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EXISTENCE RESULTS FOR ABSTRACT NEUTRAL INTEGRODIFFERENTIAL EQUATIONS

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Abstract: In this paper we prove the existence of solutions of neutral functional integrodifferential equations in a Banach space. The result is obtained by using the Schaefer fixed point theorem. As an application the controllability problem for the neutral system is discussed.

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1. Introduction

The theory of neutral delay differential equations has been extensively studied by many authors [1, 3, 12, 18-20]. Hernandez and Henriquez [13] obtained existence results for neutral functional differential equations in Banach spaces, and in [14] they have established the existence of periodic solutions for the same kind of equations. In both papers they have used the semigroup theory and the Sadovski fixed point principle. Weak solutions of integrodifferential

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equations and resolvent operators are discussed in [10, 11, 16]. Dhakne and Pachpatte [9] discussed a general class of abstract functional integrodifferential equations. Dauer and Balachandran [8] investigated the existence of solutions of nonlinear neutral integrodifferential equations in Banach spaces. Brill [6] studied the existence problem for semilinear Sobolev type evolution equation in a Banach space. Lightbourne III and Rankin III [15] discussed the same problem for functional differential equations of Sobolev type. Byszewski [7] studied the existence of solution of a semilinear functional differential equation with nonlocal condition. Lin and Liu [17] discussed the nonlocal Cauchy problem for semilinear integrodifferential equations with resolvent operators. Balachandran and Chandrasekaran [2] discussed the existence of solution of a delay differential equation with nonlocal condition. Balachandran, Park and Kwun [4] studied the same problem for nonlinear integrodifferential equations of Sobolev type with nonlocal conditions in Banach space. The purpose of this paper is to prove the existence of mild solutions for neutral functional integrodifferential equations by using the Schaefer fixed point theorem. The results generalise the results of [8, 13, 18-20].

2. Preliminaries

Consider the neutral integrodifferential equation of the form

$$\begin{aligned} \frac{d}{dt}[Ex(t) - g(t, x_t)] &= Ax(t) + \int_0^t f(s, x_s)ds, \quad t \in J = [0, b], \\ x_0 &= \phi, \quad \text{on } [-r, 0], \end{aligned} \tag{1}$$

where E and A are linear operators with domains contained in a Banach space X and ranges contained in a Banach space $Y, f, g : J \times C \to Y$ are continuous functions. Here $C = C([-r, 0], X)$ is a Banach space of all continuous functions $\phi : [-r, 0] \to X$ endowed with the norm $\|\phi\| = \sup\{|\phi(\theta)|; -r \le \theta \le 0\}$. Also for $x \in C([-r, b], X)$ we have $x_t \in C$ for $t \in [0, b], x_t(\theta) = x(t+\theta)$ for $\theta \in [-r, 0]$. The norm of X is denoted by $|\cdot|$ and Y by $\|\cdot\|$.

The operators $A : D(A) \subset X \to Y$ and $E : D(E) \subset X \to Y$ satisfy the hypothesis:

(C_1) A and E are closed linear operators.

(C_2) $D(E) \subset D(A)$ and E is bijective.

(C_3) $E^{-1} : Y \to D(E)$ is continuous.

(C_4) The resolvent $R(\lambda, AE^{-1})$ is a compact operator for some $\lambda \in \rho(AE^{-1})$, the resolvent set of (AE^{-1}).

The hypothesis (C_1) , (C_2) and the closed graph theorem imply the boundedness of the linear operator $AE^{-1} : Y \to Y$.

Lemma. (see [21]) *Let $S(t)$ be a uniformly continuous semigroup and let A be its infinitesimal generator. If the resolvent $R(\lambda : A)$ of A is compact for every $\lambda \in \rho(A)$, then $S(t)$ is a compact semigroup.*

From the above fact, AE^{-1} generates a compact semigroup $T(t), t > 0$ on Y.

Definition 2.1. A solution $x : (-r, b) \to X, b > 0$ is called a mild solution of the Cauchy problem (1) if $x_0 = \phi$, the restriction of $x(\cdot)$ to the interval $[0, b)$, is continuous, if for each $0 \le t < b$ the function $AE^{-1}T(t-s)g(s, x_s), s \in [0, t)$ is integrable, $Ex(t) \in C([0, b]; Y) \cap C^1([0, b]; Y)$ and the integral equation

$$\begin{aligned} x(t) &= E^{-1}T(t)[E\phi(0) - g(0,\phi)] + E^{-1}g(t, x_t) \\ &\quad + \int_0^t E^{-1}AE^{-1}T(t-s)g(s, x_s)ds \\ &\quad + \int_0^t E^{-1}T(t-s)\int_0^s f(\tau, x_\tau)d\tau ds, \quad t \in J, \end{aligned} \tag{2}$$

is satisfied.

(C_5) There exist constants $M_1 \ge 1$ and $M_2 > 0$ such that $\|T(t)\| \le M_1$ and $\|AE^{-1}T(t)\| \le M_2$.

(C_6)For each $t \in J$ the function $f(t, .) : C \to Y$ is continuous and for each $x \in C$ the function $f(., x) : J \to Y$ is strongly measurable.

(C_7) For each positive integer k, there exists $\alpha_k \in L^1(0, b)$ such that

$$\sup_{\|x\| \le k} \|f(t, x)\| \le \alpha_k(t), \quad \text{for } \ t \in J \text{ a.e.}$$

(C_8) The function $g : J \times C \to Y$ is completely continuous and for any bounded set K in $C([-r, b], X)$ the set $\{t \to g(t, x_t) : \ x \in K\}$ is equicontinuous in $C([0, b], X)$.

(C_9) There exists a constant G such that $\|g(t, \phi)\| \le G; \ t \in J, \ \phi \in C$.

(C_{10}) There exists an integrable function $m : [0, b] \to [0, \infty)$such that

$$\|f(t, \phi)\| \leq m(t)\Omega(\|\phi\|), \quad 0 \leq t \leq b, \quad \phi \in C,$$

where $\Omega : [0, \infty) \to (0, \infty)$ is a continuous nondecreasing function.

(C_{11})

$$\int_0^b \hat{m}(s)ds \leq \int_c^\infty \frac{ds}{s + \Omega(s)},$$

where

$$c = M_1|E^{-1}|[\|E\phi(0)\| + G] + |E^{-1}|G + |E^{-1}|M_2 Gb,$$

and

$$\hat{m}(t) = \max\{1, |E^{-1}|M_1 m(t)\}.$$

Schaefer Theorem. (see [22]) *Let E be a normed linear space. Let $F : E \to E$ be a completely continuous operator, that is, it is continuous and the image of any bounded set is contained in a compact set and let*

$$\zeta(F) = \{x \in E \quad : x = \lambda F x \ \text{ for some } \ 0 < \lambda < 1\}.$$

Then either $\zeta(F)$ is unbounded or F has a fixed point.

3. Main Results

Theorem 3.1. *If the assumptions (C_1) to (C_{11}) are satisfied then the problem (1) has a mild solution on $[-r, b]$.*

Proof. Consider a Banach space $Z = C([0, b] : X)$ with norm

$$\|x\|_1 = \sup\{|x(t) : \ -r \leq t \leq b\}.$$

To prove the existence of mild solution of (1), let us consider the equation

$$x(t) = \lambda F x(t), \quad 0 \leq \lambda < 1,$$

where

$$\begin{aligned} Fx(t) = {} & E^{-1}T(t)[E\phi(0) - g(0, \phi)] + E^{-1}g(t, x_t) \\ & + \int_0^t E^{-1}AE^{-1}T(t-s)g(s, x_s)ds \\ & + \int_0^t E^{-1}T(t-s)\int_0^s f(\tau, x_\tau)d\tau ds. \quad (3) \end{aligned}$$

Now

$$\begin{aligned} |x(t)| &= |\lambda Fx(t)| \le |Fx(t)| \\ &\le |E^{-1}|M_1[\|E\phi(0)\| + G] + |E^{-1}|G + \int_0^t |E^{-1}|M_2 G ds \\ &\quad + \int_0^t |E^{-1}|M_1 \int_0^s m(\tau)\Omega(\|x_\tau\|)d\tau ds \\ &\le |E^{-1}|M_1[\|E\phi(0)\| + G] + |E^{-1}|G + |E^{-1}|M_2 Gb \\ &\quad + |E^{-1}|M_1 \int_0^t \int_0^s m(\tau)\Omega(\|x_\tau\|)d\tau ds. \end{aligned}$$

We consider the following function μ given by

$$\mu(t) = \sup\{|x(s)| : \ -r \le s \le t\}, \quad 0 \le t \le b.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |x(t^*)|$. If $t^* \in [0, b]$ by the previous inequality we have

$$\begin{aligned} \mu(t) &\le |E^{-1}|M_1[\|E\phi(0)\| + G] + |E^{-1}|G + |E^{-1}|M_2 Gb \\ &\quad + |E^{-1}|M_1 \int_0^{t^*} \int_0^s m(\tau)\Omega(\mu(\tau))d\tau ds \\ &\le |E^{-1}|M_1[\|E\phi(0)\| + G] + |E^{-1}|G + |E^{-1}|M_2 Gb \\ &\quad + |E^{-1}|M_1 \int_0^t \int_0^s m(\tau)\Omega(\mu(\tau))d\tau ds. \end{aligned}$$

If $t^* \in [-r, 0]$, then $\mu(t) = \|\phi\|$ and the above inequality holds since $M_1 \ge 1$. Denoting $v(t)$ the right hand side of the above, we have

$$v(0) = c, \quad \mu(t) \le v(t), \quad 0 \le t \le b,$$

and

$$\begin{aligned} v'(t) &= |E^{-1}|M_1 \int_0^t m(s)\Omega(\mu(s))ds \\ &\le M_1|E^{-1}| \int_0^t m(s)\Omega(v(s))ds. \end{aligned}$$

Let

$$w(t) = v(t) + M_1|E^{-1}| \int_0^t m(s)\Omega(v(s))ds.$$

Then $w(0) = v(0),$ and $v(t) \leq w(t)$. Now

$$\begin{aligned} w'(t) &= v'(t) + M_1|E^{-1}|m(t)\Omega(v(t)) \\ &\leq M_1|E^{-1}| \int_0^t m(s)\Omega(v(s))ds + M_1|E^{-1}|m(t)\Omega(v(t)) \\ &\leq w(t) + M_1|E^{-1}|m(t)\Omega(w(t)) \\ &\leq \hat{m}(t)\{w(t) + \Omega(w(t))\}, \quad 0 \leq t \leq b. \end{aligned}$$

This implies that

$$\int_{w(0)}^{w(t)} \frac{ds}{s + \Omega(s)} \leq \int_0^b \hat{m}(s)ds < \int_c^\infty \frac{ds}{s + \Omega(s)}, \quad 0 \leq t \leq b.$$

This inequality implies that there is a constant K such that $v(t) \leq K, t \in [0, b]$ and hence $\mu(t) \leq K,\ t \in [0, b]$. Since $\|x_t\| \leq \mu(t)$, for all $t \in [0, b]$, we have

$$\|x\|_1 = \sup\{|x(t)| : -r \leq t \leq b\} \leq K,$$

where K depends only on b and on the functions $\hat{m}$, and Ω.

We shall now prove that the operator $\Phi : Z \to Z$ defined by

$$\begin{aligned} (\Phi x)(t) &= \phi(t), \quad t \in [-r, 0], \\ (\Phi x)(t) &= E^{-1}T(t)[E\phi(0) - g(0, \phi)] + E^{-1}g(t, x_t) \\ &\quad + \int_0^t E^{-1}AE^{-1}T(t-s)g(s, x_s)ds \\ &\quad + \int_0^t E^{-1}T(t-s) \int_0^s f(\tau, x_\tau)d\tau ds, \quad t \in J \end{aligned}$$

is a completely continuous operator.

Let $B_k = \{x \in Z : \|x\|_1 \leq k\}$ for some $k \geq 1$. We first show that Φ maps B_k into an equicontinuous family. Let $x \in B_k$ and $t_1, t_2 \in [0, b]$. Then if

$0 < t_1 < t_2 \leq b$,

$$\begin{aligned}
&\|(\Phi x)(t_1) - (\Phi x)(t_2)\| \\
&\quad \leq \|E^{-1}[T(t_1) - T(t_2)][E\phi(0) - g(0,\phi)]\| \\
&\qquad + \|E^{-1}[g(t_1, x_{t_1}) - g(t_2, x_{t_2})]\| \\
&\qquad + \| \int_0^{t_1} E^{-1}AE^{-1}T(t_1 - s)g(s, x_s)ds \\
&\qquad - \int_0^{t_2} E^{-1}AE^{-1}T(t_2 - s)g(s, x_s)ds\| \\
&\qquad + \| \int_0^{t_1} E^{-1}T(t_1 - s) \int_0^s f(\tau, x_\tau)d\tau ds \\
&\qquad - \int_0^{t_2} E^{-1}T(t_2 - s) \int_0^s f(\tau, x_\tau)d\tau ds\| \\
&\quad \leq |E^{-1}|\|T(t_1) - T(t_2)\|\|E\phi(0) - g(0,\phi)\| \\
&\qquad + |E^{-1}|\|g(t_1, x_{t_1}) - g(t_2, x_{t_2})\| \\
&\qquad + \int_0^{t_1} |E^{-1}|\|AE^{-1}[T(t_1 - s) - T(t_2 - s)]\|\|g(s, x_s)\|ds \\
&\qquad + \int_{t_1}^{t_2} |E^{-1}|\|AE^{-1}T(t_2 - s)\|\|g(s, x_s)\|ds \\
&\qquad + \int_0^{t_1} |E^{-1}|\|T(t_1 - s) - T(t_2 - s)\| \int_0^s \|f(\tau, x_\tau)\|d\tau ds \\
&\qquad + \int_{t_1}^{t_2} |E^{-1}|\|T(t_2 - s)\| \int_0^s \|f(\tau, x_\tau)\|d\tau ds \\
&\quad \leq |E^{-1}|\|T(t_1) - T(t_2)\|\|E\phi(0) - g(0,\phi)\| \\
&\qquad + |E^{-1}|\|g(t_1, x_{t_1}) - g(t_2, x_{t_2})\| \\
&\qquad + G \int_0^{t_1} |E^{-1}|\|AE^{-1}[T(t_1 - s) - T(t_2 - s)]\|ds \\
&\qquad + G \int_{t_1}^{t_2} |E^{-1}|\|AE^{-1}T(t_2 - s)\|ds \\
&\qquad + \int_0^{t_1} |E^{-1}|\|T(t_1 - s) - T(t_2 - s)\|\alpha_k(s)ds \\
&\qquad + \int_{t_1}^{t_2} |E^{-1}|\|T(t_2 - s)\|\alpha_k(s)ds. \qquad (4)
\end{aligned}$$

Since g is completely continuous and the compactness of $T(t)$ for $t > 0$ implies that $T(t)$ is continuous in the uniform operator topology for $t > 0$. Thus, the right hand side of (4), which is independent of $x \in B_k$, tends to zero as $t_2 - t_1 \to 0$.

Thus, Φ maps B_k into an equicontinuous family of functions. Notice that we considered here only the case $0 < t_1 < t_2$, since the other cases $t_1 < t_2 < 0$ and $t_1 < 0 < t_2$ are very similar. It is easy to see that the family ΦB_k is uniformly bounded.

Next, we show that $\overline{\Phi B_k}$ is compact. Since we have shown ΦB_k is an equicontinuous collection, by the Arzela-Ascoli Theorem it suffices to show that Φ maps B_k into a precompact set in X.

Let $0 < t \leq b$ be fixed and ϵ be a real number satisfying $0 < \epsilon < t$. For $x \in B_k$ we define

$$\begin{aligned}(\Phi_\epsilon x)(t) &= E^{-1}T(t)[E\phi(0) - g(0,\phi)] + E^{-1}g(t,x_t)\\ &\quad + \int_0^{t-\epsilon} E^{-1}AE^{-1}T(t-s)g(s,x_s)ds\\ &\quad + \int_0^{t-\epsilon} E^{-1}T(t-s)\int_0^s f(\tau,x_\tau)d\tau ds, \quad t \in J.\end{aligned}$$

Since $T(t)$ is a compact operator, the set $Y_\epsilon(t) = \{(\Phi_\epsilon x)(t) : x \in B_k\}$ is precompact in X for every ϵ, $0 < \epsilon < t$. Moreover, for every $x \in B_k$ we have

$$\begin{aligned}&\|(\Phi x)(t) - (\Phi_\epsilon x)(t)\|\\ &\leq \|\int_0^t E^{-1}AE^{-1}T(t-s)g(s,x_s)ds\\ &\quad - \int_0^{t-\epsilon} E^{-1}AE^{-1}T(t-s)g(s,x_s)ds\|\\ &\quad +\|\int_0^t E^{-1}T(t-s)\int_0^s f(\tau,x_\tau)d\tau ds\\ &\quad - \int_0^{t-\epsilon} E^{-1}T(t-s)\int_0^s f(\tau,x_\tau)d\tau ds\|\\ &\leq \int_{t-\epsilon}^t |E^{-1}|\|AE^{-1}T(t-s)\|\|g(s,x_s)\|ds\\ &\quad + \int_{t-\epsilon}^t |E^{-1}|\|T(t-s)\|\int_0^s \|f(\tau,x_\tau)\|d\tau ds\end{aligned}$$

$$\leq \int_{t-\epsilon}^{t} |E^{-1}| \|AE^{-1}T(t-s)\| \|g(s,x_s)\| ds$$
$$+ \int_{t-\epsilon}^{t} |E^{-1}| \|T(t-s)\| \alpha_k(s) ds \to 0, \quad \text{as } \epsilon \to 0.$$

Therefore there are precompact sets arbitrarily close to the set $\{(\Phi x)(t) : x \in B_k\}$. Hence, the set $\{(\Phi x)(t) : x \in B_k\}$ is precompact in X.

It remains to show that $\Phi : Z \to Z$ is continuous. Let $\{x_n\}_0^\infty \subseteq Z$ with $x_n \to x$ in Z. Then, there is an integer r such that $\|x_n(t)\| \leq r$ for all n and $t \in J$, so $x_n \in B_r$ and $x \in B_r$. By (C_5)

$$f(t, x_{n_t}) \to f(t, x_t),$$

for each $t \in J$ and since

$$\|f(t, x_{n_t}) - f(t, x_t)\| \leq 2\alpha_r(t),$$

and also g is completely continuous, we have by dominated convergence theorem

$$\|\Phi x_n - \Phi x\| = \sup_{t \in J} \|E^{-1}[g(t, x_{n_t}) - g(t, x_t)]$$
$$+ \int_0^t E^{-1}AE^{-1}T(t-s)[g(s, x_{n_s}) - g(s, x_s)]ds\|$$
$$+ \int_0^t E^{-1}T(t-s) \int_0^s [f(\tau, x_{n_\tau}) - f(\tau, x_\tau)] d\tau ds$$
$$\leq |E^{-1}| \|g(t, x_{n_t}) - g(t, x_t)\|$$
$$+ \int_0^b |E^{-1}| \|AE^{-1}T(t-s)\| \|g(s, x_{n_s}) - g(s, x_s)\| ds$$
$$+ \int_0^b |E^{-1}| \|T(t-s)\| \int_0^s \|f(\tau, x_{n_\tau}) - f(\tau, x_\tau)\| d\tau ds \to 0,$$

as $n \to \infty$.

Thus, Φ is continuous. This completes the proof that Φ is completely continuous.

Finally the set $\zeta(\Phi) = \{x \in Z \ : \ x = \lambda \Phi x, \ \ \lambda \in (0,1)\}$ is bounded, as we proved in the first step. Consequently, by Schaefer's theorem the operator Φ has a fixed point in Z. Thus, the problem (1) has at least one mild solution on $[-r, b]$. □

Consider the neutral functional integrodifferential equation of the form

$$\begin{aligned} \frac{d}{dt}[Ex(t) - g(t, x_t)] &= Ax(t) + f(t, x_t, \int_0^t h(t, s, x_s)ds), \\ & \qquad\qquad t \in J = [0, b], \\ x_0 &= \phi \quad \text{on} \quad [-r, 0]. \end{aligned} \tag{5}$$

$f : J \times C \times X \to Y$, and $h : J \times J \times C \to X$ are continuous functions.

Similar to the problem (1) we associate the problem (5) to the integral equation

$$\begin{aligned} x(t) &= E^{-1}T(t)[E\phi(0) - g(0, \phi)] + E^{-1}g(t, x_t) \\ &+ \int_0^t E^{-1}AE^{-1}T(t-s)Ag(s, x_s)ds \\ &+ \int_0^t E^{-1}T(t-s)f(s, x_s, \int_0^s h(s, \tau, x_\tau)d\tau)ds, \quad t \in J. \end{aligned} \tag{6}$$

Assume the following conditions:

(H_1) For each $(t, s) \in J \times J$ the function $h(t, s, \cdot) : C \to X$ is continuous, and for each $x \in C$ the function $h(\cdot, \cdot, x) : J \times J \to X$ is strongly measurable.

(H_2) For each $t \in J$ the function $f(t, \cdot, \cdot) : C \times X \to Y$ is continuous, and for each $(x, y) \in C \times X$ the function $f(\cdot, x, y) : J \to Y$ is strongly measurable.

(H_3) For each positive integer k, there exists $\alpha_k \in L^1(0, b)$ such that

$$\sup_{\|x\| \le k} \|f(t, x, y)\| \le \alpha_k(t), \text{ for } t \in J, \quad \text{a.e.}$$

(H_4) There exists an integrable function $m : [0, b] \to [0, \infty)$ and a constant $\beta > 0$ such that

$$|h(t, s, x)| \le \beta m(s)\Omega_0(\|x\|), \quad 0 \le s < t \le b, \quad x \in C,$$

where Ω_0:$[0, \infty) \to (0, \infty)$ is a continuous nondecreasing function .

(H_5) There exists an integrable function $p : [0, b] \to [0, \infty)$ such that

$$\|f(t, x, y)\| \le p(t)\Omega(\|x\| + |y|), \quad 0 \le t \le b, \quad x \in C, \quad y \in X,$$

where $\Omega : [0, \infty) \to (0, \infty)$ is a continuous nondecreasing function.

(H_6)

$$\int_0^b \hat{m}(s)ds < \int_c^\infty \frac{ds}{\Omega(s)+\Omega_0(s)},$$

where

$$c = |E^{-1}|M_1[\|E\phi(0)\| + G] + |E^{-1}|G + |E^{-1}|M_2Gb,$$

and

$$\hat{m}(t) = \max\{|E^{-1}|M_1p(t), \beta m(t)\}.$$

Theorem 3.2. *If the assumptions* $(C_1 - C_4, C_8, C_9)$ *and* (H_1) *to* (H_6) *are satisfied, then the problem (5) has a mild solution on* $[-r, b]$.

Proof. To prove this theorem consider the following nonlinear operator equation

$$x(t) = \lambda Fx(t), \quad 0 \le \lambda < 1,$$

where

$$\begin{aligned} Fx(t) &= E^{-1}T(t)[E\phi(0) - g(0,\phi)] + E^{-1}g(t,x_t) \\ &\quad + \int_0^t E^{-1}AE^{-1}T(t-s)Ag(s,x_s)ds \\ &\quad + \int_0^t E^{-1}T(t-s)f(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau)ds, \quad t \in J. \end{aligned} \tag{7}$$

Now

$$\begin{aligned} |x(t)| &= |\lambda Fx(t)| \le |Fx(t)| \\ &\le |E^{-1}|M_1[\|E\phi(0)\| + G] + |E^{-1}|G + |E^{-1}|M_2Gb \\ &\quad + |E^{-1}|M_1 \int_0^t p(s)\Omega[\|x_s\| + \beta\int_0^s m(\tau)\Omega_0(\|x_\tau\|)d\tau]ds. \end{aligned}$$

We consider the following function μ given by

$$\mu(t) = \sup\{|x(s)| : -r \le s \le t\}, \quad 0 \le t \le b.$$

Let $t^* \in [-r, t]$ be such that $\mu(t)$=$|x(t^*)|$. If $t^* \in [0, b]$, by the previous inequality we have

$$\begin{aligned}\mu(t) &\leq |E^{-1}|M_1[\|E\phi(0)\| + G] + |E^{-1}|G + |E^{-1}|M_2Gb \\ &\quad + |E^{-1}|M_1 \int_0^{t^*} p(s)\Omega[\mu(s) + \beta \int_0^s m(\tau)\Omega_0(\mu(\tau))d\tau]ds \\ &\leq |E^{-1}|M_1[\|E\phi(0)\| + G] + |E^{-1}|G + |E^{-1}|M_2Gb \\ &\quad + M_1|E^{-1}| \int_0^t p(s)\Omega[\mu(s) + \beta \int_0^s m(\tau)\Omega_0(\mu(\tau))d\tau]ds.\end{aligned}$$

If $t^* \in [-r, 0]$, then $\mu(t) = \|\phi\|$ and the above inequality holds since $M_1 \geq 1$. Denoting $v(t)$ the right hand side of the above , we have

$$v(0) = c, \quad \mu(t) \leq v(t), \quad 0 \leq t \leq b,$$

and

$$\begin{aligned}v'(t) &= |E^{-1}|M_1 p(t)\Omega[\mu(t) + \beta \int_0^t m(s)\Omega_0(\mu(s))ds] \\ &\leq |E^{-1}|M_1 p(t)\Omega[v(t) + \beta \int_0^t m(s)\Omega_0(v(s))ds].\end{aligned}$$

Let

$$w(t) = [v(t) + \beta \int_0^t m(s)\Omega_0(v(s))ds],$$

then $w(0) = v(0)$ and $v(t) \leq w(t)$. Now

$$\begin{aligned}w'(t) &= v'(t) + \beta m(t)\Omega_0(v(t)) \\ &\leq |E^{-1}|M_1 p(t)\Omega[v(t) + \beta \int_0^t m(s)\Omega_0(v(s))ds] + \beta m(t)\Omega_0(v(t)) \\ &\leq |E^{-1}|M_1 p(t)\Omega(w(t)) + \beta m(t)\Omega_0(w(t)) \\ &\leq \hat{m}(t)\{\Omega(w(t)) + \Omega_0(w(t))\}.\end{aligned}$$

This implies that

$$\int_{w(0)}^{w(t)} \frac{ds}{\Omega(s) + \Omega_0(s)} \leq \int_0^b \hat{m}(s)ds < \int_c^\infty \frac{ds}{\Omega(s) + \Omega_0(s)}, \quad 0 \leq t \leq b.$$

This inequality implies that there is a constant K such that $v(t) \leq K, t \in [0, b]$ and hence $\mu(t) \leq K,\ t \in [0, b]$. Since $\|x_t\| \leq \mu(t)$, for all $t \in [0, b]$, we have

$$\|x\|_1 = \sup\{|x(t)| : -r \leq t \leq b\} \leq K,$$

where K depends only on b and on the functions $\hat{m}, \Omega$ and Ω_0.

We shall show that the operator defined by

$$\begin{aligned}(\Psi x)(t) &= E^{-1}T(t)[E\phi(0) - g(0,\phi)] + E^{-1}g(t,x_t)\\ &+ \int_0^t E^{-1}AE^{-1}T(t-s)Ag(s,x_s)ds\\ &+ \int_0^t E^{-1}T(t-s)f(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau)ds, \quad \Big[t\in[0,b]\Big]\\ &= \phi(t), \quad t\in[-r,0],\end{aligned}$$

has a fixed point. This fixed point is then the solution of (5). The remaining part of the proof is similar to Theorem 3.1 and hence it is omitted. □

4. Example

Consider the following integrodifferential equation of the form

$$\begin{aligned}&\frac{\partial}{\partial t}[z(t,x) - z_{xx}(t,x) - p(x, z(t,x-r))] = \frac{\partial^2}{\partial x^2}z(t,x)\\ &\quad + \int_0^t q(s, z(x,s-r))ds, \quad 0\le x\le\pi,\ t\in J=[0,b], \qquad (8)\\ &z(0,t) = z(\pi,t) = 0, \quad t\in J,\\ &z(x,t) = \phi(x,t), \quad -r\le t\le 0,\end{aligned}$$

where ϕ is continuous and p and q are continuous functions that satisfy certain smoothness conditions.

Take $X = Y = L^2[0,\pi]$ and let

$$g(t,w_t)x = p(t,w(t-x)) \text{ and } f(t,w_t)y = q(t,w(t-y)), \quad 0\le y\le\pi.$$

Define the operators $A : D(A)\subset X\to Y$ and $E : D(E)\subset X\to Y$ by

$$Aw = w'', \text{ and } Ew = w - w'',$$

where each domain $D(A)$ and $D(E)$ is given by

$$\{w\in X : \quad w, w' \text{ are absolutely continuous}, w''\in X,\ w(0)=w(\pi)=0\}.$$

Then A and E can be written respectively as

$$\begin{aligned} Aw &= \sum_{n=1}^{\infty} n^2 (w, w_n) w_n, \quad w \in D(A), \\ Ew &= \sum_{n=1}^{\infty} (1+n^2)(w, w_n) w_n, \quad w \in D(E), \end{aligned}$$

where $w_n(x) = \sqrt{2/\pi} \sin nx$, $n = 1, 2, ...$, is the orthogonal set of vectors of A. Furthermore, for $w \in X$ we have

$$\begin{aligned} E^{-1}w &= \sum_{n=1}^{\infty} \frac{1}{1+n^2} (w, w_n) w_n, \\ AE^{-1}w &= \sum_{n=1}^{\infty} \frac{-n^2}{1+n^2} (w, w_n) w_n, \\ T(t)w &= \sum_{n=1}^{\infty} \exp(\frac{-n^2 t}{1+n^2})(w, w_n) w_n. \end{aligned}$$

It is easy to see that AE^{-1} generates a strongly continuous semigroup $T(t)$ on Y and $T(t)$ is compact such that $|AE^{-1}T(t)| \leq k_1$ for each $t > 0$.

The function $p : J \times [0, \pi] \to [0, \pi]$ is continuous and there exists a constant $k_2 > 0$ such that

$$\|p(t, w(t-y))\| \leq k_2.$$

For the function $q : J \times [0, \pi] \to [0, \pi]$ there exists an integrable function $\alpha : J \to [0, \infty)$ such that

$$|q(t, w(t-y))| \leq \alpha(t) \Omega_1(\|w\|),$$

where $\Omega_1 : [0, \infty) \to (0, \infty)$ is continuous and nondecreasing such that

$$\int_0^b \hat{n}(s) < \int_c^{\infty} \frac{ds}{s + \Omega_1(s)},$$

where

$$c = |E^{-1}|[\|E\phi(0)\| + k_1] + |E^{-1}|k_1 + |E^{-1}|k_1 k_2 b,$$

and

$$\hat{n}(t) = \max\{1, |E^{-1}|\alpha(t)\}.$$

Further, all the conditions stated in the above theorem are satisfied. Hence the equation (8) has a mild solution on $[0, b]$.

5. Application

As an application of Theorem 3.1, we shall consider the equation (1) with a control parameter such as

$$\begin{aligned} \frac{d}{dt}[Ex(t) - g(t, x_t)] &= Ax(t) + Bu(t) + \int_0^t f(s, x_s)ds, \quad t \in J = [0, b], \\ x_0 &= \phi \quad \text{on} \quad [-r, 0], \end{aligned} \tag{9}$$

where B is a bounded linear operator from a Banach space U into Y and $u \in L^2(J, U)$.

The mild solution is given by

$$\begin{aligned} x(t) &= E^{-1}T(t)[E\phi(0) - g(0, \phi)] + E^{-1}g(t, x_t) \\ &\quad + \int_0^t E^{-1}AE^{-1}T(t-s)g(s, x_s)ds \\ &\quad + \int_0^t E^{-1}T(t-s)[Bu(s) + \int_0^s f(\tau, x_\tau)d\tau]ds, \quad t \in [0, b], \\ x_0 &= \phi \quad \text{on} \quad [-r, 0]. \end{aligned}$$

Definition 5.1. System (9) is controllable to the origin on the interval J if for every continuous initial function $\phi \in C$, there exists a control $u \in L^2(J, U)$ such that the mild solution $x(t)$ of (9) satisfies $x(b) = 0$.

For the controllability of the above systems one can refer the paper [5] and the references cited therein. To establish the controllability result we need the following additional hypothesis:

(H_7) The linear operator $W : L^2(J, U) \to X$, defined by

$$Wu = \int_0^b E^{-1}T(t-s)Bu(s)ds,$$

has an inverse operator $\tilde{W}^{-1}$ defined on $L^2(J, U)/\ \ker W$ and there exist a positive constant M_3 such that $|E^{-1}B\tilde{W}^{-1}| \leq M_3$.

(H_8)
$$\int_0^b \hat{m}(s)ds < \int_c^\infty \frac{ds}{s + \Omega(s)},$$

where

$$c = |E^{-1}|M_1[\|E\phi(0)\| + G] + |E^{-1}|G + |E^{-1}|M_2Gb + |E^{-1}|M_1Nb,$$

$$\hat{m}(t) = \max\{1, |E^{-1}|M_1m(t))\},$$

and

$$
\begin{aligned}
N &= M_3[|E^{-1}|M_1[\|E\phi(0)\| + G] + |E^{-1}|G + |E^{-1}|M_2Gb \\
&\quad + |E^{-1}|M_1 \int_0^b \int_0^s p(\tau)\Omega(\|x_\tau\|)d\tau ds.
\end{aligned}
$$

Theorem 5.1. *If the hypothesis $(C_1)-(C_{10})$ and $(H_7)-(H_8)$ are satisfied, then the system (9) is controllable.*

Proof. Using the hypothesis (H_7) for an arbitrary function $x(\cdot)$, define the control

$$
\begin{aligned}
u(t) &= -\tilde{W}^{-1}\Big[E^{-1}T(t)(E\phi(0) - g(0,\phi)) + E^{-1}g(b, x_b) \\
&\quad + \int_0^b E^{-1}AE^{-1}T(b-s)g(s,x_s)ds \\
&\quad + \int_0^b E^{-1}T(b-s)\int_0^s f(\tau, x_\tau)d\tau ds\Big](t).
\end{aligned}
$$

We shall show that when using the control the operator $\Psi : C_b^0 \to C_b^0$ defined by

$$
\begin{aligned}
\Psi(t) &= E^{-1}T(t)[E\phi(0) - g(0,\phi)] + E^{-1}g(t, x_t) \\
&\quad + \int_0^t E^{-1}AE^{-1}T(t-s)g(s,x_s)ds \\
&\quad + \int_0^t E^{-1}T(t-s)[Bu(s) + \int_0^s f(\tau, x_\tau)d\tau]ds, \quad \Big[t \in [0,b]\Big], \\
&= \phi(t), \quad t \in [-r, 0],
\end{aligned}
$$

has a fixed point. This fixed point is then a solution of system (9). Substituting $u(t)$ in the above equation we get $(\Psi x)(b) = 0$, which means that the control u steers system (9) from the initial function ϕ to the origin in time b, provided we can obtain a fixed point of the nonlinear operator Ψ. The remaining part of the proof is similar to Theorem 3.1 and hence it is omitted. □

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