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SEMI-DISCRETIZED HEAT EQUATION APPROXIMATED BY LINEAR FINITE ELEMENTS METHOD

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Abstract: One dimensional heat equation with initial and boundary conditions is chosen for consideration. Firstly, replacing the time derivative by first order forward-difference approximation, the partial differential equation is reduced to second order ordinary differential equation at time levels. Secondly, a numerical solution of the ordinary differential equation is investigated by linear finite element method in local coordinate system. Finally, numerical results are obtained on the one dimensional heat equation.

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1. Introduction

For time-dependent problems, usually a two-stage formulation is followed. In the first stage, the differential equations are approximated by the finite element method to obtain a set of ordinary differential equations in time. In the second stage, the differential equations in time are solved exactly or further approximated by either variational methods or finite difference methods to obtain algebraic equations, which are then solved for the nodal values [4].

Problems of the heat-conduction equations are continuously being studied, and the majority of the problems are stated with homogeneous boundary con-

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ditions. Among these are [4,7]. E.S. Onah transformed the heat equation into a variational form by continuos time Galerkin method, and expressed in terms of linear spline basis function. On the other hand, the collacation method is applied on the heat equation with cubic splines as the basis functions. The both methods of discretization transform the heat equation to system of ordinary differential equations in time, respectively. D.A. French developed a scheme, which is using the discontinuous Galerkin method for the time discretization [2]. H. Lu at al [3], extended the Galerkin method to forward-backward heat equations. They applied variable transformations to solve a wide glass of forward-backwards heat equations [3].

We shall consider the one-dimensional initial boundary value problem of the heat conduction equation as stated in

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < X, \quad t > 0, \tag{1}$$

with the initial condition

$$u(x,0) = \Phi(x), \qquad 0 \le x \le X\,, \tag{2}$$

and the boundary conditions

$$u(0,t) = f(t) \quad \text{and} \quad u(X,t) = g(t), \quad t > 0\,, \tag{3}$$

where α is a constant and Φ, f and g are the prescribed functions of the variables.

A numerical work includes the investigation of a finite difference scheme in [2]. The aim of the present paper is to use finite difference scheme for the time derivative and obtain semi-discretized ordinary differential equation which is analysed by linear finite element method in local coordinate system. The remainder of the paper is organised as follows. In Section 2, we reach the weak form of the ordinary differential equation which is discrete at all time levels, and the finite element solution is done. In Section 3, a numerical example is considered. Results are presented by the tables and figure. The conclusion is given in Section 4

2. Discretization

In this section first, we shall reduce equation (1) to ordinary differential equation by the finite difference for the time derivative at all time levels. Second, we shall obtain a weak form of the differential equation.

We specify the grids. Let $h = \frac{b-a}{M+1}$ and $x_i = ih$ for $i = 0, 1, ..., M + 1$. Let $t_j = jk$ for $j = 0, 1, ..., N$ and k is a real number. In the equation (1) replacing the time derivative by first-order forward difference [6]

$$\frac{\partial u}{\partial t} = \frac{u(x, t + k) - u(x, t)}{k} + o(k)\,, \tag{4}$$

and applying on the equation (1) to each of the N interior points, equation (1) can be written following discrete ordinary differential equation

$$\frac{d^2u(x, t + k)}{dx} = -\alpha(\frac{u(x, t + k) - u(x, t)}{k}), \quad 0 < x < X\,. \tag{5}$$

We can arrange equation (5)

$$-\frac{d^2u(x, t + k)}{dx^2} - \alpha\frac{u(x, t + k)}{k} + \alpha\frac{u(x, t)}{k} = 0, \quad 0 < x < X\,, \tag{6}$$

where $u(x, t) = u(ih, jk) = u_{i,j}(x, t)$ and $u(x, t + k) = u_{i,j+1}(x, t)$. The values of $u(x, t)$ are known, but the values of $u(x, t + k)$ are unknown. So that, we write equation (6)

$$-\frac{d^2u}{dt^2} - \beta u + \beta u_{i,j} = 0, \quad 0 < x < X\,, \tag{7}$$

where $\beta = \frac{\alpha}{k}$. Initial condition of equation (7) is

$$u_{i,0} = \Phi_i, i = 1, 2, ..., M \quad \text{and} \quad u_{0,0} = \Phi_0, u_{M+1,0} = \Phi_{M+1} \tag{8}$$

and boundary conditions are

$$u_{o,j} = f_j \quad \text{and} \quad u_{M+1,j} = g_j \quad (j = 0, 1, 2, ..., N). \tag{9}$$

Since equation (7) is valid over the domain (0,X), it is valid, in particular, over the element Ω^e=(x_e, x_{e+1}). Multiplying equation (7) over the element with a function υ, called the test function, that is twice differentiable and satisfies the conditions (8)

$$\int\limits_{x_e}^{x_{e+1}} \left[-\upsilon\frac{d^2u}{dx^2} - \beta\upsilon u\right]dx + \beta u_{i,j}\int\limits_{x_e}^{x_{e+1}} \upsilon dx = 0\,.$$

The weak form of the equation (7) is given by

$$-\upsilon(x_{e+1})\frac{du(x_{e+1})}{dx}+\upsilon(x_e)\frac{du(x_e)}{dx}+\beta u_{i,j}\int\limits_{x_e}^{x_{e+1}}\upsilon dx + \int\limits_{x_e}^{x_{e+1}}\left[\frac{du}{dx}\frac{d\upsilon}{dx}-\beta\upsilon u\right]dx=0. \quad (10)$$

Variational Approximation of the Equation Over an Element

Now suppose that we wish to find an approximate solution of the variational problem (4) with the conditions (8) using Ritz method. Let the Ritz approximation of u on the element be given by

$$u_e(x)=\sum_{j=1}^{n}\eta_j^{(e)}\Psi_j^{(e)}(x)\,, \quad (11)$$

where η_j are the parameters to be determined, and $\Psi_j(x)$ are the approximation functions to be constructed shortly. Substituting equation (5) for $u=\Psi_j$ and $\upsilon=\Psi_i(x)$ into (4), we obtain

$$-\Psi_i(x_{e+1})\frac{du(x_{e+1})}{dx}+\Psi_i(x_e)\frac{du(x_e)}{dx}+\beta u_{i,j}\int\limits_{x_e}^{x_{e+1}}\Psi_i dx + \sum_{j=1}^{n}\eta_j\int\limits_{x_e}^{x_{e+1}}\left[\frac{d\Psi_i}{dx}\frac{d\Psi_j}{dx}-\beta\Psi_i\Psi_j\right]dx=\sum_{j=1}^{n}K_{i,j}^{(e)}\eta_j^{(e)}-F_i^{(e)}\,,$$

$i=1,2,...,n$, or

$$\left[K^{(e)}\right]\left\{\eta^{(e)}\right\}=\left\{F^{(e)}\right\}\,, \quad (12)$$

where the coefficient matrix $K_{i,j}^{e}$ is called stiffness matrix, and the column vector $F_i^{(e)}$ is called force vector, are given by

$$\begin{aligned} K_{i,j}^{(e)} &= \int_{x_e}^{x_{e+1}}\left[\frac{d\Psi_i}{dx}\frac{d\Psi_j}{dx}-\beta\Psi_i\Psi_j\right]dx, \\ F_i^{(e)} &= -\beta u_{i,j}\int\limits_{x_e}^{x_{e+1}}\Psi_i dx+\Psi_i(x_{e+1})\frac{du(x_{e+1})}{dx}-\Psi_i(x_e)\frac{du(x_e)}{dx}. \end{aligned} \quad (13)$$

Using the forward-difference approximation, we have

$$\frac{du(x_e)}{dx} = \frac{u(x_e) - u(x_{e+1})}{h_e},$$

and using the backward-difference approximation,

$$\frac{du(x_{e+1})}{dx} = \frac{u(x_{e+1}) - u(x_e)}{h_e}.$$

We can write the finite element model of equation (7) as

$$\left[K^{(e)}\right]\left\{u^{(e)}\right\} = \left\{F^{(e)}\right\}, \tag{14}$$

where $u^{(e)} = [u_{o,j+1}, u_{1,j+1}, ..., u_{M+1,j+1}]$.

Let $\overline{x}$ denote the local coordinate, whose is at node 1 of an element. The local coordinate $\overline{x}$ is related to the global coordinate x by the linear "transformation" $x = \overline{x} + x_e$. In the local coordinate system, we can select the approximations functions

$$\Psi_i^{(e)}(\overline{x}) = 1 - \frac{\overline{x}}{h_e}, \quad \Psi_2^{(e)}(\overline{x}) = \frac{\overline{x}}{h_e}, \quad 0 \le \overline{x} \le h_e, \tag{15}$$

where $h_e = x_{e+1} - x_e$.

Local coordinate systems are also convenient to use in the numerical evaluation of integrals [5]. The coefficient matrix $K_{ij}^{(e)}$ and column vector $F_i^{(e)}$ in equations (13) can be written in the local coordinate system as

$$K_{ij}^{(e)} = \int_0^{h_e} \left[\frac{d\Psi_i}{d\overline{x}}\frac{d\Psi_j}{d\overline{x}} - \beta\Psi_i\Psi_j\right] d\overline{x}$$

$$F_i^{(e)} = -\beta u_{ij} \int_0^{h_e} \Psi_i d\overline{x} + \Psi_i(x_{e+1})\frac{du(x_{e+1})}{d\overline{x}} - \Psi_i(x_e)\frac{du(x_e)}{d\overline{x}}.$$

3. Numerical Results

We present in this section numerical results obtained for the heat equation. All computations are carried out using *Maple* and it takes several minutes to get all the computed results.

t	present method h_e=0.5	difference	percentage error	present method h_e =0.1	difference	percentage error	exact
0.005	0.8601	0.0197	2.36	0.8510	0.0106	1.3	0.8404
0.01	0.7885	0.0142	1.8	0.7831	0.0078	0.8	0.7743
0.02	0.6932	0.0123	1.7	0.6846	0. 0037	0.5	0.6809
0.10	0.3067	0.0035	1.1	0.3036	0.0015	0.5	0.3021

Table 1: The result of the heat equation in example $x = 0.5$, $k = 0.001$

x	present method h_e=0.5	difference	per. error	present method h_e =0.1	difference	per. error	exact
0.1	0.00001365	0.0000007	5.1	0.00001325	0.0000003	2.3	0.00001295
0.2	0.00002595	0.0000013	5	0.00002406	0.0000005	2.4	0.00002464
0.3	0.00003606	0.0000021	5.9	0.00003487	0.0000009	2.7	0.00003391
0.4	0.00004235	0.0000024	5.9	0.00004105	0.0000015	2.8	0.00003987
0.5	0.00004458	0.0000026	6	0.00004308	0.0000016	2.7	0.00004192

Table 2: The result of the heat equation in example $t = 1$, $k = 0.01$

Example. We solve equation (1) with the initial condition and boundary conditions, which are required for calculating the value of $F_i^{(e)}$. The initial condition is

$$u(x) = 2x, \quad 0 \le x \le 1/2, \quad t = 0\,,$$

$$u(x) = 2(1-x), \quad 1/2 \le x \le 1, \quad t = 0\,,$$

and the boundary conditions are $f(t) = g(t) = 0$, $t > 0$. α is chosen 1.

The heat equation is numerically solved by present method. Numerical results are given in Table 1 and Figure 1.

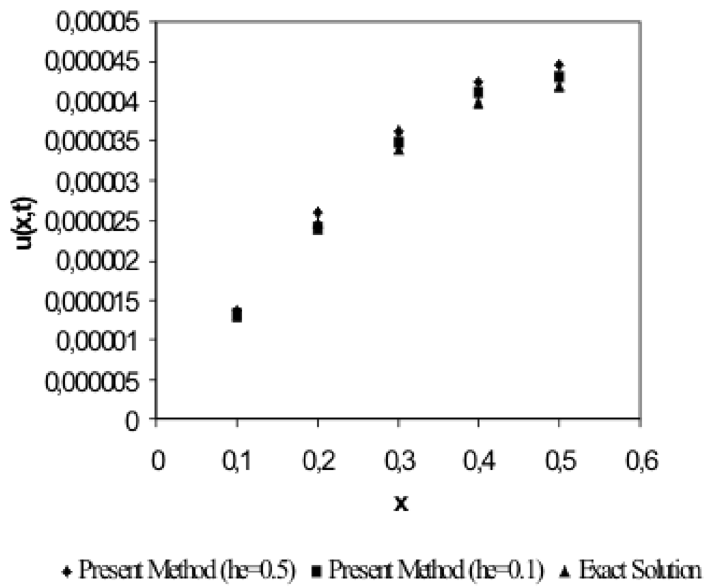

Figure 1: The comparison of approximations

4. Conclusion

The heat equation, which is time dependent problem, is numerically solved using the finite difference with finite elements. Using finite difference equation for the time derivative in the heat equation, it is reduced to second degree ordinary differential equation. This equation is solved by finite elements method also we studied in local coordinate system proving to be convenient in the derivation of the approximation functions.

Numerical example is presented in this paper to illustrate an excellent accuracy of the approximation obtained by the present method. The comparison of approximations are given by tables and the figure. Our aim is that we want to apply the present method to time dependent hyperbolic equations. We expect to obtain better results.

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