EXISTENCES, ASYMPTOTIC BEHAVIORS AND SYMMETRIC PROPERTIES OF SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS IN FLAT FLASK DOMAINS

Huei-Li Lin, Hwai-Chiuan Wang

Department of Mathematics
National Tsing Hua University
Hsinchu, Taiwan 30013, R.O.C.
1e-mail: hwang@mail.math.hthv.edu.tw

Abstract: We assert that there exists a ground state solution of equation (1) in a flat flask domain: the Esteban-Lions domain $S_0$ by adding an arbitrary small width but sufficient long corridor. We also establish the asymptotic behavior and the symmetry of solutions of equation (1) in flat flask domains.

AMS Subject Classification: 35J20, 35J25
Key Words: semilinear elliptic equation, asymptotic behavior, symmetry, flat flask domain

1. Introduction

Consider the semilinear elliptic equation

$$\begin{cases} -\Delta u + u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases}$$

(1)

where $\Omega$ is an unbounded domain in $\mathbb{R}^N$, $N \geq 2$, and the nonlinear function $f : \mathbb{R}^+ \to \mathbb{R}$ is of class $C^1$ and satisfies the following conditions:

(f1) $f(\xi) = o(\xi)$ near $\xi = 0$. 

Received: December 5, 2002

© 2003, Academic Publications Ltd.

Correspondence author
for some $1 < s < 2^* - 1$.

(f3) For some $2 < \theta \leq s + 1$ we have

$$0 \leq \theta F(\xi) < f(\xi) \xi \quad \text{for all } \xi > 0,$$

where

$$F(\xi) = \int_0^\xi f(\tau) d\tau.$$

(f4) The function

$$\xi \to \frac{f(\xi)}{\xi}$$

is strictly increasing.

We extend $f$ to be 0 for $\xi < 0$. A typical example of the nonlinearity is

$$f(\xi) = \sum_{i=1}^k a_i \xi^{p_i},$$

where $a_i > 0$ and $1 < p_i < 2^* - 1$ for each $i$. Associated with the equation (1), we consider the Sobolev space $H^1_0(\Omega)$, the potential operators $a, b_f : H^1_0(\Omega) \to \mathbb{R}$, and the energy functional $I : H^1_0(\Omega) \to \mathbb{R}$, which are given by

$$a(u) = \int_\Omega (|\nabla u|^2 + u^2), \quad b_f(u) = \int_\Omega f(u)u,$$

$$I(u) = \frac{1}{2} a(u) - \int_\Omega F(u).$$

It is known that $I$ is of $C^{1,1}$ and $I$ satisfies the mountain pass hypothesis (see Rabinowitz [10], Proposition B.10 and Lemma 6 (i)). Note that every nonzero critical point of $I$ is a nonzero solution of equation (1).

Sobolev spaces $H^1_0(\Omega)$ provide the proper functional setting for the study of the partial differential equations and Sobolev imbedding theorems (Sobolev operators) provide the connection between Sobolev spaces and Lebesgue spaces.

(i) Standard books describe that the Sobolev critical operator $I : H^1_0(\Omega) \to L^{2^*}(\Omega)$ satisfies

$$\|u\|_{L^{2^*}} \leq c \|\nabla u\|_{L^2}.$$

Let $S^c(\Omega)$ be the best constant of the Sobolev critical operator defined as

$$S^c(\Omega) = \sup \left\{ \frac{\|u\|_{L^{2^*}}}{\|\nabla u\|_{L^2}} \mid u \in H^1_0(\Omega) \setminus \{0\} \right\}.$$

Then $S^c(\Omega)$ is independent of $\Omega$. $S^c(\Omega)$ is achieved if and only if $\Omega = \mathbb{R}^N$. 
(ii) Standard books also describe that the Sobolev subcritical operator $I : H^1_0(\Omega) \to L^p(\Omega)$ satisfies
\[ \|u\|_{L^p(\Omega)} \leq c\|u\|_{H^1(\Omega)}. \]

Let $S(\Omega)$ be the best constant of the Sobolev subcritical operator defined as
\[ S(\Omega) = \sup \left\{ \frac{\|u\|_{L^p(\Omega)}}{\|u\|_{H^1(\Omega)}} \mid u \in H^1_0(\Omega) \setminus \{0\} \right\}. \]

**Open Question.** For which domain $\Omega$, $S(\Omega)$ is achieved?

For the convenience, consider the following definition:

**Definition 1.** We call that a domain $\Omega$ in $\mathbb{R}^N$ is an achieved domain if there is $u \in H^1_0(\Omega)$ such that $\frac{\|u\|_{L^p(\Omega)}}{\|u\|_{H^1(\Omega)}} = S(\Omega)$. Otherwise, we call that $\Omega$ is a nonachieved domain.

A bounded domain is achieved. For the unbounded domains, due to the lack of compactness, to solve the open question is very difficult. For convenience, we give the following definition.

**Definition 2.** A proper unbounded domain $\Omega$ in $\mathbb{R}^N$ is called an Esteban-Lions domain if there is $\chi \in \mathbb{R}^N$, $\|\chi\| = 1$ such that $n(z) : \chi \geq 0$ on $\partial \Omega$, where $n(z)$ denotes the unit outward normal to $\partial \Omega$ at the point $z$.

A typical example of an Esteban-Lions domain is the upper semi-strip domain $S_r^0$. Esteban-Lions [5] asserted that there does not exist any solutions in $H^1_0(\Omega)$ for the equation (1) in an Esteban-Lions domain $\Omega$. In particular, there does not exist any solutions in $H^1_0(S_r^0)$ for equation (1) in $S_r^0 : S_r^0$ is a nonachieved domain. Berestycki conjectured that there exists a solution of equation (1) in $S_r^0$ with a hole. Lien-Tzeng-Wang [8] asserted that there exists a ground state solution of equation (1) in $S^r$ and Chen-Chen-Wang [4] asserted that every solution $u(x, y)$ of equation (1) is radially symmetric in $x$ and axially symmetric in $y$; that is, $u(x, y - \sigma) = u(|x|, |y - \sigma|)$ for some $\sigma$. Using these results, Wang [12] asserted the Berestycki conjecture.
To describe the existence of solutions in equation (1) in a general domain is extremely difficult. We start with equation (1) in the domains, which are the perturbations of the Esteban-Lions domain $S_r^0$ by adding a domain to it.

For $r > 0$, $R > 0$, denote by $B^N(z_0; R)$ the $N$-ball, $S^r$ the strip domain, $S_r^0$ the upper semi-strip domain, and the flat flask domain $\Omega_\varepsilon$ with a small width but sufficiently long corridor, where $\delta > 0$ and $\varepsilon > 0$ are two small numbers, as follows:

\[
B^N(z_0; R) = \{ z \in \mathbb{R}^N \mid |z - z_0| < R \},
\]

\[
B_+(0; r + \delta) = \{ (x, y) \in B^N(0; r + \delta) \mid y > 0 \},
\]

\[
B_-(0; r + \delta) = \{ (x, y) \in B^N(0; r + \delta) \mid y < 0 \},
\]

\[
S^r = \{ (x, y) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid |x| < r \},
\]

\[
S^r_0 = \{ (x, y) \in S^r \mid 0 < y \},
\]

\[
E_\varepsilon = B_+(0; r + \delta) \cup \{ (x, y) \in \mathbb{R}^N \mid (x, \varepsilon y) \in B_-(0; r + \delta) \},
\]

\[
\Omega_\varepsilon = S^r_0 \cup E_\varepsilon.
\]

In this article we assert that there exists an $\varepsilon_0 > 0$, such that if $\varepsilon \leq \varepsilon_0$, then the flat flask domain $\Omega_\varepsilon$ is achieved. Furthermore, we assert that the asymptotic behavior of the solutions of equation (1) in $\Omega_\varepsilon$ and the symmetry of solutions of equation (1) in $\Omega_\varepsilon$.

2. Palais-Smale Values

In this section, we study systematically the $(PS)$-values and the index of a domain $\Omega$. We obtain the index comparison criterion (see Theorem 13.) We use this criterion to prove our existence results in next section.

The nonlinearity $f$ satisfies the followings properties.

**Lemma 3.** Assume that $f$ satisfies $(f1)-(f3)$, then:

(f5) $|f(\xi)| \leq \varepsilon |\xi| + c_1 |\xi|^s$ for any $\xi$ and $1 < s < 2^* - 1$,

(f6) $F(\xi) \geq c(r)\xi^\theta$ for $\xi \geq r > 0$, where $2 < \theta \leq s + 1$,

(f7) $F(\xi) \geq c_2\xi^\theta - c_2^2\xi^2$ for any $\xi > 0$, where $c_2 = F(1) > 0$.

**Proof.** (f5) Given $\varepsilon > 0$. By (f1), there exists $\delta > 0$ such that $|\xi| < \delta$ implies $|f(\xi)| \leq \varepsilon |\xi|$. By (f2), there exists $N > 0$ such that $|\xi| > N$ implies $|f(\xi)| \leq \varepsilon |\xi|^s$. For $\xi \in [\delta, N]$, since $f(\xi)$ is continuous on $[\delta, N]$, then there exist $c' > 0$, $c'' > 0$ such that for $\xi \in [\delta, N]$

$$|f(\xi)| \leq c' = c'' \delta^s \leq c'' |\xi|^s.$$
Thus, \(|f(\xi)| \leq \varepsilon |\xi| + c_1 |\xi|^s\) for any \(\xi\).

(f6) By (f3), we have
\[
\left( \frac{F(\xi)}{\xi^\theta} \right)' = \frac{f(\xi)\xi^\theta - F(\xi)\theta \xi^{\theta-1}}{\xi^{2\theta}} = \frac{\xi^{\theta-1} [f(\xi)\xi - \theta F(\xi)]}{\xi^{2\theta}} > 0.
\]

Therefore \(\frac{F(\xi)}{\xi^\theta}\) is a strictly increasing function: there exists \(r > 0\) such that \(\xi \geq r > 0\) implies \(F(\xi)\xi - \theta F(\xi) > 0\).

Therefore \(F(\xi) \geq c_2 \xi^\theta - c_2 \xi^2\). For \(0 < \xi < 1\), we have
\[
c_2 \xi^\theta - c_2 \xi^2 = c_2 \xi^2 \left( \xi^{\theta-2} - 1 \right) \leq 0 \leq F(\xi).
\]

Hence \(F(\xi) \geq c_2 \xi^\theta - c_2 \xi^2\) for any \(\xi > 0\), where \(c_2 = F(1) > 0\).

In the following definitions, we simply denote Palais-Smale by \((PS)\).

**Definition 4.** (i) For \(\beta \in \mathbb{R}\), a sequence \(\{u_n\}\) in \(H^1_0(\Omega)\) is a \((PS)_\beta\)-sequence for \(I\) if \(I(u_n) = \beta + o(1)\) and \(I'(u_n) = o(1)\) strongly in \(H^{-1}(\Omega)\) as \(n \to \infty\);

(ii) \(\beta \in \mathbb{R}\) is a \((PS)_\beta\)-value for \(I\) if there is a \((PS)_\beta\)-sequence for \(I\),

(iii) \(I\) satisfies the \((PS)_\beta\)-condition if every \((PS)_\beta\)-sequence for \(I\) contains a convergent subsequence,

(iv) \(I\) satisfies the \((PS)\)-condition if, for every \(\beta \in \mathbb{R}\), every \((PS)_\beta\)-sequence for \(I\) contains a convergent subsequence.

A \((PS)_\beta\)-sequence for \(I\) admits a weak limit.

**Lemma 5.** Let \(\{u_n\}\) in \(H^1_0(\Omega)\) be a \((PS)_\beta\)-sequence for \(I\), then we have the following results,

(i) There exists a \(c > 0\) such that \(\|u_n\|_{H^1} \leq c\) for each \(n\),

(ii) \(a(u_n) = b_f(u_n) + o(1)\) and \(\beta \geq 0\),
(iii) If $\beta > 0$, then there exist a subsequence $\{u_n\}$ and $c' > 0$ such that $\|u_n\|_{H^1} \geq c'$ for each $n$.

Proof. (i) Let $2 < \theta \leq s + 1$, $1 < s < 2^* - 1$. By (f3), for large $n$, 

$\varepsilon_n = \|I'(u_n)\|,$

$|\beta| + 1 + \frac{\varepsilon_n\|u_n\|}{\theta} \geq I(u_n) - \frac{1}{\theta}\langle I'(u_n), u_n \rangle = \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|_{H^1}^2$

$- \int_{\Omega} [F(u_n) - \frac{1}{\theta}f(u_n)u_n] \geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|_{H^1}^2. \tag{2}$

Thus, $\|u_n\|_{H^1} \leq c$ for each $n$.

(ii) Since $\{u_n\}$ is bounded in $H^1_0(\Omega)$, we have $o(1) = \langle I'(u_n), u_n \rangle = a(u_n) - b_f(u_n)$. By (f3), we have

$\beta = \frac{1}{2}a(u_n) - \int_{\Omega} F(u_n) + o(1) \geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_{\Omega} f(u_n)u_n + o(1) \geq 0.$

(iii) Suppose that $\beta > 0$. If $\lim_{n \to \infty} \|u_n\|_{H^1} = 0$, by (f3) and (ii), then $I(u_n) = o(1)$, which is a contradiction. Thus, there exist a subsequence $\{u_n\}$ and $c' > 0$ such that $\|u_n\|_{H^1} \geq c'$ for each $n$. \hfill \Box

A $(PS)_{\beta}$-sequence for $I$ admits a weak limit, which is a critical point for $I$.

Consider the Nehari minimization problem $\alpha_M$, and the minimax problem $\alpha_{\Gamma}$, where

$\Gamma = \{ \gamma \in C([0,1], H^1_0(\Omega)) \mid I(\gamma(0)) = 0, I(\gamma(1)) < 0 \},$

$\alpha_{\Gamma} = \inf_{\gamma \in \Gamma} \sup_{\tau \in [0,1]} I(\gamma(\tau)),$

$M = \{ u \in H^1_0(\Omega) \setminus \{0\} \mid a(u) = b_f(u) \},$

$\alpha_M = \inf_{u \in M} I(u).$

Let $K(u) = \langle I'(u), u \rangle = a(u) - b_f(u)$, then $K(u)$ enjoys several properties.

Lemma 6. (i) There exist $c, \delta > 0$ such that for $\|u\|_{H^1} = \delta$, we have $K(u) \geq c\delta^2$, and $I(u) \geq \frac{1}{2}K(u) \geq \frac{c\delta^2}{2}$.

(ii) For $\gamma \in \Gamma$, there exists a $\tau_0 \in (0,1]$ such that $\|\gamma(\tau_0)\|_{H^1} = \delta$ and $K(\gamma(\tau_0)) > 0$.

(iii) If $\gamma(\tau) \notin M$ for each $\tau \in (0,1]$, then $K(\gamma(\tau)) > 0$ for each $\tau \in (0,1]$. 
Proof. (i) By \((f5)\) we have 
\[ f(\xi) \leq \epsilon |\xi| + c_1 |\xi|^s, \text{ for any } \xi > 0. \]
Then by the Sobolev continuous imbedding, we obtain 
\[
K(u) \geq \|u\|_{H^1}^2 - \epsilon \|u\|_{H^1}^2 - c_2 \|u\|_{H^1}^{s+1} \\
= \|u\|_{H^1}^2 - \left( \epsilon + c_2 \|u\|_{H^1}^{s-1} \right) \|u\|_{H^1}^2 \\
= \left[1 - \left( \epsilon + c_2 \|u\|_{H^1}^{s-1} \right) \right] \|u\|_{H^1}^2 \\
\geq c\delta^2,
\]
where \(\|u\|_{H^1} = \delta\) such that \(1 - \left( \epsilon + c_2 \|u\|_{H^1}^{s-1} \right) \geq c\) for some constant \(c > 0\).

By \((f3)\) we have 
\[
I(u) = \frac{1}{2} \|u\|_{H^1}^2 - \int_{\Omega} F(u) \\
\geq \frac{1}{2} \|u\|_{H^1}^2 - \frac{1}{\theta} \int_{\Omega} f(u) u \\
\geq \frac{1}{2} K(u) \geq \frac{c}{2} \delta^2.
\]

(ii) For \(\gamma \in \Gamma\), let 
\[ U_1 = \{ \tau \in [0, 1] \mid \|\gamma(\tau)\|_{H^1} < \delta \}, \]
and 
\[ U_2 = \{ \tau \in [0, 1] \mid \|\gamma(\tau)\|_{H^1} > \delta \}. \]
Then \(U_1, U_2\) are open in \([0, 1]\). Since \(\gamma(0) = 0\) and \(I(\gamma(1)) < 0\), then by part (i), \(U_1, U_2\) are nonempty. Thus, \([0, 1]\) is disconnected, a contradiction. Hence there exists a \(\tau_0 \in (0, 1)\) such that \(\|\gamma(\tau_0)\|_{H^1} = \delta\) and \(K(\gamma(\tau_0)) > 0\).

(iii) Since \(\gamma \in C([0, 1], H^1_0(\Omega))\) and \(\gamma(0) = 0\), by part (ii) there exists a \(\tau_0 \in (0, 1)\) such that \(0 < \|\gamma(\tau_0)\|_{H^1} = \delta\) and \(K(\gamma(\tau_0)) > 0\). Since \(K\) is continuous, then \(K \circ \gamma\) is still continuous. Assume, by contradiction, there exists a \(\tau_1 \in (0, 1)\) such that \(K(\gamma(\tau_1)) < 0\), then we get a \(\tau_2 \in [\tau_0, \tau_1] \subset (0, 1)\) such that \(K(\gamma(\tau_2)) = 0\), that is \(\gamma(\tau_2) \in M\). This contradicts the fact that \(\gamma(\tau) \notin M\) for all \(\tau \in (0, 1]\). Hence, \(K(\gamma(\tau)) > 0\) for each \(\tau \in (0, 1]\). \(\square\)

To examine that several important \((PS)\)-values are the same, we should understand the growth condition of the energy functional \(I\). Let \(u \in H^1_0(\Omega) \setminus \{0\}\)
and \( E_+ = \{ z \in \Omega \mid u(z) > 0 \} \). If \(|E_+| = 0\), then clearly we have \( I(tu) = \frac{t^2}{2} a(u) \to \infty \) as \( t \to \infty \). Note that

\[
I(tu) = \frac{t^2}{2} a(u) - \int_{\Omega} F(tu),
\]

\[
\frac{d}{dt} I(tu) = ta(u) - \int_{\Omega} f(tu) u,
\]

\[
\frac{d^2}{dt^2} I(tu) = a(u) - \int_{\Omega} f'(tu) u^2.
\]

**Lemma 7.** For \( u \in H^1_0(\Omega) \backslash \{0\} \) such that \(|E_+| > 0\). Let

\[
\varphi(u) = \min \left\{ t > 0 \mid \frac{d}{dt} I(tu) = 0 \right\},
\]

then we have the following results:

(i) \( I(tu) \to -\infty \) as \( t \to \infty \),

(ii) \( 0 < \varphi(u) < \infty \),

(iii) \( \frac{d^2}{dt^2} I(tu) < 0 \) for \( t > \varphi(u) \),

(iv) \( I \) has exact one critical point \( \varphi(u)u \),

(v) \( \frac{d}{dt} I(tu) \geq 0 \) for \( t \leq \varphi(u) \).

**Proof.** (i) Since \(|E_+| > 0\), there exists \( n_0 > 0 \) such that \(|A| > 0\), where \( A = \left\{ z \in E_+ \mid u(z) > \frac{1}{n_0} \right\} \). By \((f6)\), for \( z \in A \) and \( \frac{1}{n_0} \geq r \), we have \( F(tu) \geq c(r)(tu)^\theta \). Now

\[
I(tu) \leq \frac{t^2}{2} a(u) - \int_A F(tu) \\
\leq \frac{t^2}{2} a(u) - c(r)t^\theta \int_A u^\theta \\
\leq t^2 \left\{ \frac{1}{2} a(u) - c(r)t^{\theta-2} \int_A u^\theta \right\} \to -\infty \text{ as } t \to \infty.
\]

(ii) By the definition of \( \varphi(u) \) and part (i).

(iii) By \((f4)\), for \( u^+ = \max \{u,0\} \), we have \( \left( \frac{f(tu^+)}{tu^+} \right)' > 0 \), that is,
$f'(tu^+)u^{+2} > \frac{f((tu^+)u^+)}{t}$, then

$$\frac{d^2}{dt^2} I(tu) = \alpha(u) - \int_\Omega f'(tu)u^2 = \int_\Omega \frac{f(\varphi(u)u^+)}{\varphi(u)} - \int_\Omega f'(tu^+)u^{+2}$$

$$< \int_\Omega \frac{f(\varphi(u)u^+)}{\varphi(u)} - \frac{f((tu^+)u^+)}{tu^+}$$

$$= \int_\Omega u^{+2} \left( \frac{f(\varphi(u)u^+)}{\varphi(u)u^+} - \frac{f((tu^+)u^+)}{tu^+} \right)$$

$$< 0 \text{ for } t > \varphi(u).$$

(iv) By the definition of $\varphi(u)$, $I(0) = 0$, part (ii) and (iii).

(v) By the definition of $\varphi(u)$ and part (iv).

Remark 1. By Lemma 7, we have:

(i) $M = \{ v = \varphi(u)u \mid u \in H^1_0(\Omega), |E_+| > 0 \}$,

(ii) $\alpha_M = \inf_{u \in H^1_0} \sup_{t \geq 0} I(tu) = \inf_{u \in H^1_0} \sup_{|E_+| > 0} I(tu) = \inf_{u \neq 0} \sup_{|E_+| > 0} I(tu)$.

Brezis-Nirenberg [3] asserted that $\alpha_\Gamma$ is a $(PS)$-value and Stuart [11] asserted that $\alpha_M$ is a $(PS)$-value. These two values are the same.

Theorem 8. If $(f1) - (f4)$ hold, then

$$\alpha_\Gamma = \alpha_M.$$ 

Proof. For $u \in H^1_0(\Omega) \setminus \{0\}$, by Lemma 7, $I$ has exactly one critical point $\varphi(u)u$ and $\frac{d^2}{dt^2} I(tu) < 0$ for $t > \varphi(u)$. We claim that $\alpha_M \geq \alpha_\Gamma$. In fact, for $u \in H^1_0(\Omega) \setminus \{0\}$ and $|E_+| > 0$, there exists $t_1 > 0$ such that $I(t_1u) < 0$. For $\varepsilon > 0$, there exists $u \in H^1_0(\Omega) \setminus \{0\} \text{ and } |E_+| > 0$ such that

$$\alpha_M + \varepsilon > \sup_{t \geq 0} I(tu) = I(\varphi(u)u) = \sup_{\tau \in [0,1]} I(\gamma(\tau)) \geq \alpha_\Gamma,$$

where $\gamma: [0,1] \to H^1_0(\Omega)$ is continuous with $\gamma(\tau) = \tau t_1 u$. Thus, $\alpha_M \geq \alpha_\Gamma$.

On the other hand, for $\gamma \in \Gamma$ there exists $\tau \in (0,1]$ such that $\gamma(\tau) \in M$. Otherwise, then $\gamma(\tau) \notin M$ for all $\tau \in (0,1]$. Therefore, by Lemma 6 (iii), we
have $K(\gamma(\tau)) > 0$. Then, using (f3) to obtain

$$I(\gamma(\tau)) = \frac{1}{2} \int_{\Omega} |\nabla \gamma(\tau)|^2 + \gamma(\tau)^2 - \int_{\Omega} F(\gamma(\tau))$$

$$> \frac{1}{2} \int_{\Omega} f(\gamma(\tau)) \gamma(\tau) - \int_{\Omega} F(\gamma(\tau))$$

$$\geq \int_{\Omega} F(\gamma(\tau)) - \int_{\Omega} F(\gamma(\tau))$$

$$\geq 0 \quad \text{for all } \tau \in [0,1].$$

This contradicts the definition of $\Gamma$. For $\gamma \in \Gamma$ there exists $\tau_0 \in (0,1]$ such that $\gamma(\tau_0) \in M$, then $\alpha_\Gamma + \varepsilon > \sup_{\tau \in [0,1]} I(\gamma(\tau))$ and thus $\alpha_\Gamma \geq \alpha_M$. Then we conclude that $\alpha_\Gamma = \alpha_M$. \qed

**Definition 9.** By Theorem 8, we conclude that the numbers $\alpha_\Gamma$ and $\alpha_M$ are the same. Any one of them is called the index of $I$ in $\Omega$ and denoted by $\alpha(\Omega)$. We call that a solution $u$ of equation (1) is a ground state solution if $I(u) = \alpha(\Omega)$, and is a higher energy solution if $I(u) > \alpha(\Omega)$.

**Remark 2.** It is easy to see that a domain $\Omega$ is achieved if and only if there is a ground state solution of equation (1) in $\Omega$.

The energy of each nonzero solution of equation (1) admits a lower bound.

**Theorem 10.** (i) Let $\{u_n\}$ in $H_0^1(\Omega)$ be a $(PS)_{\alpha(\Omega)}$-sequence for $I$ and $u$ in $H_0^1(\Omega)$ satisfying $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$. Then, $u$ is a weak solution of equation (1).

(ii) Let $\{u_n\}$ in $H_1^1(\Omega)$ be a $(PS)_{\alpha(\Omega)}$-sequence for $I$ and $u$ in $H_0^1(\Omega)$ satisfying $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $u$ is nonzero. Then $u$ is a positive ground solution of equation (1).

**Proof.** (i) Since $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$, then for each $\phi \in C_c^\infty(\Omega)$

$$\int_{\Omega} \nabla u_n \nabla \phi + u_n \phi \rightarrow \int_{\Omega} \nabla u \nabla \phi + u \phi.$$ 

By (f5), $u_n \rightarrow u$ strongly in $L^q_{\text{loc}}(\Omega)$ and a.e. for $1 \leq q < 2^*$, we obtain

$$\int_{\Omega} f(u_n) \phi \rightarrow \int_{\Omega} f(u) \phi.$$ 

Thus,

$$\langle I'(u), \phi \rangle = \int_{\Omega} \nabla u \nabla \phi + u \phi - \int_{\Omega} f(u) \phi = \lim_{n \rightarrow \infty} (\int_{\Omega} \nabla u_n \nabla \phi + u_n \phi - \int_{\Omega} f(u_n) \phi) = \lim_{n \rightarrow \infty} \langle I'(u_n), \phi \rangle = 0.$$
(ii) By (i), \( u \) is a nonzero solution of equation (1). Thus, \( u \in M \) and \( I(u) \geq \alpha(\Omega) \). Since \( I \) is weakly lower semicontinuous, we have

\[ \alpha(\Omega) \leq I(u) \leq \liminf_{n \to \infty} I(u_n) = \alpha(\Omega). \]

Let \( u^- = \max \{-u, 0\} \). Multiply equation (1) by \( u^- \) and integrate to obtain

\[ \int_{\Omega} \nabla u \nabla u^- + uu^- = \int_{\Omega} f(u) u^- . \]

Consequently, \( \int_{\Omega} |\nabla u^-|^2 + |u^-|^2 = 0 \), that is, \( u^- = 0 \). By the maximum principle, \( u \) is a positive ground state solution of equation (1). \( \square \)

**Remark 3.** If the Nehari minimization problem \( \alpha_M \) or the minimax problem \( \alpha_G \) admits a nonzero solution \( u \), then \( u \) must be a ground state solution.

If \( I \) satisfies the \((PS)_{\alpha(\Omega)}\)-condition, then the Nehari minimization problem \( \alpha_M \) and the minimax problem \( \alpha_G \) admit a ground state solution.

Let \( \{u_n\} \) be a \((PS)_{\beta}\)-sequence for \( I \), where \( \beta > 0 \), and \( t_n u_n \in M \) for \( t_n = \varphi(u_n) > 0 \), then \( \{||t_n u_n||_{H^1}\} \) is bounded above and bounded away from zero.

**Lemma 11.** Let \( \{u_n\} \) in \( H^1_0(\Omega) \) be a \((PS)_{\beta}\)-sequence for \( I \), where \( \beta > 0 \) and \( t_n u_n \in M \) for \( t_n > 0 \), then there exist a subsequence \( \{u_{n_k}\} \) and a subsequence \( \{t_k\} \) such that \( \epsilon' \leq t_k \leq \epsilon'' \) and \( C' \leq ||t_k u_{n_k}||_{H^1} \leq C'' \) for each \( n \in \mathbb{N} \), where \( \epsilon', \epsilon'', C' \) and \( C'' > 0 \).

**Proof.** (i) Claim \( t_k \leq \epsilon'' \). By Lemma 5, \( \{u_n\} \) is bounded, say \( ||u_n||_{H^1} \leq c_1 \).

\[
I(u_n) = \frac{1}{2} a(u_n) - \int_{\Omega} F(u_n) = \beta + o(1),
\]

\[
< I'(u_n), u_n > = a(u_n) - \int_{\Omega} f(u_n) u_n = o(1).
\]

Suppose \( \int_{\Omega} f(u_n) u_n = o(1) \), then \( a(u_n) = o(1) \) and

\[
\int_{\Omega} F(u_n) \leq \frac{1}{\theta} \int_{\Omega} f(u_n) u_n = o(1).
\]

Thus, \( I(u_n) = o(1) \), a contradiction. Therefore there exist a subsequence \( \{u_{n_k}\} \) and \( l > 0 \) such that

\[
l \leq \int_{\Omega} f(u_n) u_n \leq \varepsilon \int_{\Omega} |u_n|^2 + c \int_{\Omega} |u_n|^{s+1} \leq \varepsilon c_1^2 + c \int_{\Omega} |u_n|^{s+1}.
\]
Let $\varepsilon c_2^2 \leq \frac{1}{2}$ to obtain $\int_\Omega |u_n|^{s+1} \geq c_2 > 0$. Since $\theta \leq s + 1 < 2^*$, let $s + 1 = (1 - \xi) \theta + \xi 2^*$, $0 \leq \xi < 1$, then

\[
c_2 \leq \int_\Omega |u_n|^{s+1} = \int_\Omega |u_n|^{(1-\xi)\theta} |u_n|^{\xi 2^*}
\]

\[
\leq \left( \int_\Omega |u_n|^{\theta} \right)^{1-\xi} \left( \int_\Omega |u_n|^{2^*} \right)^{\xi} \leq c_3 \left( \int_\Omega |u_n|^{\theta} \right)^{1-\xi}.
\]

Thus, $\int_\Omega |u_n|^{\theta} \geq c_4 > 0$. Hence by Lemma 3, (f7) and Lemma 5, we have

\[
0 \leq I(t_n u_n) \leq \frac{c_6}{2} a(u_n) + c_5 \int_\Omega |t_n u_n|^2 - c_5 \int_\Omega |t_n u_n|^\theta
\]

\[
\leq c_6 t_n^2 - c_7 t_n^\theta,
\]

where $c_5, c_6, c_7 > 0$.

Thus, there is a $c'' > 0$ such that

\[
t_n \leq c'' \text{ for each } n \in \mathbb{N}.
\]

(ii) Claim $c' \leq t_n$. By Lemma 6 (i), there exist $c, \delta > 0$ such that for $\|u\|_{H^1} = \delta$, we have $I(u) \geq \frac{c_2^2}{2}$. Since $I(0) = 0$ and $I$ is continuous, we have $I(u) < \frac{\varepsilon c_2^2}{2}$ for $\|u\|_{H^1} < \delta_1 < \delta$. By Lemma 5, we have $0 < c_9 \leq \|u_n\|_{H^1} \leq c_{10}$ for each $n \in \mathbb{N}$. Since $I$ is achieved the maximum at $t_n u_n$ for each $n \in \mathbb{N}$, then

\[
t_n c_{10} \geq t_n \|u_n\|_{H^1} = \|t_n u_n\|_{H^1} \geq \delta_1 \text{ for each } n \in \mathbb{N}.
\]

Therefore there is $c' > 0$ such that $t_n \geq c'$. Hence, by part (i) and (ii), we have $c' \leq t_n \leq c''$ for each $n \in \mathbb{N}$.

By Theorem 10, a $(PS)_{\alpha(\Omega)}$-sequence for $I$ admits a weak limit $u$, which is a critical point for $I$. However, we should do more effort to assert that $u$ is nonzero, that is, $u$ is a positive solution of equation (1). From now on $\alpha(\Omega)$ is simply denoted by $\alpha$. Let

\[
\Omega_n = \Omega \cap B^N(0; n),
\]

\[
\tilde{\Omega}_n = \Omega \setminus B^N(0; n),
\]

\[
M_n = \{ u \in H^1_0(\Omega_n) \setminus \{0\} \mid a(u) = b_f(u) \},
\]

\[
\tilde{M}_n = \{ u \in H^1_0(\tilde{\Omega}_n) \setminus \{0\} \mid a(u) = b_f(u) \},
\]

\[
\alpha_n = \alpha(\Omega_n) = \inf_{u \in M_n} I(u),
\]

\[
\tilde{\alpha}_n = \alpha(\tilde{\Omega}_n) = \inf_{u \in \tilde{M}_n} I(u).
\]

We shall see what will happen whenever $u$ is zero.
Theorem 12. Let \( \{u_n\} \) in \( H_0^1(\Omega) \) be a \((PS)_\alpha\)-sequence for \( I \).

(i) Suppose that \( u_n \to 0 \) weakly in \( H_0^1(\Omega) \), then there is a subsequence \( \{u_n\} \) such that
\[
\int_{\Omega_n} f(u_n) u_n = o(1) \quad \text{and} \quad \int_{\Omega_n} F(u_n) = o(1);
\]

(ii) Suppose there is a subsequence \( \{u_n\} \) such that
\[
\int_{\Omega_n} f(u_n) u_n = o(1),
\]
then we have \( \alpha = \tilde{\alpha}_n \) for each \( n \in \mathbb{N} \).

(iii) Suppose that \( u_n \to 0 \) weakly in \( H_0^1(\Omega) \), then \( \alpha = \tilde{\alpha}_n \) for each \( n \in \mathbb{N} \).

Proof. (i) Since \( u_n \to 0 \) weakly in \( H_0^1(\Omega) \), then \( u_n \to 0 \) strongly \( L^q_{\text{loc}}(\Omega) \), where \( 1 \leq q < 2^* \). Thus, for each \( m \in \mathbb{N} \), \( \lim_{n \to \infty} \int_{\Omega_n} |u_n|^p = 0 \). We can take a subsequence \( \{u_{m_n}\} \) such that \( \int_{\Omega_n} |u_{m_n}|^p < \frac{1}{m} \). Therefore there is a subsequence \( \{u_n\} \) such that \( \int_{\Omega_n} |u_n|^p = o(1) \). By (f3), and Lemma 3 (f5), we have
\[
\int_{\Omega_n} f(u_n) u_n = o(1) \quad \text{and} \quad \int_{\Omega_n} F(u_n) = o(1).
\]

(ii) Let \( \{u_n\} \) in \( H_0^1(\Omega) \) be a \((PS)_\alpha\)-sequence for \( I \) such that
\[
I(u_n) = \frac{1}{2} a(u_n) - \int_{\Omega} F(u_n) = \alpha + o(1),
\]
\[
a(u_n) = b f(u_n) + o(1).
\]
Suppose there is a subsequence \( \{u_n\} \) such that
\[
\int_{\Omega_n} f(u_n) u_n = o(1). \tag{3}
\]

Let \( \xi \in C^\infty([0, \infty)) \) such that
\[
0 \leq \xi \leq 1, \quad \xi(t) = \begin{cases} 0 & \text{for } t \in [0, 1], \\ 1 & \text{for } t \in [2, \infty). \end{cases}
\]

Let \( \xi_n(z) = \xi(\frac{2z}{n}) \). Since \( \{\xi_n^2 u_n\} \) is bounded in \( H_0^1(\Omega) \),
\[
o(1) = \langle I'(u_n), \xi_n^2 u_n \rangle \\
= \int_{\Omega} \langle \xi_n^2 \nabla u_n, \xi_n \nabla u_n \rangle + 2 \xi_n u_n \nabla \xi_n \cdot \nabla u_n + \xi_n^2 u_n^2 \rangle - \int_{\Omega} \xi_n^2 f(u_n) u_n.
\]
Note that $|\nabla \xi_n(z)| \leq \frac{\xi_n}{n}$ and by (3), we have

$$\int_{\Omega} \xi_n u_n \nabla \xi_n \cdot \nabla u_n = o(1),$$  

(4)

$$\int_{\Omega} \xi_n^2 f(u_n) u_n = \int_{\Omega} f(u_n) u_n + o(1) = a(u_n) + o(1),$$  

(5)

and

$$\int_{\Omega} F(\xi_n u_n) = \int_{\Omega} F(u_n) + o(1).$$  

(6)

We conclude that

$$\int_{\Omega} \xi_n^2 (|\nabla u_n|^2 + u_n^2) = a(u_n) + o(1).$$  

(7)

Let $v_n = \xi_n u_n$. By (3) – (7), we have

$$a(v_n) = \int_{\Omega} (|\nabla v_n|^2 + v_n^2) = \int_{\Omega} \left[|\nabla \xi_n|^2 u_n^2 + \xi_n^2 (|\nabla u_n|^2 + u_n^2) + 2 \xi_n u_n \nabla \xi_n \cdot \nabla u_n \right] = a(u_n) + o(1),$$

$$b_f(v_n) = \int_{\Omega} \xi_n f(\xi_n u_n) u_n = b_f(u_n) + o(1),$$

and

$$I(v_n) = \frac{1}{2} \int_{\Omega} (|\nabla v_n|^2 + v_n^2) - \int_{\Omega} F(v_n)$$

$$= \frac{1}{2} \int_{\Omega} \left[|\nabla \xi_n|^2 u_n^2 + \xi_n^2 (|\nabla u_n|^2 + u_n^2) + 2 \xi_n u_n \nabla \xi_n \cdot \nabla u_n \right]$$

$$- \int_{\Omega} F(v_n)$$

$$= \frac{1}{2} a(u_n) - \int_{\Omega} F(u_n) + o(1)$$

$$= \alpha + o(1).$$

Let $h_n(t) = I(tv_n)$ for each $n \in \mathbb{N}$ and $t \geq 0$, then $h_n'(t) = t \int_{\Omega} (|\nabla v_n|^2 + v_n^2) - \int_{\Omega} f(tv_n)v_n$ and $h_n'(1) = \langle I'(v_n), v_n \rangle = o(1)$. Claim $|h_n'(t)| \leq |h_n'(1)| + t \epsilon$ for any $\epsilon > 0$ and large $n$. Case 1: for $t_n \leq t \leq 1$, by Lemma 7, then $|h_n'(t)| \leq |h_n'(1)|$. Case 2: for $1 < t \leq t_n$, since $a(v_n) = b_f(v_n) + o(1)$, $h_n'(1) = o(1)$ and Lemma 7 (v), then for any $\epsilon > 0$ we have $0 \leq h_n'(1) = a(v_n) - b_f(v_n) < \epsilon$ for
large \( n \) and \( h'_n(t) \geq 0 \). Thus, by \((f4)\),

\[
h'_n(t) - h'_n(1) = t \int_\Omega (|\nabla v_n|^2 + v_n^2) - \int_\Omega f(t v_n) v_n - h'_n(1) < t \int_\Omega \frac{f(v_n)v_n^2}{v_n} + t \varepsilon - t \int_\Omega \frac{f(t v_n)v_n^2}{t v_n} = t \int_\Omega v_n^2 - \frac{f(v_n^+)}{v_n^+} - \frac{f(t v_n^+)}{t v_n^+} + t \varepsilon < t \varepsilon \text{ for large } n.\]

Therefore \( |h'_n(t)| \leq |h'_n(1)| + t \varepsilon \) for any \( \varepsilon > 0 \) and large \( n \). Let \( \{t_n\} \) be a sequence such that \( t_n v_n \in \tilde{M}_{2} \). Thus, by the mean value theorem, Cases 1 and 2 and from Lemma 11, a subsequence \( \{t_n\} \) is bounded, we have

\[
\lim_{n \to \infty} |I(v_n) - I(t_n v_n)| = \lim_{n \to \infty} |h_n(1) - h_n(t_n)| = \lim_{n \to \infty} |h'_n(\theta_n)| |1 - t_n|, \text{ where } \theta_n \text{ is between } 1 \text{ and } t_n \leq c \lim_{n \to \infty} |h'_n(1)| = 0.
\]

Hence \( I(t_n v_n) = I(v_n) + o(1) = \alpha + o(1) \). Since \( t_n v_n \in \tilde{M}_2 \), we have \( \tilde{\alpha}_n \leq I(t_n v_n) \) for each \( n \in \mathbb{N} \). Taking limit to obtain \( \lim_{n \to \infty} \tilde{\alpha}_n \leq \alpha \). Since \( \Omega \supset \tilde{\Omega}_n \supset \tilde{\Omega}_{n+1} \), we have \( \alpha \leq \tilde{\alpha}_n \leq \tilde{\alpha}_{n+1} \) for each \( n \in \mathbb{N} \). Then we conclude that \( \alpha = \tilde{\alpha}_n \) for each \( n \in \mathbb{N} \).

(iii) By part (i) and part (ii).

We have the index comparison criterion.

**Theorem 13.** If \( \alpha < \tilde{\alpha}_n \) for some \( n \in \mathbb{N} \), then there is a positive ground state solution of equation (1) in \( \Omega \).

**Proof.** Let \( \{u_n\} \in H^1_0(\Omega) \) be a \((PS)_{\alpha(\Omega)}\)-sequence for \( I \) and \( u \in H^1_0(\Omega) \) satisfying \( u_n \rightharpoonup u \) weakly in \( H^1_0(\Omega) \). By Theorem 12, \( u \) is nonzero. By Theorem 10 (ii), \( u \) is a positive ground state solution of equation (1).

3. Existence of Solutions

In this section, we apply Theorem 13 to prove the following existence results:

**Theorem 14.** There exists an \( \varepsilon_0 > 0 \), such that if \( \varepsilon \leq \varepsilon_0 \), then there exists a positive ground state solution of equation (1) in \( \Omega_\varepsilon \).
Proof. The decomposition Lemma 4.1 in Lien-Tzeng-Wang [8] can be modified to hold for the equation (1). As a consequence, we have that $\alpha(S^r)$ admits a minimizer. Since $S^r \subseteq S^{r+\delta}$ and $\alpha(S^r)$ admits a minimizer, by Lien-Tzeng-Wang [8], Lemma 2.7, we have $\alpha(S^{r+\delta}) < \alpha(S^r)$. Since $E_{\varepsilon} \subset S^{r+\delta}$ and $\lim_{\varepsilon \to 0} \alpha(E_{\varepsilon}) = \alpha(S^{r+\delta})$, there exists $\varepsilon_0 > 0$, such that if $\varepsilon \leq \varepsilon_0$, then $\alpha(E_{\varepsilon}) < \alpha(S^r)$. Fix $\varepsilon$, $\varepsilon \leq \varepsilon_0$, there exists a large $N \in \mathbb{N}$ such that

$$\alpha(\tilde{\Omega}_{\varepsilon,N}) = \alpha(S^r_N) = \alpha(S^r).$$

Thus,

$$\alpha(\Omega_{\varepsilon}) \leq \alpha(E_{\varepsilon}) < \alpha(S^r) = \alpha(\tilde{\Omega}_{\varepsilon,N}).$$

By Theorem 13, there exists a positive ground state solution $u$ of equation (1) in $\Omega_{\varepsilon}$. \hfill \Box

4. Asymptotic Behavior of Solutions

In this section, we present two asymptotic behavior of each solution of equation (1) in the flat flask domains $\Omega_{\varepsilon}$, where $\varepsilon \leq \varepsilon_0$: one is basic and the other is advanced.

**Theorem 15.** (Basic) If $u \in H^1_0(\Omega_{\varepsilon})$ is a weak solution of equation (1) in $\Omega_{\varepsilon}$, then we have the following results.

(i) $u \in L^q(\Omega_{\varepsilon})$ for all $q \in [2, \infty)$ such that $\|u\|_{L^q(\Omega_{\varepsilon})} \leq c_q p(\|u\|_{H^1(\Omega_{\varepsilon})})$, where $c_q$ is a constant dependent on $q$ and $p(\|u\|_{H^1(\Omega_{\varepsilon})})$ is a polynomial of $\|u\|_{H^1(\Omega_{\varepsilon})}$ in real power,

(ii) $u \in C^{1,\theta}(\Omega_{\varepsilon}) \cap W^{2,q'}(\Omega_{\varepsilon})$ for some $q' > N$ and $\theta = 2 - \frac{N}{q'} - \left[2 - \frac{N}{q}\right]$, such that $\|u\|_{L^\infty(\Omega_{\varepsilon})} \leq \|u\|_{C^{1,\theta}(\Omega_{\varepsilon})} \leq c_{q'} \|u\|_{W^{2,q'}(\Omega_{\varepsilon})} \leq c_{q'} p(\|u\|_{H^1(\Omega_{\varepsilon})})$,

(iii) $\lim_{y \to \infty} u(x, y) = 0$ uniformly in $x$, where $(x, y) \in \Omega_{\varepsilon}$,

(iv) $u \in C^{2,\theta}(\Omega_{\varepsilon})$.

**Proof.** The following estimate is essentially due to Brezis-Kato [2], based on Moser’s [9] iteration technique. We can not find this form in any literatures, so we prove, as follows.

(i) Let $\varphi = \varphi_{d, l} = u \min\{|u|^{2d}, l^2\}$, then $\nabla \varphi = \min\{|u|^{2d}, l^2\} \nabla u + \chi_{\{u \leq l\}} 2d |u|^{2d} \nabla u$. 


Suppose $u \in H^1_0(\Omega_c)$, we have
\[
\int_{\Omega_c} |\nabla \varphi|^2 = \int_{\Omega_c} \left| \min\{|u|^{2d}, l^2\} \nabla u + \chi_{\{|u|^{d} \leq l\}} 2d|u|^{2d} \nabla u \right|^2 \\
\leq 2 \left( \int_{\Omega_c} \left| \min\{|u|^{2d}, l^2\} \nabla u \right|^2 + \chi_{\{|u|^{d} \leq l\}} 2d|u|^{2d} \nabla u \right)^2 \\
= 2 \left( l^4 \int_{\{|u|^{d} \leq l\}} |\nabla u|^2 + l^4 \int_{\{|u|^{d} \geq l\}} |\nabla u|^2 \right) \\
+ 8d^2 l^4 \int_{\{|u|^{d} \leq l\}} |\nabla u|^2 \\
\leq c \int_{\Omega_c} |\nabla u|^2 < \infty.
\]
Thus, $\varphi \in H^1_0(\Omega_c)$. For $d \geq 0$, $l \geq 1$, we have
\[
\int_{\{|u|^{d} \leq l\}} \left| \nabla (|u|^{d+1}) \right|^2 \\
\leq \int_{\{|u|^{d} \leq l\}} \left| \nabla (|u|^{d+1}) \right|^2 + l^2 \int_{\{|u|^{d} \geq l\}} |\nabla u|^2 \\
= \int_{\Omega_c} \left| \nabla (u \min\{|u|^{d}, l\}) \right|^2 \\
= \int_{\Omega_c} \left| \min\{|u|^{d}, l\} \nabla u + \chi_{\{|u|^{d} \leq l\}} d |u|^{d} \nabla u \right|^2 \\
\leq 2 \int_{\Omega_c} |\nabla u|^2 \min\{|u|^{2d}, l^2\} \\
+ 2d^2 \int_{\{|u|^{d} \leq l\}} |\nabla u|^2 |u|^{2d}.
\]
Multiply equation (1) with $\varphi$ to obtain
\[
\int_{\Omega_c} \nabla u \cdot \nabla \varphi + \int_{\Omega_c} u \varphi = \int_{\Omega_c} f(u) \varphi.
\]
Thus,
\[
\int_{\Omega_c} |\nabla u|^2 \min\{|u|^{2d}, l^2\} + 2d \int_{\{|u|^{d} \leq l\}} |\nabla u|^2 |u|^{2d} \\
+ \int_{\Omega_c} u^2 \min\{|u|^{2d}, l^2\} \leq c \int_{\Omega_c} |u|^{2d+2} + \int_{\Omega_c} |u|^{2d+s+1}.
\]
Suppose $u \in L^{2d+s+1}(\Omega_c)$. Since $2 \leq 2d+2 < 2d+s+1$, write $\frac{1}{2d+2} = \frac{2}{2d} + \frac{1-\alpha}{2d+s+1}$, where $0 < \alpha \leq 1$. Then
\[
||u||_{L^{2d+2}(\Omega_c)} \leq ||u||_{L^2(\Omega_c)} ||u||_{L^{2d+s+1}(\Omega_c)}^{1-\alpha} \leq c ||u||_{L^{2d+s+1}(\Omega_c)}^{1-\alpha}.
\]
Then from \( (9) \) we have
\[
\int_{\Omega_{\varepsilon}} |\nabla u|^2 \min\{|u|^{2d}, l^2\} + 2d \int_{\{u|^{d} \leq l\}} |\nabla u|^2 |u|^{2d} \leq c_d p(||u||_{L^{2d+1}}),
\]
where \( p(||u||_{L^{2d+1}}) \) is a polynomial of \( ||u||_{L^{2d+1}} \). By \( (8) \) and \( (10) \) we have
\[
\int_{\{u|^{d} \leq l\}} |\nabla(|u|^{d+1})|^2 \leq c_d p(||u||_{L^{2d+1}}),
\]
where \( c_d \) is a constant dependent on \( d \). Let \( l \to \infty \) to obtain that
\[
\int_{\Omega_{\varepsilon}} |\nabla(|u|^{d+1})|^2 \leq c_d p(||u||_{L^{2d+1}}).
\]
By the Sobolev inequality, we have
\[
||u||_{L^{2(d+1)+2*}(\Omega_{\varepsilon})} = ||u|^{d+1}||_{L^{2*}(\Omega_{\varepsilon})} \leq c \left\| \nabla(|u|^{d+1}) \right\|_{L^2(\Omega_{\varepsilon})} \leq c_d p(||u||_{L^{2d+1}}).
\]
Thus,
\[
||u||_{L^{2d+1+2*}(\Omega_{\varepsilon})} \leq c_d p(||u||_{L^{2d+1}}).
\]
Now iterate, letting \( d_0 = 0 \) and \( 2d_i + s + 1 = (d_{i-1} + 1) \frac{2N}{s} \), for \( i \geq 1 \). Since \( s + 1 < 2^s \), then we have \( d_1 > 0 \) and \( d_i \geq (\frac{2^s}{2})^{t-1}d_1 \). Hence \( \lim_{i \to \infty} d_i = \infty \) and
\[
||u||_{L^{2d_i+2*+1}} = ||u||_{L^{(2d_{i-1}+1)+2*}} \leq c_d p(||u||_{L^{2d_{i-1}+1}}).
\]
By iterating, we conclude that
\[
||u||_{L^{2d_i+1}} \leq c_d p(||u||_{L^{s+1}}) \leq c_d p(||u||_{H^1(\Omega_{\varepsilon})}).
\]
By the interpolation property, for every \( q \in [2, \infty) \), \( u \in L^{q}(\Omega_{\varepsilon}) \) satisfies
\[
||u||_{L^q(\Omega_{\varepsilon})} \leq c_q p(||u||_{H^1(\Omega_{\varepsilon})}).
\]
(ii) Since \( u \in H^1_0(\Omega_{\varepsilon}) \) is a weak solution of equation \((1)\), by part \( (i) \), \( f(u) \in L^q(\Omega_{\varepsilon}) \) for all \( q \in [2, \infty) \). By Gilbarg-Trudinger \([6]\), Theorem 9.15 and Lemma 9.17, the Dirichlet problem
\[
\begin{cases}
-\Delta v + v = f(u) \text{ in } \Omega_{\varepsilon}, \\
v \in W^{1,q}_0(\Omega_{\varepsilon}) \cap H^1_0(\Omega_{\varepsilon}),
\end{cases}
\]
has a unique strong solution \( v \in W^{2,q}(\Omega_{\varepsilon}) \cap W^{1,q}_{0}(\Omega_{\varepsilon}) \cap H^{1}_{0}(\Omega_{\varepsilon}) \) and
\[
\|v\|_{W^{2,q}(\Omega_{\varepsilon})} \leq c_{q} \|f(u)\|_{L^{q}(\Omega_{\varepsilon})}.
\]
Thus, \( u \) and \( v \) satisfy weakly
\[
\begin{aligned}
-\Delta v + v &= f(u) \text{ in } \Omega_{\varepsilon}, \\
-\Delta u + u &= f(u) \text{ in } \Omega_{\varepsilon}.
\end{aligned}
\]
By Gilbarg-Trudinger [6], Corollary 8.2, \( u = v \in W^{2,q}(\Omega_{\varepsilon}) \cap W^{1,q}_{0}(\Omega_{\varepsilon}) \cap H^{1}_{0}(\Omega_{\varepsilon}) \) and
\[
\|u\|_{W^{2,q}(\Omega_{\varepsilon})} \leq c_{q} \|f(u)\|_{L^{q}(\Omega_{\varepsilon})} \leq c_{q}p(\|u\|_{H^{1}(\Omega_{\varepsilon})}),
\]
for all \( q \in [2, \infty) \). Fix \( q' \in [2, \infty) \) such that \( q' > N \) and \( \theta = 2 - \frac{N}{q'} - \left[ 2 - \frac{N}{q} \right] \).
By Brezis [1], p. 168, we have \( u \in C^{1,\theta}(\overline{\Omega_{\varepsilon}}) \) and
\[
\|u\|_{L^{\infty}(\Omega_{\varepsilon})} \leq \|u\|_{C^{1,\theta}(\overline{\Omega_{\varepsilon}})} \leq c_{q'}\|u\|_{W^{2,q'}(\Omega_{\varepsilon})} \leq c_{q'}p(\|u\|_{H^{1}(\Omega_{\varepsilon})}).
\]

(iii) Since \( u \in H^{1}_{0}(\Omega_{\varepsilon}) \) is a weak solution of equation (1), by part (ii), \( u \in C^{1,\theta}(\overline{\Omega_{\varepsilon}}) \cap W^{2,q'}(\Omega_{\varepsilon}) \). For each \( t > 0 \), apply Brezis [1], p. 168 to obtain
\[
\|u\|_{L^{\infty}(\Omega_{\varepsilon})} \leq \|u\|_{W^{2,q'}(\Omega_{\varepsilon})},
\]
where \( \Omega_{t} = \{ z \in \Omega_{\varepsilon} \mid |y| > t \} \). Since \( \|u\|_{W^{2,q'}(\Omega_{t})} = o(1) \) as \( t \to \infty \), we have \( \lim_{y \to \infty} u(x, y) = 0 \) uniformly in \( x \), where \( (x, y) \in \Omega_{\varepsilon} \).

(iv) Since \( f \in C^{1}(\mathbb{R}^{+}) \) and \( f(0) = 0 \) for \( \xi < 0 \), then \( f \) is locally Lipschitz continuous. By part (ii), \( u \in C^{1,\theta}(\overline{\Omega_{\varepsilon}}) \). Thus, \( f \circ u \in C^{0,\theta}(\Omega_{\varepsilon}) \). By Gilbarg-Trudinger [6, Theorem 9.19], we have \( u \in C^{2,\theta}(\Omega_{\varepsilon}) \).

In order to assert the advanced asymptotic behavior of solutions, let \( \lambda_{1} \) be the first eigenvalue and \( \phi_{1} \) the corresponding first positive eigenfunction of the Dirichlet problem \( -\Delta \phi_{1} = \lambda_{1} \phi_{1} \text{ in } B^{N-1}(0, r), \phi_{1} = 0 \text{ on } \partial B^{N-1}(0, r) \). Then, we have:

**Theorem 16.** (Advanced) Let \( u \) be a positive solution of equation (1) in \( \Omega_{\varepsilon} \). Then for any \( 0 < \delta < \min\{1 + \lambda_{1}, 2\} \) there exist \( c_{1} > 0, c_{2} > 0 \) and \( R > s > 0 \) such that for \( z = (x, y) \in S_{R}^{s} \),
\[
c_{1}\phi_{1}(x)e^{-\sqrt{1+\lambda_{1}}\delta y} \leq u(z) \leq c_{2}\phi_{1}(x)e^{-\sqrt{1+\lambda_{1}}\delta y}.
\]

**Proof.** (i) For \( 0 < \delta < \min\{1 + \lambda_{1}, 2\} \), we choose an \( R_{1} > s > 0 \) such that
\[
\delta - \frac{\sqrt{1+\lambda_{1}+\delta}}{y} \geq 0, \quad \text{for } y \geq R_{1}.
\]
Define
\[ w_\delta(z) = \phi_1(x)e^{-\sqrt{1+\lambda_1+\delta}y}, \quad \text{for } z = (x, y) \in S^r_{R_1}, \]
and \( c_1 = \inf_{z \in S^r_{R_1}} \frac{u(x, R_1)}{w_\delta(x, R_1)} \). Thus, \( \frac{u(x, R_1)}{w_\delta(x, R_1)} \geq \frac{u(x, R_1)}{\phi_1(x)} \). Since \( u(x, y) \) is radially symmetric in \( x \) (see Theorem 19 below), then
\[
\inf_{z \in S^r_{R_1}} \frac{u(x, R_1)}{w_\delta(x, R_1)} = \inf_{x \in L} \frac{u(x, R_1)}{\phi_1(x)} \geq \inf_{x \in L} \frac{u(x, R_1)}{\phi_1(x)}, \tag{11}
\]
where \( L \) is a fixed diameter of \( B^{N-1}(0; r) \). For each \( x_0 \in \partial L \subset \partial B^{N-1}(0; r) \), take a small ball \( B^0 \) in \( B^{N-1}(0; r) \) such that \( x_0 \in \partial B^0 \). Note that \( \phi_1(x) > 0 \) for each \( x \in B^0 \), \( \phi_1(x_0) = 0 \), and \( \phi_1(x) \in C^2 \left( B^{N-1}(0; r) \right) \). Then by the Hopf Boundary Point Lemma, we have \( \frac{\partial u}{\partial \nu}(x_0) < 0 \), where \( \nu \) is the outward unit normal vector of \( B^0 \) at \( x_0 \). For each \( z_1 = (x_0, R) \in \partial L \times \mathbb{R} \subset \partial \Omega_{\varepsilon} \), take a small ball \( B^1 \) in \( \Omega_{\varepsilon} \) such that \( z_1 \in \partial B^1 \). Note that \( u(z) > 0 \) for each \( z \in B^1 \) and \( u(z_1) = 0 \). By Theorem 15 (ii) and (iv), \( u(z) \in C^2(\Omega_{\varepsilon}) \cap C^{1,\theta}(\Omega_{\varepsilon}) \). Thus, by the Hopf Boundary Point Lemma, we have \( \frac{\partial u}{\partial \nu}(z_1) < 0 \), where \( \nu = (\nu, 0) \) is the outward unit normal vector of \( B^1 \) at \( z_1 \). Let \( u_1(x) = u(x, R_1) \), then
\[ \frac{\partial u_1}{\partial \nu}(x_0) = \nabla u_1(x_0) \cdot \nu = \nabla u(z_1) \cdot \nu = \frac{\partial u}{\partial \nu}(z_1) < 0. \]
By the L'Hospital rule, we have
\[ \lim_{x \to x_0 \text{ normally}} x \in \mathbb{B}_{N-1}(0, r) \frac{u(x, R_1)}{\phi_1(x)} = \lim_{h \to 0, x_0 \text{ normally}} u(x_0 + h\nu) = \frac{\partial u}{\partial \nu}(x_0) > 0. \]
Define
\[ \frac{u(x_0, R_1)}{\phi_1(x)} = \lim_{x \to x_0 \text{ normally}} x \in \mathbb{B}_{N-1}(0, r) \frac{u(x, R_1)}{\phi_1(x)} \]
then \( \frac{u(x, R_1)}{\phi_1(x)} : \overline{L} \to \mathbb{R} \) is continuous and \( \frac{u(x, R_1)}{\phi_1(x)} > 0 \) for \( x \in \overline{L} \). Thus, by (11), \( c_1 > 0 \). Let \( v_1(z) = c_1 w_\delta(z) \) for \( z \in \overline{S}^r_{R_1} \), then \( v_1(z) \leq u(z) \) for \( z \in \overline{S}^r_{R_1} \) and \( y = R_1 \). For \( z \in \overline{S}^r_{R_1} \), we have
\[
\Delta (v_1 - u)(z) - (v_1 - u)(z) = (\Delta v_1(z) - v_1(z)) + (-\Delta u(z) + u(z)) = v_1(z) \left( \delta - \frac{\sqrt{1+\lambda_1+\delta} (N-1)}{y} \right) + f(u) \geq 0.
\]
By the maximum principle, we have $v_1(z) \leq u(z)$ for $z \in S_{R_1}^r$.

(ii) For $0 < \delta < \min \{1 + \lambda_1, 2\}$, by (f5) and Theorem 15 (iii), we choose an $R > R_1 > 0$ such that $f(u) \leq \frac{\delta}{2} u$, for $y \geq R$. Define

$$w_{-\delta}(z) = \phi_1(x)e^{-\sqrt{1+\lambda_1-\delta y}}, \quad \text{for } z = (x, y) \in S_{R_1}^r,$$

$$\frac{1}{c_2} = \sup_{z \in S_{R_1}^r} \frac{w_{-\delta}(z)}{u(z)} > 0.$$

It is similar to part (i), $\frac{1}{c_2} < \infty$. Let $v_2(z) = c_2 w_{-\delta}(z)$ for $z \in S_{R_1}^r$, then $v_1(z) \leq u(z)$ for $z \in S_{R_1}^r$ and $y = R$. For $z \in S_{R_1}^r$, we have

$$-\Delta(u - v_2)(z) + (u - v_2)(z) = (-\Delta u(z) + u(z)) + (\Delta v_2(z) - v_2(z))$$

$$= f(u(z)) + (-\delta - \frac{\sqrt{1+\lambda_1-\delta(N-1)}}{y})v_2(z) \leq \frac{\delta}{2}(u - v_2)(z).$$

Therefore

$$-\Delta(u - v_2)(z) + (1 - \frac{\delta}{2})(u - v_2)(z) \leq 0.$$

As in part (i), we obtain $u(z) \leq v_2(z)$ for $z \in S_{R_1}^r$. \qed

5. Symmetry of Solutions

We apply the “moving plane” method in Li-Ni [7] to prove the symmetry of each solution of equation (1) in the flat flask domain $\Omega_\varepsilon$, where $\varepsilon \leq \varepsilon_0$, as follows.

Let $u(x, y)$ be a solution of equation (1) in $\Omega_\varepsilon$. By Theorem 15 and the Schauder regularity, $u$ is a $C^2$ solution. Define

$$T_\lambda = \{(x, y) = (x_1, x_2, \cdots, x_{N-1}, y) \in \Omega_\varepsilon \mid x_1 = \lambda\},$$

$$\Sigma_\lambda = \Omega_\varepsilon \cap \{(x, y) \mid x_1 < \lambda\}.$$ 

For any $(x, y) = (x_1, x_2, \cdots, x_{N-1}, y) \in \Omega_\varepsilon$, set

$$(x^\lambda, y) = (2\lambda - x_1, x_2, \cdots, x_{N-1}, y),$$

that is to say, $(x^\lambda, y)$ is the reflection of $(x, y)$ with respect to $T_\lambda$. Let $\Lambda$ be the collection of all $\lambda \in (-s, 0)$ such that the following statements hold:

$$\begin{cases}
    u(x, y) < u(x^\lambda, y) \quad \text{for all } (x, y) \in \Sigma_\lambda, \\
    u_{x_1}(x, y) > 0 \quad \text{on } T_\lambda.
\end{cases}$$
Lemma 17. For some \( 0 < \delta < s, (-s, -s + \delta) \subset \Lambda \).

Proof. Given \( \lambda \in (-s, 0) \), set \( v^\lambda(x, y) = u(x, y) - u(x^\lambda, y) \) for \((x, y) \in \Sigma_\lambda \), then \( v^\lambda(x, y) = 0 \) for \((x, y) \in T_\lambda \), and \( v^\lambda(x, y) \) satisfies

\[
\Delta v^\lambda(x, y) + c_\lambda(x, y)v^\lambda(x, y) = 0,
\]

(12)

where \( c_\lambda(x, y) = \frac{f(u(x, y)) - f(u(x^\lambda, y))}{u(x, y) - u(x^\lambda, y)} - 1 = f'(\zeta_\lambda) - 1 \), where \( \zeta_\lambda \) is in between \( u(x, y) \) and \( u(x^\lambda, y) \). By Lemma 3 (f5), \( \lim_{y \to 0^+} f'(y) = 0 \). Take \( y_0 \) such that if \( 0 < y \leq y_0 \), then \( f'(y) < 1 \). By Theorem 15 (iii), \( \lim_{y \to \infty} u(x, y) = 0 \) uniformly in \( x \), we can choose \( \delta, s > \delta > 0 \) such that \( s - \delta < y < s \) implies \( u(x, y) \leq y_0 \) uniformly in \( y \). Claim that if \( -s < \lambda < -s + \delta \), then \( v^\lambda(x, y) \leq 0 \) in \( \Sigma_\lambda \). Otherwise, suppose there exists \( \lambda \) such that \( -s < \lambda < -s + \delta \), \( v^\lambda(x, y) > 0 \) for some \((x, y) \in \Sigma_\lambda \). Since \( \lim_{y \to \infty} v^\lambda(x, y) = 0 \) uniformly in \( x \), \( v^\lambda(x, y) \) achieves its maximum at \((x_\lambda, y_\lambda) \in \Sigma_\lambda \), then

\[
\nabla v^\lambda(x_\lambda, y_\lambda) = 0, \quad \{v^\lambda_{ij}(x_\lambda, y_\lambda)\} \leq 0.
\]

But

\[
\Delta v^\lambda(x_\lambda, y_\lambda) \leq 0, \quad (f'(\zeta_\lambda) - 1)v^\lambda(x_\lambda, y_\lambda) = c_\lambda(x, y)v^\lambda(x_\lambda, y_\lambda) < 0,
\]

which contradicts to equation (12). Thus, for \( -s < \lambda < -s + \delta \), \( v^\lambda(x, y) \leq 0 \) in \( \Sigma_\lambda \). Applying the maximum principle and Hopf Boundary Point Lemma, for \( -s < \lambda < -s + \delta \), \( v^\lambda(x, y) < 0 \) in \( \Sigma_\lambda \) and \( v^\lambda_{x_1}(x, y) > 0 \) for \((x, y) \in T_\lambda \). Hence \( u_{x_1}(x, y) > 0 \) for \((x, y) \in T_\lambda \). Then, \( (-s, -s + \delta) \subset \Lambda \). \( \square \)

Lemma 18. If \((-s, \lambda) \subset \Lambda \), then there exists \( \tau > 0 \) such that \([\lambda, \lambda + \tau) \subset \Lambda \).

Proof. Suppose not, there exists a decreasing sequence \( \lambda_k \to \lambda \) and a sequence \( \{(x_k, y_k)\} \) of points in \( \Sigma_{\lambda_k} \) such that \( v^\lambda_k(x_k, y_k) = u(x_k, y_k) - u(x^\lambda_{k}, y_k) > 0 \). By the compactness of \( \overline{\mathbb{B}^{N-1}(0; s)} \), there is a subsequence \( \{(x_k, y_k)\} \) such that \( x_k \to \bar{x} \in \overline{\mathbb{B}^{N-1}(0; s)} \). Then we have two possibilities as shown in Case 1 and 2.

Case 1. \( y_k \to \infty \). From \( \lim_{y_k \to \infty} u(x_k, y_k) = 0 \), we may assume

\[
v^\lambda_k(x_k, y_k) = \max_{(x, y) \in \Sigma_{\lambda_k}} v^\lambda_k(x, y),
\]

\[
\nabla v^\lambda_k(x_k, y_k) = 0, \quad \{v^\lambda_{ij}(x_k, y_k)\} \leq 0.
\]
As in Lemma 17, we obtain a contradiction.

Case 2. $y_k \to y$. We have $(x_k, y_k) \to (\overline{x}, \overline{y}) \in \overline{\Sigma}_\lambda$. Thus, $v^\lambda(\overline{x}, \overline{y}) \geq 0$. Clearly $(\overline{x}, \overline{y}) \notin \Sigma_\lambda$ since $v^\lambda(x, y) < 0$ in $\Sigma_\lambda$. If $(\overline{x}, \overline{y}) \in T_\lambda$ then $u_{x_1}(\overline{x}, \overline{y}) < 0$, which contradicts to $\lambda \in \Lambda$. Moreover, $(\overline{x}, \overline{y}) \notin \partial \Omega \cap \Sigma_\lambda$ since if $(\overline{x}, \overline{y}) \in \partial \Omega \cap \Sigma_\lambda$ then $0 = u(\overline{x}, \overline{y}) \geq u(\overline{x}, \overline{y}) > 0$, a contraction. We conclude that Case 2 is impossible. □

**Theorem 19.** Let $u(x, y)$ be a solution of equation (1) in $\Omega_\varepsilon$. Then $u$ is radially symmetric in $x$: that is to say, there is a function $h : [0, t] \times (-t, \infty) \to \mathbb{R}$ such that $u(x, y) = h(|x|, y)$.

**Proof.** Let $\mu = \sup\{\lambda \in (-s, 0) \mid (\overline{x}, \overline{y}) \subset \Lambda\}$. Then $\mu \notin \Lambda$. If not, by Lemma 18 we would have $[\mu, \mu + \varepsilon) \subset \Lambda$, which contradicts to the definition of $\mu$. We claim that $\mu = 0$. Suppose not, $\mu \in (-s, 0)$. By continuity we have $u(x, y) \leq u(x, y)$ for all $(x, y) \in \Sigma_\mu$, then by the maximum principle we have $u(x, y) \equiv u(x, y)$ for all $(x, y) \in \Sigma_\mu$, which is impossible. Thus, $\mu = 0$. By reversing the $x_1$ axis, we conclude that $u(x, y)$ is symmetric with respect to the hyperplane $T_0$ and $u_{x_1}(x, y) < 0$ for $x_1 > 0$. Since the $x_1$ direction can be chosen arbitrarily, we conclude that $u(x, y)$ is radially symmetric in $x$. □

**References**


