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NORM CONTINUITY FOR A FUNCTIONAL DIFFERENTIAL EQUATION WITH FRACTIONAL POWER

Miklavž Mastinšek
EPF-University of Maribor
Razlagova 14
2000 Maribor, SLOVENIA
e-mail: mastinsek@uni-mb.si

Abstract: A functional differential equation $du/dt = -Au(t) + bA^\alpha u(t) + (a * Au)(t)$ is studied, where $-A$ is the infinitesimal generator of a bounded analytic semigroup in Hilbert space X, A^α is the fractional power of A and the convolution term contains a square integrable real function a. Eventual norm continuity is obtained for $0 \leq \alpha < 1$.

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1. Introduction

We consider the stability properties for the functional differential equation (FDE) of the form:

$$\begin{aligned} &u'(t) = -Au(t) + bA^\alpha u(t-h) + \int_{-h}^0 a(r)Au(t+r)dr, \quad t > 0\,, \\ &u(0) = \phi^0\,, \\ &u(r) = \phi^1(r) \quad r \in [-h, 0)\,, \end{aligned} \tag{1.1}$$

where $-A$ is the infinitesimal generator of an analytic semigroup of linear operators $S(t)$ on a Hilbert space X and A^α is the fractional power of A for $0 \leq \alpha < 1$. Morover $a(.)$ is a measurable square integrable real function, $b \in R$ and the initial value $\phi = (\phi^0, \phi^1)$ belongs to the product space $Z := F \times L^2(-h, 0 : D(A))$

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with F denoting a suitable intermediate space between $D(A)$ and X given by (1.6) bellow.

The exsistence of solution of FDE (1.1) has been studied in Di Blasio et al [1]. The following result was obtained: For an arbitrary initial value $\phi = (\phi^0, \phi^1) \in Z$ there is a unique solution $u = u(t)$ of FDE (1.1) on the interval $[-h, T], T > 0$ such that $u \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;X)$,

$$||u||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;X)} \leq c_1||\phi||_Z\,, \tag{1.2}$$

and the constant c_1 depends on $T, ||a||, b, \alpha$. Moreover, the solution semigroup $T(t)$ can be defined in the product space $Z = F \times L^2(-h, 0; D(A))$ in the following way:

$$T(t)\phi = (u(t), u_t(.))\,, \quad t \geq 0\,, \tag{1.3}$$

for every $\phi \in Z$ and u_t given by $u_t(s) = u(t+s), s \in [-h, 0]$. This semigroup is strongly continuous in Z, hence there exist real constants M and ω such that

$$||T(t)|| \leq Me^{\omega t}\,, \quad t \geq 0\,. \tag{1.4}$$

It is known, when the semigroup $T(t)$ is eventually norm continuous (i.e. it is continuous in the uniform operator topology for $t \geq t_0 > 0$), then the asymptotic behaviour of the semigroup $T(t)$ can be obtained through the location of the spectrum of its infinitesimal generator (see e.g. [3], p. 281).

There are various conditions such that $T(t)$ is norm continuous for $t \geq t_0$. For the case where $a(.)$ is continuous see e.g. Di Blasio et al [2] and Jeong [5]. In [5] it was proved that the solution semigroup is not eventually norm continuous when $\alpha = 1$. The objective of this paper is to establish norm continuity for $t > h$ also for the case, where $b \neq 0, 0 \leq \alpha < 1$ and $a(.)$ is not necessarily continuous. We consider equations where a fractional power A^α acts in the discrete delay term.

Notation. Throughout X is a Hilbert space with norm $||.||$, R is the set of real numbers and $L(X)$ is the space of bounded linear operators in X. Throughout $-A$ is the infinitesimal generator of a bounded analytic semigroup $\{S(t); t \geq 0\}$ on X with 0 in the resolvent set and A^α is the fractional power of A for $0 \leq \alpha < 1$. Thus, there are constants M_0 and M such that it holds:

$$||S(t)|| \leq M_0, \quad t \geq 0 \quad \text{and} \quad ||A^\alpha S(t)|| \leq \frac{M}{t^\alpha} \quad \text{for} \quad t > 0\,. \tag{1.5}$$

For the characterization of fractional powers see e.g. [8] Chapter 2.6. For a closed linear injective operator A in X its domain $D(A)$ is regarded as a Banach space with the norm $||x||_{D(A)} = ||Ax||$. The Lions real interpolation space F between $D(A)$ and X is given by:

$$F = \left\{x \in X; \int_0^\infty ||AS(t)x||^2 dt < \infty\right\}, \quad \text{and norm} \\ ||x||_F = (||x||^2 + \int_0^\infty ||AS(t)x||^2 dt)^{\frac{1}{2}}\,. \tag{1.6}$$

$C(t_0, T; Y)$ is the space of continuous functions from an interval $[t_0, T]$ to a Banach space Y and $L^2(t_0, T; Y)$ denotes the space of measurable square integrable functions from $[t_0, T]$ to Y. As usual for a reflexive space Y $W^{1,2}(t_0, T; Y)$ denotes the space of absolutely continuous functions on $[t_0, T]$ such that their derivative belongs to $L^2(t_0, T; Y)$. We note that there is a continuous embedding:

$$L^2(0, T, D(A)) \cap W^{1,2}(0, T; X) \to C(0, T; F)\,, \tag{1.7}$$

so that there exists a constant c_0 such that

$$||u||_{C(0,T;F)} \leq c_0 ||u||_{L(0,T;D(A)) \cap W^{1,2}(0,T;X)}\,, \tag{1.8}$$

for each $u \in L^2(0, T; D(A)) \cap W^{1,2}(0, T; X)$ (see e.g. [1], [6]).

2. Norm Continuity

Lemma 1. *Let $u \in L^2(0, T; D(A))$ and let $g : [0, T] \to X$ be defined by:*

$$g(t) := \int_0^\infty A^\alpha S(t-s) u(s) ds\,, \quad t \in [0, T]\,. \tag{2.1}$$

Then $g \in L^2(0, T; D(A))$ and the following inequality holds:

$$||g||_{L^2(0,T;D(A))} \leq ||A^\alpha S(.)||_{L^1(0,T;L(X))} ||u||_{L^2(0,T;D(A))}\,. \tag{2.2}$$

Proof. Note that $||A^\alpha S(t)|| \leq \frac{M}{t^\alpha}$ for $t > 0$ and $0 \leq \alpha < 1$. Thus, we may conclude $A^\alpha S(.) \in L^1(0, T; L(X))$ for any $T > 0$. Moreover by assumption $u \in L^2(0, T; D(A))$, and hence $A^\alpha S(t-.)Au(.)$ is a strongly measurable function for every $t \in [0, T]$. By the Bochner Theorem and by the known estimates on convolutions of real L^1 and L^2 functions we conclude that the integral

$$h(t) := \int_0^\infty A^\alpha S(t-s) Au(s) ds$$

exists for a.e. $t \in [0, T]$ and that the following ineqation holds:

$$||g||_{L^2(0,T;D(A))} \leq ||A^\alpha S(.)||_{L^1(0,T;X)} ||Au||_{L^2(0,T;X)}\,.$$

Since A is a linear closed operator in X, then it follows $h(t) = Ag(t)$ for a.e. $t \in [0,T]$ (see e.g. Propossition C4 in [3]). Hence, $g \in L^2(0,T;D(A))$ and (2.2) holds. □

We consider now the continuity of the solution semigroup $T(t)$ in the uniform operator topology for the case $0 \le \alpha < 1$ and $t > h$. As noted, if $\alpha = 1$ the solution semigroup is not eventually norm continuous (see e.g. [5]). Actually we can prove that the solution semigroup of FDE (1.1) is norm continuous uniformly in t on any closed interval $[t_0, T]$ for $t_0 > h$. More precisely:

Theorem 1. *Let $\{T(t); t \ge 0\}$ be the solution semigroup of (1.1) and suppose that $0 < h < t_0 \le \tau \le \tau' \le T$, with t_0 and T fixed. Then for every $\epsilon > 0$ there is a $\Delta > 0$ such that*

$$||T(\tau')\phi - T(\tau)\phi||_Z < \epsilon ||\phi||_Z \tag{2.3}$$

holds for every $\phi \in Z$, whenever $(\tau' - \tau) < \Delta$. The solution semigroup $\{T(t); t \ge 0\}$ is continuous in the uniform operator topology at every $t > h$.

Proof. We consider explicitly the discrete delay term in (1.1) with the fractional power A^α. For the distributed delay term we refer to [7], where the case $b = 0$ was treated. Let $u \in L^2(-h, T; D(A))$ be the solution of equation (1.1) and $0 < h < \tau_0 \le \tau \le \tau' \le T$. We have to show that

$$\begin{aligned} &||T(\tau')\phi - T(\tau)\phi||_Z \\ &\qquad = ||u(\tau') - u(\tau)||_F + ||u_{\tau'} - u_\tau||_{L^2(-h,0;D(A))} \le \epsilon ||\phi||_Z \end{aligned} \tag{2.4}$$

holds for every $\phi \in Z$, whenever $(\tau' - \tau) \le \Delta$. First, let us consider the L^2 norm of the second term in (2.4). By definition we have:

$$||u_{\tau'} - u_\tau||^2_{L^2(-h,0;D(A))} = \int_{-h}^{0} ||Au(\tau' - \tau) - Au(\tau + r)||^2 dr\,. \tag{2.5}$$

Denoting $\tau' - \tau = \Delta$, $t = \tau + r$ and $t_0 = \tau_0 - h$ we get:

$$\begin{aligned} &= \int_{\tau-h}^{\tau} ||Au(t+\Delta) - Au(t)||^2 dt \le \int_{t_0}^{\tau} ||Au(t+\Delta) - Au(t)||^2 dt \\ &= ||u(.+\Delta) - u(.)||^2_{L^2(t_0, T-\Delta; D(A))} \end{aligned}$$

In order to estimate the last integral we first re-write FDE (1.1) as the integral equation and denote

$$\begin{aligned} u(t) = S(t)\phi^0 + b\int_0^t A^\alpha S(t-s)u(s-h)ds \\ + \int_0^h S(t-s)\int_{-h}^{0} a(r)Au(s+r)drds \end{aligned} \tag{2.6}$$

$$u(t) := u_1(t) + u_2(t) + u_3(t) \tag{2.7}$$

We shall give the L_2 norm of the difference of each of the three terms in (2.7)

i) Let $t_0 > 0$ and $\epsilon_1 > 0$ be arbitrary fixed numbers. By assumption A is the infinitesimal generator of an analytic semigroup $S(t)$ and $\phi^0 \in F$. Hence, we have:

$$\begin{aligned} &||u_1(.+\Delta) - u_1(.)||^2_{L^2(t_0,T;D(A))} \\ &\qquad = \int_{t_0}^{T} ||AS(t+\Delta)\phi^0 - AS(t)\phi^0||^2 dt \\ &\qquad \leq ||S(t_0+\Delta) - S(t_0)||^2 \int_{t_0}^{T} ||AS(t-t_0)\phi^0||^2 dt \\ &\qquad \leq \epsilon_1^2 ||\phi^0||_F^2 \leq \epsilon_1^2 ||\phi||_F^2 , \end{aligned} \tag{2.8}$$

for Δ sufficiently small.

ii) For the second term in (2.7) we write

$$\begin{aligned} &u_2(t+\Delta) - u_2(t) \\ &\quad = b\int_t^{t+\Delta} A^\alpha S(t+\Delta-s)u(s-h)ds \\ &\quad + b\int_0^t [A^\alpha S(t+\Delta-s) - A^\alpha S(t-s)]u(s-h)ds \\ &\quad =: bw_1(t) + bw_2(t) . \end{aligned} \tag{2.9}$$

Let us rewrite $w_1(t)$ by substitution $t+\Delta-s=r$:

$$w_1(t) := \int_0^\Delta A^\alpha S(r)u(t+\Delta-r-h)dr .$$

By the argument given in the proof of Lemma 1 above we get

$$||w_1||_{L^2(0,T-\Delta;D(A))} \leq ||A^\alpha S(.)||_{L^2(0,\Delta;L(X))} ||u||_{L^2(-h,T;D(A))} .$$

From the second estimate (1.5) it follows

$$||w_1||_{L^2(0,T-\Delta;D(A))} \leq \frac{M}{1-\alpha}\Delta^{1-\alpha} ||u||_{L^2(-h,T;D(A))} . \tag{2.10}$$

For the term w_2 we note that the difference $A^\alpha S(t+\Delta-.)-A^\alpha S(t-.)$ is an element of the vector space $L^1(0,T-\Delta;L(X))$ for any $T-\Delta>0$. Hence, by Lemma 1 we get

$$||w_2||_{L^2(0,T-\Delta;D(A))} \leq ||A^\alpha S(.+\Delta)-A^\alpha S(.)||_{L^1(0,t-\Delta;L(X))}||u||_{L^1(-h,T;D(A))}\,.$$

By the continuity of the shift operator in L^1 it holds:

$$||A^\alpha S(.+\Delta)-A^\alpha S(.)||_{L^1(0,T-\Delta;L(X))} < \epsilon_\alpha\,,$$

for any $\epsilon_\alpha>0$ and Δ sufficiently small . Hence, it follows

$$||w_2||_{L^2(0,T-\Delta;D(A))} \leq \epsilon_\alpha ||u||_{L^2(-h,T;D(A))}\,. \tag{2.11}$$

By (2.10), (2.11) and (1.2) it follows that

$$||u_2(.+\Delta)-u_2(.)||_{L^2(t_0,T-\Delta;D(A))} \leq (\frac{M}{1-\alpha}\Delta^{1-\alpha}+\epsilon_\alpha)||u||_{L^2(-h,T;D(A))} \leq \epsilon_2||\phi||_Z\,,$$

for ϵ_2 and Δ sufficiently small.

iii) For the third term u_3 in (2.7) we refer to [7], where it was proved that

$$||u_3(.+\Delta)-u_3(.)||_{L^2(0,T-\Delta;D(A))} \leq \epsilon_3||\phi||_Z\,. \tag{2.13}$$

Therefore by estimates (2.8)-(2.13) and (2.5) we may conclude that for any $\epsilon>0$ the inequalities:

$$||u_{\tau'}-u_\tau||_{L^2(0,T-\Delta;D(A))} \leq ||u(t+\Delta)-u(t)||_{L^2(0,T-\Delta;D(A))} \leq \epsilon||\phi||_Z$$

hold for every pair $\tau,\tau'\geq t_0$ such that $\tau'-\tau=\Delta$ is sufficiently small. By using the continuous embedding (1.7), (1.8) the estimate for $||u(\tau')-u(\tau)||_F$ in (2.4) can be obtained exactly in the same way as it is shown for the case $b=0$ in [7], hence we omit the details. Therefore, from the estimate (2.4) we may conclude that the solution semigroup is norm continuous uniformly over any closed interval $[t_0,T]$ and $t_0>0$.This shows that it is norm continuous at every $t>h$. □

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