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DEVELOPMENT AND ANALYSIS OF QUADRATURE AND GALERKIN METHODS FOR APPROXIMATE SOLUTION TO THE INTEGRAL FORMULATION OF VOLTERRA'S POPULATION EQUATION WITH DIFFUSION AND NOISE*

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Abstract: In this research, an equivalent integral formulation of Volterra's population equation with diffusion and noise is numerically studied. The noise term in time and space is represented through a Brownian sheet. Two independent numerical methods are developed to solve this integral equation, a quadrature method and a semi-discrete Galerkin procedure. Error analyses for the two methods are performed which prove convergence of the approximations to the exact solution. Three numerical examples are given which confirm the results of the error analyses.

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1. Introduction

There are many equations which describe the growth of populations. A functional equation used to describe growth of a single, isolated population is

$$\frac{u'(t)}{u(t)} = F(u)(t), \qquad t \geq 0\,, \tag{1}$$

where $u(t)$ is the size of the population at time t, and $u'(t)/u(t)$ is the per capita growth rate. The most well known form for F is the linear expression for logistic growth, $F(u)(t) = a - bu(t)$, where $a > 0$ is the birth rate coefficient and $b > 0$ is the intra-species competition coefficient. If past population sizes have an effect on the present growth rate, Volterra's [16, 17] equation results:

$$u'(t) = au(t) - bu^2(t) - u(t)\int_0^t f(t-s)u(s)ds, \tag{2}$$

where the delay kernel $f(s)$ can be considered to be a "hereditary" term which could result from age structure, for example. For a discussion of equation (2), see Miller [12] or Cushing [4].

If the population is spatially distributed, the species lives in a bounded domain $\Omega \subset \mathbb{R}^n$, the population diffuses spatially, and it is confined to a region Ω, then the model has the form

$$\begin{cases} u_t(t,x) = u_{xx}(t,x) + au(t,x) - bu^2(t,x) \\ \qquad\qquad - u(t,x)\int_0^t f(t-s)u(s,x)ds, \quad (x,t) \in \Omega \times [0,T], \\ \partial u/\partial\eta = 0, \qquad x \in \partial\Omega, t \geq 0, \\ u(x,0) = u_0(x), \qquad x \in \Omega, \end{cases} \tag{3}$$

where $u_0 \in C^1(\bar{\Omega})$ is the initial population size, and $\partial/\partial\eta$ denotes the exterior normal derivative to the boundary $\partial\Omega$. Properties of the solution to (3) have been studied by Schiaffino [14], Redlinger [13], and Yamada [19].

However, equation (3) neglects the random effects on the population size that could result from the environment for example. Thus, in a more general form, and the form of interest in this investigation, the model becomes

$$\begin{cases} u_t(t,x) = u_{xx}(t,x) + \alpha(t,x,u(t,x),v(t,x)) \\ \qquad\qquad +\beta(t,x,u(t,x))W(t,x), \\ u_x(t,0) = u_x(t,1) = 0, \qquad 0 < t \leq T, \\ u(0,x) = u_0(x), \qquad 0 \leq x \leq 1, \end{cases} \tag{4}$$

where $v(t,x) = \int_0^t f(t-s)u(s,x)ds$, $W(t,x)$ is the Brownian sheet, $u(0,x) \geq 0$, and $(t,x) \in \mathbb{D}_T = [0,T] \times [0,1]$. For example,

$$\alpha(t,x,u(t,x),v(t,x)) = au(t,x) - bu^2(t,x) - u(t,x)v(t,x)$$

in equation (4). The last term in equation (4) models the random influence on the population size. Henceforth, we refer to equation (4) as Volterra's population equation with diffusion and noise.

Manthey and Stiewe [11] and Lewin [8] considered the equivalent integral formulation of (4) given by

$$\begin{aligned} u(t,x) = \int_0^1 & G(t,x,y)u_0(y)dy \\ & + \int_0^t \int_0^1 G(t-s,x,y)\beta(s,y,u(s,y))dW(s,y) \\ & + \int_0^t \int_0^1 G(t-s,x,y)\alpha(s,y,u(s,y),v(s,y))dyds\,, \\ & \qquad\qquad 0 \leq x \leq 1, \quad 0 < t \leq T\,, \end{aligned} \tag{5}$$

where G is Green's function solution to the corresponding deterministic linear partial differential equation

$$\begin{cases} G_t(t,x,y) = G_{xx}(t,x,y) & 0 < x < 1, 0 < t \leq T, \\ G_x(t,0,y) = G_x(t,1,y) = 0, & 0 < t \leq T, \\ G(0,x,y) = \delta(x-y). \end{cases}$$

It is also noted in [11] that the stochastic integral in (5) is only well-defined in space dimension one due to the strong singularity of G at time zero.

Notice that G can be found through separation of variables and has the form

$$G(t,x,y) = 1 + 2\sum_{m=1}^{\infty} e^{-(m\pi)^2 t}\cos(m\pi x)\cos(m\pi y). \tag{6}$$

Note also that G could be found by applying the method of images to $\widetilde{G}$ that solves the above parabolic problem on an infinite domain, i.e.

$$\widetilde{G}(t,x,y) = \frac{1}{\sqrt{2\pi t}}\exp\Big(\frac{-(x-y)^2}{4t}\Big).$$

Qualitative behavior of the solution to (5) was studied by Manthey and Maslowski [10] for the case $\beta = 1$, α depending only on t and u, and Dirichlet boundary conditions. Manthey and Stiewe [11] proved pathwise existence and uniqueness of a non-negative continuous solution to (5) under the conditions

C-1. $\alpha(t,x,0,\Phi) \geq 0$ and $\beta(t,x,0) = 0$ for all $(t,x,\Phi) \in \mathbb{D}_T \times \mathbb{L}_2(\mathbb{D}_T)$.

C-2. Let $\alpha : \mathbb{D}_T \times \mathbb{R} \times \mathbb{L}_2(\mathbb{D}_T) \to \mathbb{R}$ and $\beta : \mathbb{D}_T \times \mathbb{R} \to \mathbb{R}$. For each $N > 0$ there exists positive constants $L_1(N)$ and $L_2(N)$ such that for all $(t,x) \in \mathbb{D}_T$

$$\big|\alpha(t,x,u,\Phi) - \alpha(t,x,v,\Psi)\big|^2 \leq L_1(N)(\big|u-v\big|^2 + \parallel \Phi - \Psi \parallel_t^2),$$

and

$$\big|\beta(t,x,u) - \beta(t,x,v)\big|^2 \leq L_2(N)\big|u-v\big|^2,$$

if $|u|$, $|v|$, $\parallel \Phi \parallel$, and $\parallel \Psi \parallel$ are all bounded by N.

C-3. There exists a constant $C_\alpha > 0$ such that for all $(t,x) \in \mathbb{D}_T$, $u \geq 0$, and $\Phi \geq 0$

$$\alpha(t,x,u,\Phi) \leq C_\alpha(1+u).$$

C-4. There exists a constant K such that for all $(t,x,u,\Phi) \in \mathbb{D}_T \times \mathbb{R} \times \mathbb{L}_2(\mathbb{D}_T)$

$$\big|\alpha(t,x,u,\Phi)\big| + \big|\beta(t,x,u)\big| \leq K.$$

C-5. There exists a constant R such that for all $(t,x) \in \mathbb{D}_T$

$$\big|\alpha(t,x,0,0)\big| + \big|\beta(t,x,0)\big| \leq R.$$

C-6. $\big|\beta(t,x,u) - \beta(t,x,v)\big|^2 \leq L_\beta(\big|u-v\big|^2)$.

In considering the Brownian sheet, some of its properties are given by Walsh [18], Cabaña [3], and Allen, Novosel, and Zhang [1]. The following properties are important in this investigation and are given here for convenience.

1. If $S = \big\{(t,x,) : a \leq t < b,\ c \leq x < d\big\}$ is a rectangle in $\mathbb{R}^2$, then

$$\int_c^d \int_a^b dW(t,x) = \int_c^d \int_a^b \frac{\partial^2 W}{\partial t \partial x}(t,x) = W(S),$$

   where $W(S) = W(b,d) - W(a,d) - W(b,c) + W(a,c)$ is Gaussian with mean zero and variance $|S|$, and $|S|$ is the area of S.

2. $\displaystyle\int_0^T \int_a^b \mathbf{1}_S dW(t,x) = W(S)$, for $S \subset (0,T) \times (a,b)$.

3. If f is nonanticipatory and measurable with $\mathbb{E}\Big(\int_0^T\int_a^b f^2(t,x)dxdt\Big) < \infty$, then the Itô isometry property holds:

$$\mathbb{E}\Big(\int_0^T\int_a^b f(t,x)dW(t,x)\Big)^2 = \mathbb{E}\Big(\int_0^T\int_a^b f^2(t,s)dxdt\Big). \tag{7}$$

4. With a partition of $[0,T]\times[0,1]$ given by the rectangles $[t_i,t_{i+1}]\times[x_j,x_{j+1}]$, where $\Delta t = T/N$, $\Delta x = 1/M$, $t_i = i\Delta t$, $x_j = j\Delta x$, for $i = 0,1,\ldots,N$, and $j = 0,1,\ldots,M$, the approximation

$$\frac{\partial^2\widehat{W}}{\partial t\partial x}(t,x) = \frac{1}{\sqrt{\Delta t\Delta x}}\sum_{i=1}^{M}\sum_{j=1}^{N}\eta_{ij}\mathbf{1}_{[t_i,t_{i+1})}(t)\mathbf{1}_{[x_j,x_{j+1})}(x) \approx \frac{\partial^2 W}{\partial t\partial x}(t,x)\,, \tag{8}$$

where $\eta_{ij} \in N(0,1)$, satisfies

$$\mathbb{E}\Big(\int_0^T\int_0^1 f(t,x)dW(t,x) - \int_0^T\int_0^1 f(t,x)d\widehat{W}(t,x)\Big)^2 \le 2T\gamma^2((\Delta t)^{2\beta} + (\Delta x)^{2\alpha})$$

if there are constants $\alpha,\beta \in (0,1]$, and $\gamma > 0$ such that

$$\big|f(t,x) - f(u,y)\big| \le \gamma(\big|t-u\big|^\beta + \big|x-y\big|^\alpha)$$

for (t,x), $(u,y) \in [0,T]\times[0,1]$, and f is a nonrandom Hölder continuous function on $[0,T]\times[0,1]$ (see [1] for a proof of this result).

Recently, Gyongy [5, 6] and Gyongy and Nualart [7] developed implicit and explicit difference methods for numerical solution to stochastic parabolic partial differential equations driven by white noise. It is shown that the implicit difference scheme is almost surely convergent when the nonlinear terms are Lipschitz continuous. A similar convergence result holds for the explicit scheme provided that the mesh sizes in time and space satisfy an additional condition. However, it is not clear how these methods can be applied to problem (4) of the present investigation. In particular, it is not clear how the integral delay term $v(t,x)$ can be approximated using the finite-difference procedures described in

[5, 6, 7]. In addition, the convergence results of the difference schemes are not applicable to problem (4) due to the presence of $v(t,x)$.

In the present investigation, two new explicit numerical methods are derived, analyzed, and computationally tested for solving Volterra's population equation with diffusion and delay. These methods are not difference approximations of (4) but approximately solve the integral formulation of Volterra's population equation (5). Numerical solution of equation (5) rather than equation (4) is studied in the present investigation as Manthey and Stiewe [11] proved existence of a unique solution to (5). Existence of a unique solution facilitates the error analyses of the numerical methods. The first numerical method applies quadrature approximations in x and t. It is shown that this method converges in the mean square sense provided that $\Delta t \to 0$, $\Delta x \to 0$, and $(\Delta x)^2/\Delta t \to 0$. The second method uses a semi-discrete Galerkin procedure with trigonometric basis functions in x to approximately solve integral formulation (5). For this method, it is shown that the approximations converge in a mean square sense provided that $\Delta t \to 0$ and $\Delta x \to 0$, where Δt and Δx define an approximate Brownian sheet. Computational examples given in the Section 4 show that the two numerical methods can accurately approximate solutions to equation (5).

2. Two Numerical Methods Considered

2.1. Quadrature Solution

Two explicit numerical methods are considered to approximately solve (5). Two methods are useful, for example, for comparison purposes as exact solutions to (5) are very difficult to obtain. The first is a quadrature scheme, which is explicit in time. If M and N are positive integers let

$$\begin{cases} \Delta x = 1/L, \Delta t = T/N\,, \\ x_j = j\Delta x, y_j = j\Delta x, z_j = \Delta x, \qquad j = 1, 2, \ldots, L\,, \\ t_i = i\Delta t, s_i = i\Delta t, \qquad i = 1, 2, \ldots, N. \end{cases} \tag{9}$$

A reasonable quadrature scheme [2] to solve stochastic integral equation (5) is

$$\begin{aligned} u_{i,j} =& F_{ij} + \sum_{n=0}^{i-1}\sum_{k=0}^{L-1} G(t_i - t_n, x_j, y_k)\alpha(t_n, y_k, u_{n,k}, v_{n,k})\Delta x\Delta t \\ &+ \sum_{n=0}^{i-1}\sum_{k=0}^{L-1} G(t_i - t_n, x_j, y_k)\beta(t_n, y_k, u_{n,k})\eta_{nk}\sqrt{\Delta x\Delta t}\,, \end{aligned} \tag{10}$$

where $u_{i,j}$ is the approximation to $u(t_i, x_j)$, $\eta_{nk} \in N(0,1)$, $v_{nk} = \sum_{l=0}^{n-1} f(t_n - t_l)u_{lk}\Delta t$, and $F_{ij} = \int_0^1 G(t_i, x_j, y)u_0(y)dy$. It is assumed that F_{ij} is exactly determined as G is given by (6) and u_0 is a known function.

2.2. Semi-Discrete Galerkin Method

In this method a solution to (5) is assumed of the form

$$u(t,x) = \sum_{n=0}^{\infty} u_n(t)\cos(n\pi x). \tag{11}$$

Note that the pathwise continuity of u implies that it can be written in this way. In addition, the Fourier cosine series is chosen here, because the form of the Green's function (6) indicates this choice may yield simplifications. Thus, from (11), it is seen that

$$u_n(0) = \begin{cases} \int_0^1 u(0,x)dx\,, & n = 0\,, \\ 2\int_0^1 u(0,x)\cos(n\pi x)dx\,, & n = 1, 2, 3, \ldots\,. \end{cases} \tag{12}$$

The functions $u_n(t)$ are now approximated using a semi-discrete Galerkin procedure. Substituting (11) into (5), multiplying by $\cos(m\pi x)$, and integrating with respect to x over $[0,1]$ gives

$$\begin{aligned} c_m u_m(t) =& \int_0^t \int_0^1 \widehat{G}_m(t-s, y)\alpha(s, y, u(s,y), v(s,y))dyds \\ &+ \int_0^1 \widehat{G}_m(t, y)u(0,y)dy \\ &+ \int_0^t \int_0^1 \widehat{G}_m(t-s, y)\beta(s, y, u(s,y))dW(y,s)\,, \end{aligned} \tag{13}$$

where $c_m = \int_0^1 \cos^2(m\pi x)dx$ and

$$\widehat{G}_m(t,y) = \int_0^1 G(t,x,y)\cos(m\pi x)dx = e^{-(m\pi)^2 t}\cos(m\pi y).$$

Thus, since $\int_0^1 \widehat{G}_m(t,y)u(0,y)dy = u_m(0)c_m e^{-(m\pi)^2 t}$, for $m = 0, 1, 2, \ldots$, then the coefficients of the series at time t_i are given by

$$\begin{aligned} u_m(t_i) =& u_m(0)e^{-(m\pi)^2 t_i} \\ &+ \frac{1}{c_m}\int_0^{t_i}\int_0^1 \widehat{G}_m(t_i - s, y)\alpha(s, y, u(s,y), v(s,y))dyds \\ &+ \frac{1}{c_m}\int_0^{t_i}\int_0^1 \widehat{G}_m(t_i - s, y)\beta(s, y, u(s,y))dW(s,y). \end{aligned} \tag{14}$$

It is assumed that the integrations in the first integral are exactly evaluated with respect to the variable y. Now, using approximation (8) and a rectangular rule approximation for the second term, the following explicit approximation procedure is obtained:

$$\begin{aligned} u_{m,i} =& u_{m,0}e^{-(m\pi)^2 t_i} \\ &+ \frac{1}{c_m}\sum_{l=0}^{i-1}\int_0^1 \widehat{G}_m(t_i - t_l, y)\alpha(t_l, y, \hat{u}(t_l,y), \hat{v}(t_l,y))dy \\ &+ \frac{1}{c_m}\sum_{l=0}^{i-1}\sum_{k=0}^{L-1}\sqrt{\Delta x \Delta t}\eta_{l,k}\widehat{G}_m(t_i - t_l, y_k)\beta(t_l, y_k, \hat{u}(t_l, y_k)), \end{aligned} \tag{15}$$

where we have written $u_{m,i}$ for $u_m(t_i)$, $t_i = s_i = i\Delta t$, $x_j = y_j = j\Delta x$, $\eta_{l,k} \in N(0,1)$. In addition, $v(t_l, y)$ is approximated by

$$v(t_l, y) \approx \hat{v}(t_l, y) = \sum_{r=0}^{l-1} f(t_l - t_r)\hat{u}(t_r, x)\Delta t.$$

Thus, (11) is approximated at discrete time t_l by

$$u(t_l, y) = \sum_{m=0}^{\infty} u_m(t_l)\cos(m\pi y) \approx \sum_{m=0}^{L} u_{m,l}\cos(m\pi y) = \hat{u}(t_l, y).$$

3. Error Analysis of the Two Methods

For the error analysis of the two methods, let L and N be positive integers and set

$$\begin{cases} \Delta x = 1/L, \Delta t = T/N\,, \\ x_j = j\Delta x, y_j = j\Delta x, z_j = \Delta x, \qquad j = 1, 2, \ldots, L\,, \\ t_i = i\Delta t, s_i = i\Delta t, \qquad i = 1, 2, \ldots, N\,. \end{cases} \tag{16}$$

The conditions from Manthey and Stiewe [11] in Section 2.1 are assumed in this analysis. However, to perform the error analysis, additional conditions are also assumed. Along with the conditions C-1 through C-6 given in the introduction, the following conditions are also assumed:

C-7 $\beta^2(t,x,u) \le c_\beta(1+u^2), \qquad (t,x) \in \mathbb{D}_T.$

C-8 $\left|\alpha(t,x,u,\phi)-\alpha(t,x,v,\psi)\right|^2 \le L_\alpha\left[\Big(u(t,x)-v(t,x)\Big)^2+\Big(\phi(t,x)-\psi(t,x)\Big)^2\right].$

C-9 $f(t) \ge 0, f \in C^1[0,T], \text{and} \int_0^T f^2(t)dt = \hat{f} < \infty.$

C-10 $u_0 \in C^2[0,1].$

C-11 $\left|\alpha(t+\Delta t, x+\Delta x, u(t,x), v(t,x)) - \alpha(t,x,u(t,x),v(t,x))\right|^2 \le c_{\alpha_1}\Delta t^2 + c_{\alpha_2}\Delta t \Delta x + c_{\alpha_3}\Delta x^2.$

C-12 $\left|\beta(t+\Delta t, x+\Delta x, u(t,x)) - \beta(t,x,u(t,x))\right|^2 \le c_{\beta_1}\Delta t^2 + c_{\beta_2}\Delta t \Delta x + c_{\beta_3}\Delta x^2.$

The following inequalities are useful in the analyses of these methods. Proofs of these results use comparison arguments with corresponding integrals and are not given here.

$$\sum_{n=1}^{\infty} \frac{1-e^{-(n\pi)^2\Delta t}}{(n\pi)^2} \le \Delta t^{1/2}\,, \tag{17}$$

$$\sum_{n=1}^{\infty} \frac{\big(\cos(n\pi(x+\Delta x)) - \cos(n\pi x)\big)^2}{(n\pi)^2} \le 2\Delta x\,, \tag{18}$$

$$\sum_{n=1}^{\infty} e^{-(n\pi)^2\Delta t} \le \Delta t^{-1/2}\,, \tag{19}$$

$$\sum_{n=1}^{\infty} \frac{\big(1-e^{-(n\pi)^2\Delta t}\big)^2}{(n\pi)^4} \le \Delta t^{3/2}\,. \tag{20}$$

The following propositions are useful in the error analysis:

Proposition 3.1. *The expression* $\max_{\mathbb{D}_T}\Big(\mathbb{E}u^2\Big)$ *is bounded for* T *sufficiently small.*

Proof. Using (5), the Cauchy Schwarz inequality, and the Itô isometry property (7),

$$\mathbb{E}u^2(t,x) \le 3R + 3\int_0^t\int_0^1 G^2(t-s,x,y)\mathbb{E}\beta^2(s,y,u(s,y))dyds$$
$$+3\int_0^t\int_0^1 G^2(t-s,x,y)dyds \times \int_0^t\int_0^1 \mathbb{E}\alpha^2(s,y,u(s,y),v(s,y))dyds,$$

where

$$\begin{aligned}
R =& \max_{\mathbb{D}_T}\left(\int_0^1 G(t,x,y)u_0(y)dy\right)^2 \\
=& \max_{\mathbb{D}_T}\left(\int_0^1 \Big(1+2\sum_{m=1}^{\infty} e^{-(m\pi)^2 t}\cos(m\pi x)\cos(m\pi y)\Big)u_0(y)dy\right)^2 \\
\le& \max_{\mathbb{D}_T}\left(\sum_{m=0}^{\infty} e^{-2(m\pi)^2 t}\cos^2(m\pi x)\right)\left(\sum_{m=0}^{\infty} u_{0,m}^2\right)
\end{aligned}$$

$$\begin{aligned}
\le& \max_{\mathbb{D}_T}\left(1+\int_0^{\infty} e^{-2(x\pi)^2 t}dx\right)\left(2\int_0^1 u^2(0,x)dx\right) \\
<& \infty \qquad \text{for} \quad t>0\,,
\end{aligned}$$

and where $u_{0,0} = \int_0^1 u_0(y)dy$, $u_{0,m} = 2\int_0^1 \cos(m\pi y)u_0(y)dy$ for $m = 1,2,\dots,$ and we have used Parseval's identity in the last step. Note that for $t = 0$, $R = \max_{0\le x\le 1} u_0^2(x)$. Continuing from above,

$$\mathbb{E}u^2(t,x) \le 3R$$
$$+\,3\,\widehat{G}\,T\max_{\mathbb{D}_T}\mathbb{E}\left(\alpha^2(s,y,u(s,y),v(s,y)) + \beta^2(s,y,u(s,y))\right)$$

where

$$\begin{aligned}
\widehat{G} =& \int_0^t\int_0^1 G^2(t-s,x,y)dyds \\
=& t + \sum_{m=1}^{\infty}\frac{1}{(m\pi)^2}\Big(1-e^{-2(m\pi)^2 t}\Big)\cos^2(m\pi x) \\
\le& T + T^{1/2}\,,
\end{aligned}$$

and where in the last step (17) was used. Notice that

$$\begin{aligned}\mathbb{E}v^2(s,y) =&\mathbb{E}\left(\int_0^s f(s-z)u(z,y)dz\right)^2\\ \le&\mathbb{E}\Big(\int_0^s f^2(s-z)dz\int_0^s u^2(z,y)dz\Big)\\ \le&\hat{f}\,T\max_{\mathbb{D}_T}\Big(\mathbb{E}u^2\Big)\,, \end{aligned} \tag{21}$$

where $\hat{f}=\int_0^T f^2(t)dt$. Combining this result with C-3, yields

$$\mathbb{E}\alpha^2(s,y,u(s,y),v(s,y)) \le c_\alpha\Big(1+\big(1+\hat{f}T\big)\max_{\mathbb{D}_T}\mathbb{E}u^2\Big). \tag{22}$$

From C-7, the similar result

$$\mathbb{E}\beta^2(s,y,u(s,y)) \le c_\beta\Big(1+\max_{\mathbb{D}_T}\mathbb{E}u^2\Big) \tag{23}$$

is obtained so that finally

$$\max_{\mathbb{D}_T}\mathbb{E}u^2(t,x) \le \frac{3\,R+3\,\widehat{G}\big(c_\alpha\,T+c_\beta\big)}{1-3\,\widehat{G}\big(c_\alpha\,T+c_\alpha\,T^2\,\hat{f}\,c_\beta\big)}\,,$$

which is finite for T sufficiently small. □

Proposition 3.2.

$$\mathbb{E}\Big(u(t,x+\Delta x)-u(t,x)\Big)^2 \le \Big(k_1+k_2\max_{\mathbb{D}_T}\mathbb{E}u^2\Big)\Delta x$$

Proof. Again, using (5), the Cauchy Schwarz inequality, and the Itô isometry property (7),

$$\begin{aligned}&\mathbb{E}\Big(u(t,x)+\Delta x)-u(t,x)\Big)^2\\ &\le 3\left(\int_0^1\Big(G(t,x+\Delta x,y)-G(t,x,y)\Big)u_0(y)dy\right)^2\\ &+3\int_0^t\int_0^1\Big(G(t-s,x+\Delta x,y)-G(t-s,x,y)\Big)^2dyds\\ &\times\int_0^t\int_0^1\mathbb{E}\alpha^2(x,y,u(x,y),v(x,y))dyds\\ &+3\int_0^t\int_0^1\Bigg(G(t-s,x+\Delta x,y)-G(t-s,x,y)\Big)^2\mathbb{E}\beta^2(x,y,u(s,y))dyds\,,\end{aligned}$$

where

$$\int_0^t \int_0^1 \Big(G(t-s, x+\Delta x, y) - G(t-s, x, y)\Big)^2 dyds$$
$$= \sum_{m=1}^{\infty} \frac{1-e^{-2(m\pi)^2 t}}{(m\pi)^2} \Big(\cos(m\pi(x+\Delta x)) - \cos(m\pi x)\Big)^2 \leq 2\Delta x \quad (24)$$

by (18), and

$$\left(\int_0^1 \Big(G(t, x+\Delta x, y) - G(t, x, y)\Big) u_0(y) dy\right)^2$$
$$= \left(2\sum_{m=1}^{\infty} e^{-(m\pi)^2 t} \Big(\cos(m\pi(x+\Delta x)) - \cos(m\pi x)\Big) u_{0,m}\right)^2$$
$$\leq 4\sum_{m=1}^{\infty} e^{-2(m\pi)^2 t} \frac{\Big(\cos(m\pi(x+\Delta x)) - \cos(m\pi x)\Big)^2}{(m\pi)^4} \times \sum_{m=0}^{\infty} u_{0,m}^2 (m\pi)^4$$
$$= 4\sum_{m=1}^{\infty} \frac{e^{-2(m\pi)^2 t}}{(m\pi)^4} \Big(\cos(m\pi(x+\Delta x)) - \cos(m\pi x)\Big)^2$$
$$\times\, 2\int_0^1 \big(u_0''(y)\big)^2 dy \leq 16\Delta x \int_0^1 \big(u_0''(y)\big)^2 dy\,,$$

where (18) was used in the last step. Notice from (22) that

$$\mathbb{E}\int_0^t \int_0^1 \alpha^2(s, y, u(s,y), v(s,y)) dyds$$
$$\leq c_\alpha\, T\Big(1 + \big(1 + \hat{f}T\big)\Big) \max_{\mathbb{D}_T} \mathbb{E}u^2\,, \quad (25)$$

and from (23) and (24),

$$\int_0^t \int_0^1 \left(G(t-s, x+\Delta x, y) - G(t-s, x, y)\right)^2 \mathbb{E}\beta^2(x, y, u(s,y)) dyds$$
$$\leq 2c_\beta \big(1 + \max_{\mathbb{D}_T} \mathbb{E}u^2\big)\Delta x\,, \quad (26)$$

so that finally

$$\begin{aligned}\mathbb{E}\Big(u(t,x{+}\Delta x)-u(t,x)\Big)^2 \\ \le & 48\Delta x\int_0^1 \big(u_0(y)\big)^2 dy + 6\Delta x c_\alpha\, T\Big(1+\big(1+\hat f\,T\big)\,\max_{\mathbb{D}_T}\mathbb{E}u^2\Big) \\ & + 6c_\beta\Delta x\left(1+\max_{\mathbb{D}_T}\mathbb{E}u^2\right) \\ \le & \Big(k_1 + k_2\max_{\mathbb{D}_T}\mathbb{E}u^2\Big)\Delta x. \qquad \square\end{aligned}$$

Proposition 3.3.

$$\mathbb{E}\Big(u(t+\Delta t,x)-u(t,x)\Big)^2 \le \Big(k_3 + k_4\max_{\mathbb{D}_T}\mathbb{E}u^2\Big)\Delta t^{1/2}$$

Proof. The proof is similar in structure to the previous proof. See [15] for details. □

3.1. Analysis of the Quadrature Method

Recall the difference scheme to solve (5):

$$\begin{aligned}u_{i,j} = & F_{ij} + \sum_{n=0}^{i-1}\sum_{k=0}^{M-1} G(t_i-t_n,x_j,y_k)\alpha(t_n,y_k,u_{n,k},v_{n,k})\Delta x\Delta t \\ & + \sum_{n=0}^{i-1}\sum_{k=0}^{M-1} G(t_i-t_n,x_j,y_k)\beta(t_n,y_k,u_{n,k})\eta_{nk}\sqrt{\Delta x\Delta t},\end{aligned} \tag{27}$$

where $F_{ij} = \int_0^1 G(t_i,x_j,y)u_0(y)dy$ is assumed to be integrated exactly, and $\eta_{nk}\in N(0,1)$.

Let $\varepsilon_{i,j} = u(t_i,x_j) - u_{i,j}$. Convergence is proved in the mean square sense. Thus,

$$\begin{aligned}
\mathbb{E}(\varepsilon_{i+1,j}^2) &= \mathbb{E}\Bigg(\Big(u(t_{i+1},x_j)-u_{i+1,j}\Big)^2\Bigg)\\
=&\mathbb{E}\Bigg(\sum_{n=0}^{i}\sum_{k=0}^{L-1}\int_{t_n}^{t_{n+1}}\int_{y_k}^{y_{k+1}} G(t_{i+1}-s,x_j,y)\alpha(s,y,u(s,y),v(s,y))\\
&\quad - G(t_i-s_n,x_j,y_k)\alpha(s_n,y_k,u_{n,k},v_{n,k})dyds\\
&\quad + \sum_{n=0}^{i}\sum_{k=0}^{L-1}\int_{t_n}^{t_{n+1}}\int_{y_k}^{y_{k+1}} G(t_{i+1}-s,x_j,y)\beta(s,y,u(s,y))\\
&\quad - G(t_i-s_n,x_j,y_k)\beta(s_n,y_k,u_{n,k})dW(y,s)\Bigg)^2\\
\leq& B_1+B_2+B_3+B_4,
\end{aligned}$$

where

$$\begin{aligned}
B_1 = 4T\sum_{n=0}^{i}\sum_{k=0}^{L-1}\int_{t_n}^{t_{n+1}}\int_{y_k}^{y_{k+1}}\Big(G(t_{i+1}-s,x_j,y)-G(t_{i+1}-s_n,x_j,y_k)\Big)^2\\
\times\,\mathbb{E}\alpha^2(s,y,u(s,y),v(s,y))dyds\,,
\end{aligned}$$

$$\begin{aligned}
B_2 = 4T\sum_{n=0}^{i}\sum_{k=0}^{L-1}\int_{t_n}^{t_{n+1}}\int_{y_k}^{y_{k+1}} G^2(t_{i+1}-s_n,x_j,y_k)\\
\times\,\mathbb{E}\Big(\alpha(s,y,u(s,y),v(s,y))-\alpha(s_n,y_k,u_{n,k},v_{n,k})\Big)^2 dyds\,,
\end{aligned}$$

$$\begin{aligned}
B_3 = 4\sum_{n=0}^{i}\sum_{k=0}^{L-1}\int_{t_n}^{t_{n+1}}\int_{y_k}^{y_{k+1}}\Big(G(t_{i+1}-s,x_j,y)-G(t_{i+1}-s_n,x_j,y_k)\Big)^2\\
\times\,\mathbb{E}\beta^2(s,y,u(s,y))dyds,
\end{aligned}$$

$$\begin{aligned}
B_4 = 4\sum_{n=0}^{i}\sum_{k=0}^{L-1}\int_{t_n}^{t_{n+1}}\int_{y_k}^{y_{k+1}} G^2(t_{i+1}-s_n,x_j,y_k)\\
\times\,\mathbb{E}\Big(\beta(s,y,u(s,y))-\beta(s_n,y_k,u_{n,k})\Big)^2 dyds.
\end{aligned}$$

Used in the above are the Cauchy Schwarz inequality and the Itô isometry property (7).

These four terms can now be analyzed, noting that the analysis of B_3 is similar to that of B_1 and the analysis of B_4 is similar to that of B_2. The term B_2 is considered here. An analysis of the term B_1 is given in [15]. We have that

$$\begin{aligned}B_2 = 4T \sum_{n=0}^{i} \sum_{k=0}^{L-1} \int_{t_n}^{t_{n+1}} \int_{y_k}^{y_{k+1}} & G^2(t_{i+1} - s_n, x_j, y_k) \\ & \times \mathbb{E}\bigg(\alpha(s, y, u(s,y), v(s,y)) - \alpha(s_n, y_k, u_{n,k}, v_{n,k}) \bigg)^2 dyds \\ \leq B_{21} + B_{22} + B_{23},&\end{aligned}$$

where

$$\begin{aligned}B_{21} =& 12T \sum_{n=0}^{i} \sum_{k=0}^{L-1} \int_{t_n}^{t_{n+1}} \int_{y_k}^{y_{k+1}} G^2(t_{i+1} - t_n, x_j, y_k) \\ & \times \mathbb{E}\bigg(\alpha(s, y, u(s,y), v(s,y)) - \alpha(t_n, y_k, u(s,y), v(s,y)) \bigg)^2 dyds\end{aligned}$$

$$\begin{aligned}B_{22} = 12T \sum_{n=0}^{i} \sum_{k=0}^{L-1} \int_{t_n}^{t_{n+1}} \int_{y_k}^{y_{k+1}} G^2(t_{i+1} - t_n, x_j, y_k) \\ \times \mathbb{E}\bigg(\alpha(t_n, y_k, u(s,y), v(s,y)) - \alpha(t_n, y_k, u(t_n,y_k), v(t_n,y_k)) \bigg)^2 dyds\end{aligned}$$

$$\begin{aligned}B_{23} = 12T \sum_{n=0}^{i} \sum_{k=0}^{L-1} \int_{t_n}^{t_{n+1}} \int_{y_k}^{y_{k+1}} G^2(t_{i+1} - t_n, x_j, y_k) \\ \times \mathbb{E}\bigg(\alpha(t_n, y_k, u(t_n,y_k), v(t_n,y_k)) - \alpha(t_n, y_k, u_{n,k}, v_{n,k}) \bigg)^2 dyds.\end{aligned}$$

For B_{21}, the condition C-11 is applied and thus,

$$\begin{aligned}B_{21} \leq& 12T \max\{c_{\alpha_1}, c_{\alpha_2}, c_{\alpha_3}\}\big(\Delta t^2 + \Delta t \Delta x + \Delta x^2\big) \\ & \times \sum_{n=0}^{i} \sum_{k=0}^{L-1} \int_{t_n}^{t_{n+1}} \int_{y_k}^{y_{k+1}} G^2(t_{i+1} - t_n, x_j, y_k) dyds\,,\end{aligned}$$

where

$$
\begin{aligned}
&\sum_{n=0}^{i}\sum_{k=0}^{L-1}\int_{t_n}^{t_{n+1}}\int_{y_k}^{y_{k+1}} G^2(t_{i+1}-t_n,x_j,y_k)dyds \\
&\le 2\sum_{n=0}^{i}\sum_{k=0}^{L-1}\int_{t_n}^{t_{n+1}}\int_{y_k}^{y_{k+1}} \Big(G(t_{i+1}-t_n,x_j,y_k)-G(t_{i+1}-t_n,x_j,y)\Big)^2 dyds \\
&\quad + 2\sum_{n=0}^{i}\sum_{k=0}^{L-1}\int_{t_n}^{t_{n+1}}\int_{y_k}^{y_{k+1}} G^2(t_{i+1}-t_n,x_j,y)dyds \\
&\le 2c_7\frac{\Delta x^2}{\Delta t} + 2\sum_{n=0}^{i}\Delta t\Big(1+2\sum_{m=1}^{\infty} e^{-2(m\pi)^2(t_{i+1}-t_n)}\cos(m\pi x_j)\Big) \\
&\le 2c_7\frac{\Delta x^2}{\Delta t} + 2T + 2\sum_{m=1}^{\infty}\frac{1-e^{-2(m\pi)^2 t_{i+1}}}{(m\pi^2)} \\
&\le 2c_7\frac{\Delta x^2}{\Delta t} + 2T + 2\sqrt{2T},
\end{aligned}
\tag{28}
$$

where (17) was used. Thus,

$$
B_{21} \le 12T\max\{c_{\alpha_1},c_{\alpha_2},c_{\alpha_3}\}\big(\Delta t^2+\Delta t\Delta x+\Delta x^2\big) \times \left(2c_7\frac{\Delta x^2}{\Delta t}+2T+2\sqrt{2T}\right).
$$

Now, according to C-8 and (28),

$$
\begin{aligned}
B_{22} \le& 12TL_\alpha \max_{\mathbb{D}_T}\mathbb{E}\Big(\big(u(s,y)-u(t_n,y_k)\big)^2+\big(v(s,y)-v(t_n,y_k)\big)^2\Big) \\
&\times \sum_{n=0}^{i}\sum_{k=0}^{L-1}\int_{t_n}^{t_{n+1}}\int_{y_k}^{y_{k+1}} G^2(t_{i+1}-t_n,y_j,y_k)dyds \\
\le& 12TL_\alpha \max_{\mathbb{D}_T}\mathbb{E}\Big(\big(u(s,y)-u(t_n,y_k)\big)^2+\big(v(s,y)-v(t_n,y_k)\big)^2\Big) \\
&\times \Big(2c_7\frac{\Delta x^2}{\Delta t}+2T+2\sqrt{2T}\Big).
\end{aligned}
$$

Propositions (3.2) and (3.3) therefore give

$$
\mathbb{E}\Big(u(s,y)-u(t_n,y_k)\Big)^2 \le \Big(k_5+k_6\max_{\mathbb{D}_T}\mathbb{E}u^2\Big)\Big(\Delta x+\Delta t^{1/2}\Big).
$$

To consider the second term above, notice that

$$\mathbb{E}\Big(v(s,y)-v(t_n,y_k)\Big)^2 \leq 2\mathbb{E}\Big(v(s,y)-v(s,y_k)\Big)^2 + 2\mathbb{E}\Big(v(s,y_k)-v(t_n,y_k)\Big)^2,$$

where by C-9, the Cauchy Schwarz inequality, and Proposition (3.2),

$$\begin{aligned} 2\mathbb{E}\Big(v(s,y)-v(s,y_k)\Big)^2 =& 2\mathbb{E}\Big(\int_0^s f(s-z)\big(u(z,y)-u(z,y_k)\big)dz\Big)^2 \\ \leq& 2\hat{f}T\Big(k_1+k_2\max_{\mathbb{D}_T}\mathbb{E}u^2\Big)\Delta x\,, \end{aligned}$$

and where

$$\begin{aligned} &2\mathbb{E}\Big(v(s,y_k)-v(t_n,y_k)\Big)^2 \\ =&2\mathbb{E}\Big(\int_0^s f(s-z)u(z,y_k)dz-\int_0^{t_n} f(t_n-z)u(z,y_k)dz\Big)^2 \\ \leq&4\mathbb{E}\Big(\int_0^{s_n}\big(f(s-z)-f(t_n-z)\big)u(z,y_k)dz\Big)^2 \\ &+\Big(\int_0^{t_n} f(t_n-z)u(z,y_k)dz\Big)^2 \\ \leq&4T^2\max_{[0,T]}\big|f'(t)\big|^2\Delta t^2\max_{\mathbb{D}_T}\mathbb{E}u^2+4\Delta t^2\max_{[0,T]}f^2(t)\max_{\mathbb{D}_T}\mathbb{E}u^2 \end{aligned} \tag{29}$$

by the Cauchy Schwarz inequality. Thus,

$$\begin{aligned} B_{22} \leq& 12TL_\alpha\Big(k_7+k_8\max_{\mathbb{D}_T}\mathbb{E}u^2\Big)\Big(\Delta x+\Delta t^{1/2}\Big) \\ &\times\Big(2c_7\frac{\Delta x^2}{\Delta t}+2T+2\sqrt{2T}\Big) \\ \leq&\Big(k_9+k_{10}\max_{\mathbb{D}_T}\mathbb{E}u^2\Big)\Big(\Delta x+\Delta t^{1/2}+\frac{\Delta x^2}{\Delta t}\Big). \end{aligned}$$

The bound on B_{23} is obtained similarly yielding

$$\begin{aligned} B_{23} \leq&\Big(\frac{\Delta x^2}{\Delta t^{1/2}}+\Delta t^{1/2}\Big)\Big(k_{13}+k_{14}\max_{\mathbb{D}_T}\mathbb{E}u^2\Big) \\ &+TL_\alpha\Big(k_{15}\frac{\Delta x^2}{\Delta t}+k_{16}\Big)\max_{n,k}\mathbb{E}\big(u(t_n,y_k)-u_{nk}\big)^2. \end{aligned}$$

Finally, using the above results, B_2 is bounded by:

$$\begin{aligned} B_2 \leq & B_{21} + B_{22} + B_{23} \\ \leq & \Big(\Delta t^{1/2} + \Delta x + \frac{\Delta x^2}{\Delta t}\Big)\Big(k_{17} + k_{18} \max_{\mathbb{D}_T} \mathbb{E} u^2\Big) \\ & + T L_\alpha \Big(k_{15} \frac{\Delta x^2}{\Delta t} + k_{16}\Big) \max_{n,k} \mathbb{E}\Big(u(t_n, y_k) - u_{nk}\Big)^2. \end{aligned}$$

The above results yield the following theorem.

Theorem 3.1. *The approximation $u_{i,j}$ to $u(t_i, x_j)$ given by (27) satisfies the following inequality provided that T is sufficiently small:*

$$\max_{i,j} \mathbb{E}\Big(u(t_{i+1}, x_j) - u_{i+1,j}\Big)^2 \leq \frac{\Big(C_1 + C_2 \max_{\mathbb{D}_T} \mathbb{E} u^2\Big)\Big(\Delta t^{1/2} + \Delta x + \frac{\Delta x^2}{\Delta t}\Big)}{1 - T L_\alpha \Big(C_3 \frac{\Delta x^2}{\Delta t} + C_4\Big)}.$$

Thus, $\max_{1 \leq i \leq N, 1 \leq j \leq L} \mathbb{E}\Big(u(t_{i+1}, x_j) - u_{i+1,j}\Big)^2 \longrightarrow 0$ *as* Δx, Δt, *and* $\frac{\Delta x^2}{\Delta t} \longrightarrow 0$.

3.2. Analysis of the Semi-Discrete Galerkin Method

Consider the exact Fourier expansion for u (i.e. the solution to (4)):

$$u(t, x) = \sum_{m=0}^{\infty} u_m(t) \cos(m \pi x). \tag{30}$$

For convenience in the analysis of this method, an approximation to the exact u using the first M terms of the series is set equal to

$$u_a(t, x) = \sum_{m=0}^{M} u_m(t) \cos(m \pi x), \tag{31}$$

and the Galerkin approximation is set equal to

$$\hat{u}(t_i, x) = \sum_{m=0}^{M} u_{mi} \cos(m \pi x), \tag{32}$$

where u_{mi} satisfy equation (15). As the tail of the series for u goes to zero (to be proved later), $\hat{u}$ will be compared with u_a in the analysis. It is assumed in the analysis of this method that $\beta(t,x,u(t,x)) = \beta(t,x)$, that is, β is independent of u. First, the following useful result is proved.

Lemma 3.1. $\sum_{m=0}^{\infty} \mathbb{E}u_m^2(t) < \infty$ *for* $0 \le t \le T$.

Proof. From (14) we see that for $m \ge 1$,

$$\begin{aligned}\mathbb{E}u_m^2(t) = \mathbb{E}\Bigg(& u_m(0)e^{-(m\pi)^2 t} + 2\int_0^t \int_0^1 \widehat{G}_m(t-s,y)\beta(s,y)dW(s,y) \\ &+ 2\int_0^t \int_0^1 \widehat{G}_m(t-s,y)\alpha(s,y,u(x,y),v(s,y))dyds\Bigg)^2 \\ &\le 3u_m^2(0)e^{-2(m\pi)^2 t} \\ &+ 12\mathbb{E}\int_0^t \int_0^1 \widehat{G}_m^2(t-s,y)dyds \\ &\times \int_0^t \int_0^1 \alpha^2(s,y,u(x,y),v(s,y))dyds \\ &+ 12\mathbb{E}\int_0^t \int_0^1 \widehat{G}_m^2(t-s,y)\beta^2(s,y)dyds\end{aligned}$$

where $\widehat{G}_m(t-s,y) = e^{-(m\pi)^2(t-s)}\cos(m\pi y)$ so using (22) and (23),

$$\begin{aligned}\mathbb{E}u_m^2(t) \le & 3u_m^2(0)e^{-2(m\pi)^2 t} + 6c_\beta\Big(1 + \max_{\mathbb{D}_T}\mathbb{E}u^2\Big)\int_0^t e^{-2(m\pi)^2(t-s)}ds \\ &+ 6c_\alpha T\Big(1 + \big(1+\hat{f}T\big)\max_{\mathbb{D}_T}\mathbb{E}u^2\Big)\int_0^t e^{-2(m\pi)^2(t-s)}ds\,,\end{aligned}$$

where $\int_0^t e^{-2(m\pi)^2(t-s)}ds \le \dfrac{1}{2(m\pi)^2}$. Thus, the result follows. □

The preceding lemma allows us to compare $\hat{u}(t,x)$ with $u_a(t,x)$ instead of with the exact solution $u(t,x)$, since it implies that

$$\mathbb{E}\int_0^1 \Big(u_a(t,x) - u(t,x)\Big)^2 dx = \sum_{m=M+1}^{\infty} \mathbb{E}u_m^2(t) \to 0\,,$$

as $M \to \infty$.

In this analysis it is shown that the expectation

$$\mathbb{E}\int_0^1 \Big(u_a(t_i,x) - \hat{u}(t_i,x)\Big)^2 dx = \sum_{m=0}^{M} c_m \mathbb{E}\Big(u_m(t_i) - u_{mi}\Big)^2$$

goes to zero under certain conditions as $\Delta t, \Delta x \to 0$. Throughout this analysis, let

$$c_m = \int_0^1 \cos^2(m\pi x)dx = \begin{cases} 1 & \text{for } m = 0\,, \\ 1/2 & \text{for } m = 1, 2, \dots\,, \end{cases} \tag{33}$$

and

$$v_a(t,x) = \int_0^t f(t-s)u_a(s,x)ds.$$

Now, with $\varepsilon_{mi} = c_m \mathbb{E}\big(u_m(t_i) - u_{mi}\big)^2$, one obtains

$$\begin{aligned}
\varepsilon_{mi} = \mathbb{E}\Bigg(&\sum_{n=0}^{i-1}\int_{t_n}^{t_{n+1}}\int_0^1 \widehat{G}_m(t_i - s, y)\alpha(s, y, u_a(s,y), v_a(s,y)) \\
&- \widehat{G}_m(t_i - t_n, y)\alpha(t_n, y, \hat{u}(t_n, y), \hat{v}(t_n, y))dyds \\
&\sum_{n=0}^{i-1}\sum_{k=0}^{L-1}\int_{t_n}^{t_{n+1}}\int_{y_k}^{y_{k+1}} \widehat{G}_m(t_i - s, y)\beta(s, y) \\
&- \widehat{G}_m(t_i - t_n, y_k)\beta(t_n, y_k)dW(s,y)\Bigg)^2 \\
\leq\; & 4(D_1 + D_2 + D_3 + D_4)
\end{aligned}$$

using the Cauchy Schwartz inequality and the Itô isometry property, where

D_1, D_2, D_3, and D_4 depend on m and are given by

$$D_1 = T\sum_{n=0}^{i-1}\int_{t_n}^{t_{n+1}}\int_0^1\Big(\widehat{G}_m(t_i-s,y)-\widehat{G}_m(t_i-t_n,y)\Big)^2 \times \mathbb{E}\alpha^2(s,y,u_a(s,y),v_a(s,y))dyds$$

$$D_2 = T\sum_{n=0}^{i-1}\int_{t_n}^{t_{n+1}}\int_0^1 \widehat{G}_m^2(t_i-t_n,y) \times \mathbb{E}\Bigg(\alpha(s,y,u_a(s,y),v_a(s,y))-\alpha(t_n,y,\hat{u}(t_n,y),\hat{v}(t_n,y))\Bigg)^2 dyds$$

$$D_3 = \sum_{n=0}^{i-1}\sum_{k=0}^{L-1}\int_{t_n}^{t_{n+1}}\int_{y_k}^{y_{k+1}}\Big(\widehat{G}_m(t_i-s,y)-\widehat{G}_m(t_i-t_n,y_k)\Big)^2\beta^2(s,y)dyds$$

$$D_4 = \sum_{n=0}^{i-1}\sum_{k=0}^{L-1}\int_{t_n}^{t_{n+1}}\int_{y_k}^{y_{k+1}}\widehat{G}_m^2(t_i-t_n,y_k)\Big(\beta(s,y)-\beta(t_n,y_k)\Big)^2 dyds.$$

An analysis of these four terms (see [15]) results in the inequality:

$$\sum_{m=0}^{M}\varepsilon_{mi} \le k_{19}\Delta t^{1/2}+k_{20}\Delta x+Tk_{21}\Big(\sum_{k=0}^{M}\varepsilon_{kn}\Big).$$

The following theorem has been proved for the Galerkin method.

Theorem 3.2. *The approximation u_a to u given by (31) satisfies the following inequality, where $\hat{u}$ is given by (32):*

$$\mathbb{E}\int_0^1\Big(u_a(t_i,x)-\hat{u}(t_i,x)\Big)^2dx = \sum_{m=0}^{M}\varepsilon_{mi} \le \frac{k_{19}\Delta t^{1/2}+k_{20}\Delta x}{1-Tk_{21}},$$

which goes to zero as Δt, and $\Delta x \longrightarrow 0$ for T sufficiently small.

In the next section, three numerical examples are described. These examples show that these two numerical procedures can accurately approximate the solution of (5).

4. Computational Examples

Example Problem 1. Consider first a simple numerical example to test the two numerical methods. Let

$$\begin{cases} \alpha(t,x,u(t,x),v(t,x)) = 0\,, \\ \beta(t,x,u(t,x)) = 1\,, \\ u_0(x) = 0\,. \end{cases}$$

The exact solution for this problem is

$$u(t,x) = \int_0^t \int_0^1 G(t-s,x,y)dW(s,y), \tag{34}$$

where G is given by equation (6). Manthey [9] studied properties of the stochastic integral solution to (4) with the conditions above for an infinite domain (i.e. $-\infty < x < \infty$):

$$u(t,x) = \int_0^t \int_{\mathbb{R}} G(t-s,x,y)dW(s,y).$$

For this solution, the mean is 0, and the variance is easily computed as

$$\begin{aligned} \sigma^2(t) &= \mathbb{E}(u^2(t,x)) \\ &= \mathbb{E}\Big(\Big[\int_0^t \int_0^1 G(t-s,x,y)dW(s,y)\Big]^2\Big) \\ &= \int_0^t \int_0^1 G^2(t-s,x,y)dyds \\ &= t + 2\sum_{m=1}^{\infty} \frac{1-e^{-2(m\pi)^2 t}}{2(m\pi)^2}\cos^2(m\pi x), \end{aligned}$$

where the Itô isometry property has been used.

With $dx = 0.1$, $dt = 0.05$, and $T = 1$, Figure 1 shows the variance of the two methods as a function of position x for a fixed time $t = 1$. Notice that the two methods agree well for this problem and agree well with the exact known values.

Example Problem 2. In a second numerical example, let

$$\begin{cases} \alpha(t,x,u(t,x),v(t,x)) = u(t,x) - 1 - (12x^2 - 12x + 2)e^t\,, \\ \beta(t,x,u(t,x)) = 1\,, \\ u_0(x) = 1 + x^2(1-x)^2\,. \end{cases}$$

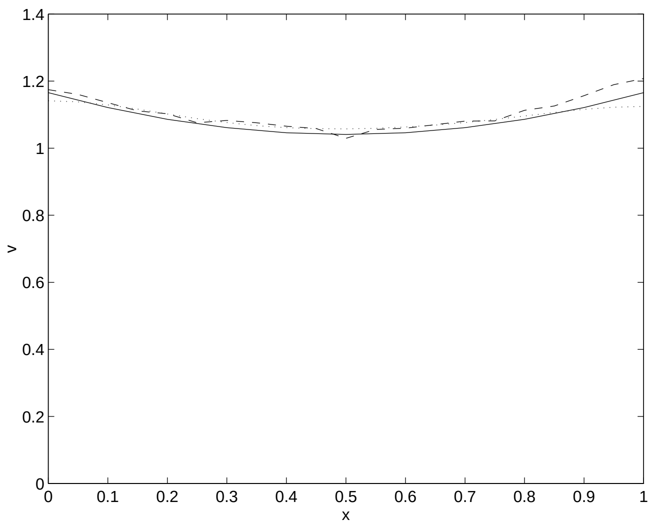

Figure 1: Variance estimation of $u(1, x)$ for Example Problem 1 ($--$ quadrature method, $\cdots$ semi-discrete galerkin method, — exact variance).

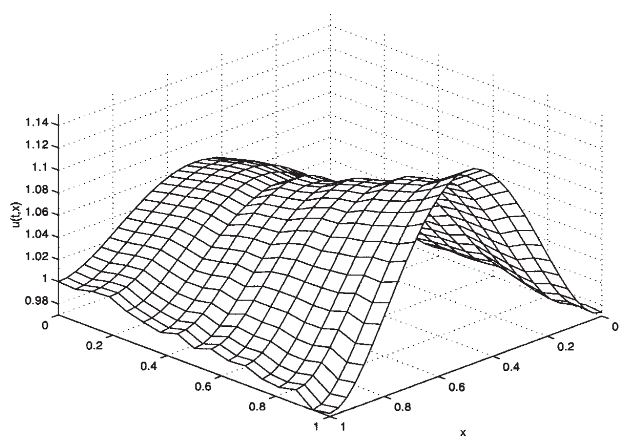

Figure 2: The expectation of $u(t, x)$ for the quadrature method (using 10,000 sample paths for Example Problem 2.)

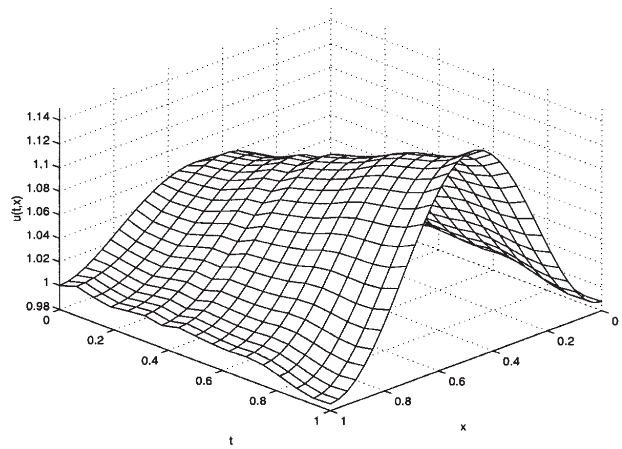

Figure 3: The expectation of $u(t, x)$ using the semi-discrete Galerkin method (with 10,000 sample paths for Example Problem 2).

Notice that the exact solution to the deterministic equation (for $\beta = 0$) is

$$u(t, x) = 1 + x^2(1 - x)^2 e^t,$$

for $u_0(x) = 1 + x^2(1 - x)^2$. The mean solution of the stochastic problem will equal the exact solution of the deterministic equation for this problem. However, the variance is not known exactly for this problem.

With $dx = 0.03125$, $dt = 0.0625$, and $T = 1$, Figure 2 shows the mean of $10,000$ sample paths from the quadrature method. Figure 3 shows the mean of $10,000$ sample paths from the semi-discrete Galerkin method. Figure 4 shows the variance of the two methods for a fixed time $t = 1$. Again the two methods agree very well.

Example Problem 3. As a final numerical example, consider the nonlinear problem:

$$\begin{cases} \alpha(t, x, u(t, x), v(t, x)) = \frac{1}{2}u(t, x) - \frac{1}{2}u^2(t, x)\,, \\ -u(t, x)\int_0^t f(t - s)u(s, x)ds\,, \\ \beta(t, x, u(t, x)) = 1 + u(t, x)\,, \\ u_0(x) = \cos(\pi x) + 1\,, \\ f(t) = e^{-t}\,. \end{cases} \tag{35}$$

The mean of 10,000 sample paths using the quadrature method is plotted in Figure 5, while the mean of 10,000 sample paths using the semi-discrete

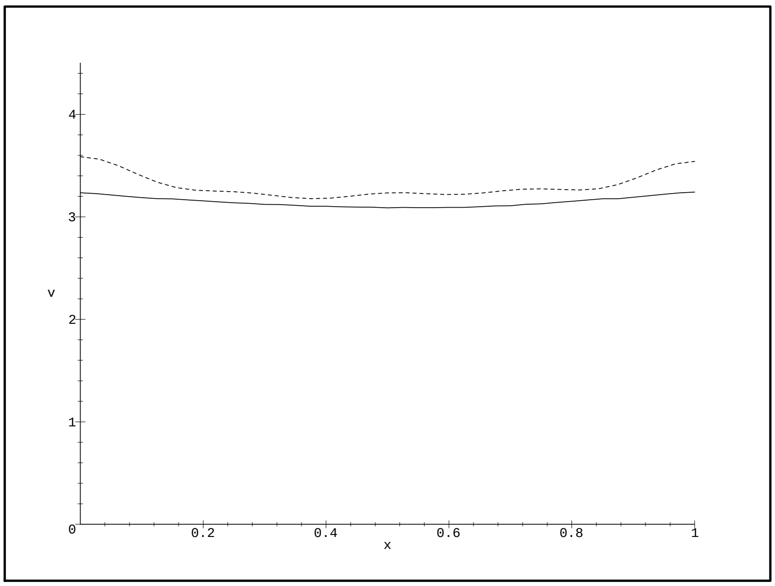

Figure 4: Variance in $u(1,x)$ for Example Problem 2 (— quadrature method, --- semi-discrete Galerkin method).

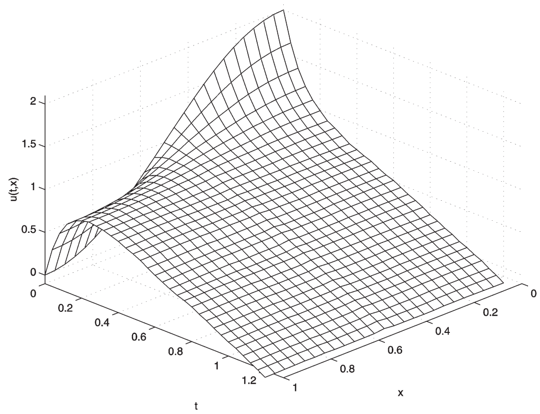

Figure 5: The expectation of $u(t,x)$ for the quadrature method (using 10,000 sample paths for Example Problem 3).

Galerkin method is plotted in Figure 6. The variances of the two methods are again compared in Figure 7 at $t = 1$.

For this problem, the exact solution $u(t,x)$ is not known even for the case $\beta = 0$. However, notice that a constant steady state exists (when $\beta = 0$) and is equal to $\bar{u} = \frac{1}{2}(\frac{1}{2} + \int_0^\infty f(t)dt)^{-1} = \frac{1}{3}$, but that the mean of the stochastic solution dies out when $t \approx 1.2$. Thus, the presence of the randomly varying term causes a persistent population to go extinct in this example. This is also the case when spatial dependence is not considered.

5. Conclusions

The Volterra population equation is a well-known equation used to describe the behavior of populations in time. In this investigation, numerical methods for approximating Volterra's population equation with diffusion and noise were studied. Two independent numerical methods were developed in this investigation: a Galerkin method and a quadrature method. Error analyses were performed on both methods proving convergence of the methods. Computational experiments were performed for three different examples and the numerical solutions for these examples were compared for the two methods. Good agreement was obtained between the two methods.

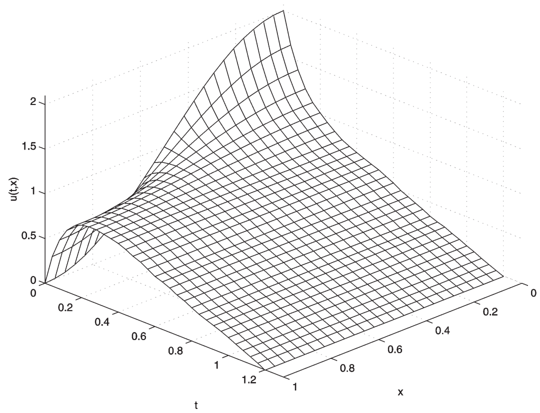

Figure 6: The expectation of $u(t,x)$ for the semi-discrete Galerkin method (using 10,000 sample paths for Example Problem 3).

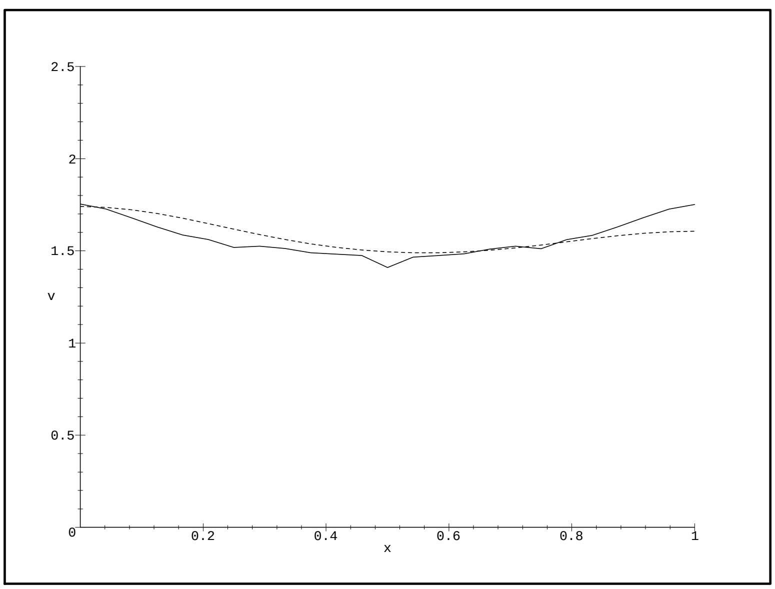

Figure 7: Variance in $u(1,x)$ for Example Problem 3 (— quadrature method, --- semi-discrete galerkin method).

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