken in the norm of } V_0\,. \tag{13}$$

For $u \in \Omega$,

$$F'(u) \text{ is a continuous linear transformation from } V_\rho \text{ to } V_0. \tag{14}$$

Theorem 3. *Suppose $g \in V_0$, $\mu > 0$ and for $u \in \Omega$, the derivative $F'(u)$ has the property that if $Q > 0$, there is $h \in V_r$ such that*

$$\|F'(u)h - g\|_0 \leq Q^{-\mu}, \quad \|h\|_r \leq Q. \tag{15}$$

Suppose in addition that F is coercive in the sense that

$$\|F'(u)h\|_0 \;\geq\; \|h\|_0, \quad h \in V_r,\; u \in \Omega\,, \tag{16}$$

and

$$\mu \;>\; \rho/(r-\rho). \tag{17}$$

There is $\lambda > 0$ and $u \in \Omega \cap V_\rho$ such that

$$F(u) = \lambda g.$$

The following argument uses heavily parts of some arguments in [6].

Proof. Suppose $u \in \Omega$. Define $Q_n = 2^n$, $n = 1, 2, \ldots$. Pick $h_1, h_2, \cdots \in V_r$ such that

$$\|F'(u)h_n - g\|_0 \;\leq Q_n^{-\mu},\; \|h_n\|_r \;\leq\; Q_n,\; n = 1, 2, \ldots. \tag{18}$$

Note that

$$\begin{aligned}\|h_{n+1} - h_n\|_0 \;&\leq\; \|F'(u)(h_{n+1} - h_n)\|_0 \\ &= \|(F'(u)h_{n+1} - g) - (F'(u)h_n - g)\|_0 \leq Q_{n+1}^{-\mu} + Q_n^{-\mu} \;\leq\; 2(2^{-n\mu}), \\ &\qquad\qquad n = 1, 2, \ldots\end{aligned}$$

Thus, $h_1, h_2, \ldots$ is a Cauchy sequence in V_0 but also

$$\|h_{n+1} - h_n\|_r \;\leq\; Q_{n+1} + Q_n \;\leq\; 2^{n+2}\,,$$

and so, using (11)

$$\|h_{n+1} - h_n\|_\rho \;\leq \|h_{n+1} - h_n\|_0^{1-\rho/r}\|h_{n+1} - h_n\|_r^{\rho/r}, \quad n = 1, 2, \ldots\,,$$

and hence

$$\begin{aligned}\|h_{n+1} - h_n\|_\rho \;&\leq\; (22^{-n\mu})^{1-\rho/r}(2^{n+2})^{\rho/r} \\ &\leq 4(2^{-n\mu(1-\rho/r)}2^{n\rho/r}) = 4(2^{(\rho/r)(\mu+1)-\mu})^n, \quad n = 1, 2, \ldots.\end{aligned}$$

Since by hypothesis, $\mu > \rho/(r-\rho)$, it follows that

$$q = \mu - (\rho/r)(\mu + 1) \;> 0.$$

Hence,

$$\|h_{n+1} - h_n\|_\rho \;\leq\; 4(2^{-q})^n,\; n = 1, 2, \ldots.$$

Thus, $h_1, h_2, \ldots$ converges in the norm of V_ρ to an element $h \in V_\rho$. Note that

$$\|h\|_\rho \ \leq \|h_0\|_\rho \ + \sum_{k=1}^{\infty} \|h_{k+1} - h_k\|_\rho \ \leq 2 + 4/(1-2^{-q}) = M,$$

the last equality serving as definition for M. Using (18) and (1) we have

$$F'(u)h = g.$$

In summary, there is $M > 0$ so that if $u \in \Omega$, then there is $h \in V_\rho$ so that

$$F'(u)h = g \text{ and } \|h\|_\rho \leq M.$$

Furthermore, since $0 < l < \rho$ and F is continuous as a function from $V_l \to V_0$, the theorem follows using Theorem 2 ($H = V_\rho,\ K = V_l$). □

A comparison between the preceding and Section 5 of [6] follows. The expression $M5.6$ refers to 5.6 of [6], for example.

The spaces in (10) are often called a scale of Sobolev spaces. The space V_ρ is H in the first section. Our Theorem 3 most directly compares to Moser's theorem on page 285 of [6] (the only theorem in Section 5 of [6]). As part of the hypothesis of this theorem, there are assumptions $M5.1, ..., M5.6$. Condition (12) is part of this hypothesis. Condition $M5.3$ is that for some $0 < s < r$

$$\|F(u)\|_s \leq MK \quad \text{ if } \quad \|u\|_r < M,\ u \in \Omega.$$

This assumption has no counterpart in Theorem 3.

In [6], for some $s \in (0, r)$ and for each $u \in \Omega$, the derivative function $F'(u)$ is from V_r to V_s. For Theorem 3 we require that $F'(u)$ be a continuous linear transformation from V_ρ to V_0. These last two requirements seem to not be directly comparable but the one we use in Threorem 3 seems a mild requirement for applications to PDE.

Our conditions (15) and (16) compare with $M5.4$ in that they both deal with approximate solutions of certain linear equations. The condition $M5.4$ is required to hold for all $g \in V_s$ satisfying $\|g\|_s \leq 1,\ \|g\|_s \leq K$, whereas (15) is required to hold only for the target function $g \in V_0$ in Theorem 3. However, in applications, in order to prove condition (15) it might be necessary to assume that $g \in V_s$ for some $s \in (0, r)$.

Our condition (16) vs. $M5.4$ and $M5.21$ require more explanation. In $M5.4$, it is required that, for a given $W, Q > 0$, there be $h \in V_r$ so that

$$\|F'(u)h - g\|_0 \ \leq \ WQ^{-\mu},\ \|h\|_r \ \leq WQ\,,$$

holds and (16) holds *just for that element* h *satisfying* (15). In the present work, (15) and (16) together are stronger than that required in the theorem in [6] on page 285. However, in what seems to be some of the main applications in [6], a positivity condition, $M5.21$ is required (which is much stronger than (16)); $M5.21$ is a pair of conditions, the first, of which in itself is stronger than (16) and the second is a positivity condition in the V_s norm, (there is no counterpart of this in the present work). In short, in so far as what may be called coercive systems in [6], the present positivity conditions seem much weaker.

Condition $M5.6$ is a quadratic conditon: There is $\beta \in (0,1)$ so that

$$\|F(u+h)-F(u)-F'(u)h\|_0 \leq M\|v\|_0^{2-\beta}\|v\|_r^{\beta}, \qquad u \in \Omega,\ v \in V_r,\ h \in V_r. \tag{19}$$

This condition has no counterpart in the present work. To accomodate $M5.6$ in [6], an additonal restriction must be placed on μ, thus making the existence of appropriate approximate solutions more difficult to prove.

We venture an opinion on the origin of the complexity of the hypothesis in [6] (and the corresponding greater complexity of the argument). Moser's argument uses Newton's method and a great deal of hard analysis to overcome the problem of 'loss of derivatives' due to the fact that, roughly speaking, if $g \in K$, $x \in H$, $F'(x)h = g$, $g_1 = F(x+h)$, then g_1 may not have approximation properties as good as those of g. Such problems seem present in other works on inverse function theorems (cf. [4], [5]) and seem to this writer as almost inevitable, when ordinary (that is discrete) Newton's method is used to find solutions to nonlinear systems.

The above comparison might strike the reader as arcane and complicated, a fair assesment in the opinion of this writer. To put things more in perspective, one might note that approximate solution conditions such as $M5.4$ or (15) together with (16) seem often to lead to actual solutions h to $F'(u)h = g$ (in a lower order Sobolev space to be sure) for some $u \in \Omega$ and a given element g (solutions h having a certain bound independent of the element $u \in \Omega$ giving rise to the problem). Such problems might in specific instances be best atacked directly, i.e., without considering approximate solutions in the 'higher' space V_r, limiting oneself to spaces having the role of V_ρ, V_l. The next paragraph gives some more speculation in this direction.

For some problems, given u, g, the linear problem of solving for h in

$$F'(u)h = g \tag{20}$$

is an elliptic linear partial differential equation. For such problems it is not uncommon that h has more derivatives than does g (the precise number more usually being the order of the problem). Furthermore, estimates on derivatives of h can be made in terms of estimates on derivatives of g. For problems, which are not elliptic the situation is not as straightforward, of course, but for the most part more derivatives for g yields more derivatives for the solution h. In fact it is not uncommon to be able to specify enough derivatives for g in order that there exists a solution h with desired bounds.

4. Relationship with Continuous Newton's Method

A feature of the present work is that it is motivated by *continuous* Newton's method as indicated in [2]. We indicate below how this leads to a way to avoid the 'loss of derivatives' phenomenom indicated above. Once $g \in K$ is chosen, *all* linear systems to be solved in the argument have g as right hand side. How does it happen that this is sufficient? For motivation we examine continuous Newton's method in case F above is C^1 and $F'(0)$ has a continuous inverse, which is defined on all of K. Continuous Newton's method in this case takes the form of finding a function z on $[0,\infty)$ so that

$$z(0) = 0, \quad z'(t) = -(F'(z(t))^{-1}(F(z(t)) - g), \quad t \geq 0. \tag{21}$$

Rewrite (21) as

$$z(0) = 0, \quad F'(z)z' = -(F(z) - g)\,,$$

and observe that this in turn may be rewritten

$$z(0) = 0,\ (F(z) - g)' = -(F(z) - g).$$

This last problem has the solution

$$(F(z(t)) - g) = e^{-t}(F(z(0)) - g) = -e^{-t}g, \quad t \geq 0.$$

This suggests rewriting (21) as

$$z(0) = 0,\ z'(t) = F'(z(t))^{-1}g, \quad t \in [0,1), \tag{22}$$

with a change of scale from $[0,\infty)$ to $[0,1)$. Consideration on how (22) might be discretized led to inspiration for the present note. To make our scheme work we only needed to have, given x in the interior of B_r, solutions h to

$$F'(x)h = g\,,$$

for the fixed element $g \in K$, which is the target for F itself. An iteration suggested by the above is

$$x \;\rightarrow\; x + th, \text{ where } F'(x)h = g.$$

This iteration was very much in mind during the development of Theorems 1 and 2. One can use the idea of this iteration to give a somewhat constructive argument for Theorem 2 and in fact find a function $z : [0, r/M) \rightarrow H$, continuous as a function from $[0, r/M) \rightarrow J$ so that

$$F(z(t)) = tg, \quad t \in [0, r/M).$$

References [8], [9] gives some more perspective on continuous Newton's method. In the setting of [9] it is shown that the domains of attraction for continuous Newton's method are not fractal, whereas this is not the case for conventional Newton's method.

5. Theorem 1 vs. Theorem 2

As indicated, Theorem 2 follows from Theorem 1. One purpose of the added generality of the first theorem is the following: In cases, in which H, J, K are function spaces, for a given x, ϵ in Theorem 1, $h \in H$ might be chosen as a member of a finite element subspace of H, which is defined on an appropriate grid. For a smaller choice of ϵ, an approximation on a finer grid might be needed to give the main inequality in the theorem. It may be that the added generality of the first theorem over the second opens a small window of opportunity for applications.

6. Late Added Note

M.G. Crandall [3] has found a version of Theorem 2, in which the hypothesis of weak continuity of F on H replaces the hypothesis of F being continuous on J, the space, in which H is compactly embedded. He offers an alternate argument for Theorem 2, which is simplier in some substantial ways.

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