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ON A NEW EXTENSION OF HARDY-HILBERT'S INEQUALITY AND ITS APPLICATIONS

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Abstract: This paper deals with an extension of Hardy-Hilbert's inequality with a best constant factor, which involves the β function. As applications, we obtain its equivalent form.

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1. Introduction

If $a_n, b_n \geq 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1$, and $0 < \sum_{n=0}^{\infty} a_n^p < \infty$, $0 < \sum_{n=0}^{\infty} b_n^q < \infty$, then the famous Hardy-Hilbert's inequality (Hardy et al [1]) may be written in the following form:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} < \frac{\pi}{\sin(\pi/p)}\Big\{\sum_{n=0}^{\infty} a_n^p\Big\}^{1/p}\Big\{\sum_{n=0}^{\infty} b_n^q\Big\}^{1/q}, \tag{1.1}$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible. And its equivalent

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form is (Yang et al [2]):

$$\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{\infty}\frac{a_m}{m+n+1}\Big)^p<\Big[\frac{\pi}{\sin(\pi/p)}\Big]^p\sum_{n=0}^{\infty}a_n^p, \tag{1.2}$$

where the constant factor $\big[\pi/\sin(\pi/p)\big]^p$ is still the best possible.

In recent years, inequality (1.1) had been strengthened by Yang [3] as

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1}<\Big\{\sum_{n=0}^{\infty}\Big[\frac{\pi}{\sin(\pi/p)}-\frac{\ln 2-\gamma}{(2n+1)^{1+1/p}}\Big]a_n^p\Big\}^{1/p}$$
$$\times\Big\{\sum_{n=0}^{\infty}\Big[\frac{\pi}{\sin(\pi/p)}-\frac{\ln 2-\gamma}{(2n+1)^{1+1/q}}\Big]b_n^q\Big\}^{1/q}, \tag{1.3}$$

where $\ln 2-\gamma=0.1159315^+$ ($\gamma=0.57721566^+$ is Euler constant). Yang [4] gave a strengthened of (1.1) other than (1.3).

By introducing a parameter λ, and the β function, Yang et al [2] gave a generalization of (1.1), as follows:

If $2-\min\{p,q\}<\lambda\le 2, 0<\sum_{n=0}^{\infty}(n+1/2)^{1-\lambda}a_n^p<\infty$, and $0<\sum_{n=0}^{\infty}(n+1/2)^{1-\lambda}b_n^q<\infty$, then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}}<B\Big(\frac{p-2+\lambda}{p},\frac{q-2+\lambda}{q}\Big)$$
$$\times\Big\{\sum_{n=0}^{\infty}\big(n+\frac{1}{2}\big)^{1-\lambda}a_n^p\Big\}^{1/p}\Big\{\sum_{n=0}^{\infty}\big(n+\frac{1}{2}\big)^{1-\lambda}b_n^q\Big\}^{1/q}, \tag{1.4}$$

where the constant factor $B\big(\frac{p-2+\lambda}{p},\frac{q-2+\lambda}{q}\big)$ is the best possible($B(u,v)$ is the β function). For $\lambda=1$, inequality (1.4) reduces to (1.1).

The main objective of this paper is to estimate a new coefficient as

$$\omega_\lambda(r,n)=\big(n+\frac{1}{2}\big)^{\lambda(1-1/r)}\sum_{n=0}^{\infty}\frac{1}{(m+n+1)^{\lambda}}\big(m+\frac{1}{2}\big)^{-1+\lambda/r}$$
$$(n\in N_0,\quad r>1,\quad 0<\lambda\le r), \tag{1.5}$$

and give a new inequality related to the double series

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}},$$

with a best constant factor, other than (1.4).

For this, we introduce some lemmas.

2. Some Lemmas

First, we need the formula of the β function (see [5])

$$B(u,v)=\int_0^\infty \frac{t^{-1+u}}{(1+t)^{u+v}}dt=B(v,u)\quad (u,v>0), \tag{2.1}$$

and the following inequality(Kuang et al [6]):

If $f^{(4)}\in C[0,\infty), \int_0^\infty f(x)dx<\infty$, and $(-1)^n f^{(n)}(x)>0, f^{(n)}(\infty)=0 (n=0,1,2,3,4)$, then

$$\sum_{m=0}^{\infty} f(m)<\int_0^\infty f(x)dx+\frac{1}{2}f(0)-\frac{1}{12}f^{'}(0). \tag{2.2}$$

Lemma 2.1. *If $n\in N_0, r>1$, and $0<\lambda\le r$, define $R_\lambda(r,n)$as*

$$R_\lambda(r,n)=\int_{-1/2}^{0}\frac{1}{(x+n+1)^\lambda}\big(x+\frac{1}{2}\big)^{-1+\lambda/r}dx$$
$$-\frac{4r-\lambda}{3r(n+1)^\lambda 2^{\lambda/r}}-\frac{\lambda}{6(n+1)^{\lambda+1}2^{\lambda/r}}. \tag{2.3}$$

Then we have $R_\lambda(r,n)>0$.

Proof. Integration by parts, we obtain

$$\int_{-1/2}^{0}\frac{1}{(x+n+1)^\lambda}\big(x+\frac{1}{2}\big)^{-1+\lambda/r}dx=\frac{r}{\lambda}\int_{-1/2}^{0}\frac{1}{(x+n+1)^\lambda}d\big(x+\frac{1}{2}\big)^{\lambda/r}$$
$$=\frac{r}{\lambda(x+n+1)^\lambda}\big(x+\frac{1}{2}\big)^{\lambda/r}|_{-1/2}^{0}+r\int_{-1/2}^{0}\frac{1}{(x+n+1)^{\lambda+1}}\big(x+\frac{1}{2}\big)^{\lambda/r}dx$$
$$=\frac{r}{\lambda(n+1)^\lambda 2^{\lambda/r}}+\frac{r^2}{\lambda+r}\int_{-1/2}^{0}\frac{1}{(x+n+1)^{\lambda+1}}d\big(x+\frac{1}{2}\big)^{1+\lambda/r}$$
$$>\frac{r}{\lambda(n+1)^\lambda 2^{\lambda/r}}+\frac{r^2}{2(\lambda+r)(n+1)^{\lambda+1}2^{\lambda/r}}. \tag{2.4}$$

Hence by (2.3), since $r>1$, and $0<\lambda\le r$, we have

$$R_\lambda(r,n)$$
$$>\big[\frac{r}{\lambda}-\frac{4r-\lambda}{3r}\big]\frac{1}{(n+1)^\lambda 2^{\lambda/r}}+\big[\frac{r^2}{2(\lambda+r)}-\frac{\lambda}{6}\big]\frac{1}{(n+1)^{\lambda+1}2^{\lambda/r}}$$

$$= \Big[\frac{(r-\lambda)(3r-\lambda)}{3r\lambda}\Big]\frac{1}{(n+1)^{\lambda}2^{\lambda/r}} + \Big[\frac{3r^2-\lambda r-\lambda^2}{6(\lambda+r)}\Big]\frac{1}{(n+1)^{\lambda+1}2^{\lambda/r}} > 0. \quad (2.5)$$

The lemma is proved. □

Lemma 2.2. *If $n \in N_0$, $r > 1$, and $0 < \lambda \le r$, and $\omega_\lambda(r,n)$ is defined by (1.5), we have*

$$\omega_\lambda(r,n) < B\Big(\frac{\lambda}{r}, \lambda(1-\frac{1}{r})\Big). \quad (2.6)$$

Proof. For fixed n, setting

$$f(x) = \frac{1}{(x+n+1)^\lambda}\Big(x+\frac{1}{2}\Big)^{-1+\lambda/r}, \quad x \in (-\frac{1}{2}, \infty),$$

by (2.2), we have

$$\sum_{m=0}^{\infty}\frac{1}{(m+n+1)^\lambda}\Big(m+\frac{1}{2}\Big)^{-1+\lambda/r} < \int_0^\infty \frac{1}{(x+n+1)^\lambda}\Big(x+\frac{1}{2}\Big)^{-1+\lambda/r}dx$$
$$+\frac{1}{2^{\lambda/r}(n+1)^\lambda} + \frac{1}{12}\Big[\frac{2\lambda}{(n+1)^{\lambda+1}2^{\lambda/r}} + \frac{4(r-\lambda)}{r(n+1)^\lambda 2^{\lambda/r}}\Big]$$
$$= \int_{-1/2}^\infty \frac{1}{(x+n+1)^\lambda}\Big(x+\frac{1}{2}\Big)^{-1+\lambda/r}dx$$
$$-\Big[\int_{-1/2}^0 \frac{1}{(x+n+1)^\lambda}\Big(x+\frac{1}{2}\Big)^{-1+\lambda/r}dx$$
$$-\frac{4r-\lambda}{3r(n+1)^\lambda 2^{\lambda/r}} - \frac{\lambda}{6(n+1)^{\lambda+1}2^{\lambda/r}}\Big]. \quad (2.7)$$

Setting $u = (x+\frac{1}{2})/(n+\frac{1}{2})$, by (2.1), we find

$$\int_{-1/2}^\infty \frac{1}{(x+n+1)^\lambda}\Big(x+\frac{1}{2}\Big)^{-1+\lambda/r}dx$$
$$= \Big(n+\frac{1}{2}\Big)^{\lambda(-1+1/r)}\int_0^\infty \frac{u^{-1+\lambda/r}}{(1+u)^\lambda}du$$
$$= \Big(n+\frac{1}{2}\Big)^{\lambda(-1+1/r)}B\Big(\frac{\lambda}{r}, \lambda(1-\frac{1}{r})\Big). \quad (2.8)$$

Hence by (2.7), (2.3) and (2.8), we have

$$\omega_\lambda(r,n) < B\big(\frac{\lambda}{r}, \lambda(1-\frac{1}{r})\big) - \big(n+\frac{1}{2}\big)^{\lambda(1-1/r)} R_\lambda(r,n).$$

By Lemma 2.1, we have (2.6). The lemma is proved. □

Lemma 2.3. *If* $p>1, \frac{1}{p}+\frac{1}{q}=1, 0<\lambda\le\min\{p,q\}$, *and* $0<\epsilon<\lambda(q-1)$, *then we have*

$$I=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(m+n+1)^\lambda}\big(m+\frac{1}{2}\big)^{\lambda-1-\frac{\lambda+\epsilon}{p}}\big(n+\frac{1}{2}\big)^{\lambda-1-\frac{\lambda+\epsilon}{q}}$$

$$> \frac{2^\epsilon}{\epsilon}\int_0^\infty \frac{1}{(1+u)^\lambda}u^{\frac{\lambda}{p}-1-\frac{\epsilon}{q}}du - \big(\frac{\lambda}{p}-\frac{\epsilon}{q}\big)^{-2}. \tag{2.9}$$

Proof. We have $\lambda-1-\frac{\lambda+\epsilon}{r}<0 (r=p,q)$, and $(\frac{\lambda}{p}-\frac{\epsilon}{q})>0$. Hence we have

$$I > \int_0^\infty \big(x+\frac{1}{2}\big)^{\lambda-1-\frac{\lambda+\epsilon}{p}}\big[\int_0^\infty \frac{1}{(x+y+1)^\lambda}\big(y+\frac{1}{2}\big)^{\lambda-1-\frac{\lambda+\epsilon}{q}}dy\big]dx$$

$$= \int_0^\infty \big(x+\frac{1}{2}\big)^{-1-\epsilon}\big[\int_{1/(2x+1)}^\infty \frac{1}{(1+u)^\lambda}u^{\frac{\lambda}{p}-1-\frac{\epsilon}{q}}du\big]dx$$

setting $u=\frac{y+1/2}{x+1/2}$

$$= \int_0^\infty \big(x+\frac{1}{2}\big)^{-1-\epsilon}\big[\int_0^\infty \frac{1}{(1+u)^\lambda}u^{\frac{\lambda}{p}-1-\frac{\epsilon}{q}}du\big]dx$$

$$- \int_0^\infty \big(x+\frac{1}{2}\big)^{-1-\epsilon}\big[\int_0^{1/(2x+1)} \frac{1}{(1+u)^\lambda}u^{\frac{\lambda}{p}-1-\frac{\epsilon}{q}}du\big]dx$$

$$> \frac{2^\epsilon}{\epsilon}\int_0^\infty \frac{1}{(1+u)^\lambda}u^{\frac{\lambda}{p}-1-\frac{\epsilon}{q}}du - \int_0^\infty \big(x+\frac{1}{2}\big)^{-1}\big[\int_0^{1/(2x+1)} u^{\frac{\lambda}{p}-1-\frac{\epsilon}{q}}du\big]dx$$

$$= \frac{2^\epsilon}{\epsilon}\int_0^\infty \frac{1}{(1+u)^\lambda}u^{\frac{\lambda}{p}-1-\frac{\epsilon}{q}}du - \big(\frac{\lambda}{p}-\frac{\epsilon}{q}\big)^{-2}. \tag{2.10}$$

The lemma is proved. □

3. Main Results and Applications

Theorem 3.1. *If* $a_n, b_n \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \lambda \leq \min\{p, q\}$, *and*

$$0 < \sum_{n=0}^{\infty} \big(n + \frac{1}{2}\big)^{(1-\lambda)(p-1)} a_n^p < \infty,\ 0 < \sum_{n=0}^{\infty} \big(n + \frac{1}{2}\big)^{(1-\lambda)(q-1)} b_n^q < \infty,$$

then we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^{\lambda}} < B\big(\frac{\lambda}{p}, \frac{\lambda}{q}\big)$$

$$\times\{\sum_{n=0}^{\infty} \big(n + \frac{1}{2}\big)^{(1-\lambda)(p-1)} a_n^p\}^{1/p}\{\sum_{n=0}^{\infty} \big(n + \frac{1}{2}\big)^{(1-\lambda)(q-1)} b_n^q\}^{1/q}, \tag{3.1}$$

where the constant factor $B\big(\frac{\lambda}{p}, \frac{\lambda}{q}\big)$ *is the best possible. In particular, if* $1 < \lambda = p \leq q$, *we have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^{p}}$$
$$< \frac{1}{p-1}\{\sum_{n=0}^{\infty} \big(n + \frac{1}{2}\big)^{-(p-1)^2} a_n^p\}^{1/p}\{\sum_{n=0}^{\infty} \big(n + \frac{1}{2}\big)^{-1} b_n^q\}^{1/q}. \tag{3.2}$$

Proof. By Holder's inequality and (1.5) for r=p,q, we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^{\lambda}} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \Big[\frac{a_m}{(m+n+1)^{\lambda/p}} \cdot \frac{(m+1/2)^{(q-\lambda)/q^2}}{(n+1/2)^{(p-\lambda)/p^2}}\Big]$$

$$\times\Big[\frac{b_n}{(m+n+1)^{\lambda/q}} \cdot \frac{(n+1/2)^{(p-\lambda)/p^2}}{(m+1/2)^{(q-\lambda)/q^2}}\Big]$$

$$\leq \Big\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m^p}{(m+n+1)^{\lambda}} \cdot \frac{(m+1/2)^{(q-\lambda)p/q^2}}{(n+1/2)^{(p-\lambda)/p}}\Big\}^{1/p}$$

$$\times\Big\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{b_n^q}{(m+n+1)^{\lambda}} \cdot \frac{(n+1/2)^{(p-\lambda)q/p^2}}{(m+1/2)^{(q-\lambda)/q}}\Big\}^{1/q}$$

$$= \Big\{ \sum_{m=0}^{\infty} \omega_\lambda(p,m)\big(m+\frac{1}{2}\big)^{(1-\lambda)(p-1)} a_m^p \Big\}^{1/p}$$

$$\times \Big\{ \sum_{n=0}^{\infty} \omega_\lambda(q,n)\big(n+\frac{1}{2}\big)^{(1-\lambda)(q-1)} b_n^q \Big\}^{1/q}.$$

Hence by (2.6), we have (3.1).

For $0 < \epsilon < \lambda(q-1)$, setting $\overline{a}_m$ and $\overline{b}_n$ as

$$\overline{a}_m = \big(m+\frac{1}{2}\big)^{\lambda-1-\frac{\lambda+\epsilon}{p}}, \quad \overline{b}_n = \big(n+\frac{1}{2}\big)^{\lambda-1-\frac{\lambda+\epsilon}{q}},$$

then we have

$$\sum_{n=0}^{\infty} \big(n+\frac{1}{2}\big)^{(1-\lambda)(p-1)} \overline{a}_n^p = \sum_{n=0}^{\infty} \big(n+\frac{1}{2}\big)^{(1-\lambda)(q-1)} \overline{b}_n^q$$

$$= \sum_{n=0}^{\infty} \big(n+\frac{1}{2}\big)^{-1-\epsilon} = 2^{1+\epsilon} + \sum_{n=1}^{\infty} \big(n+\frac{1}{2}\big)^{-1-\epsilon}$$

$$< 2^{1+\epsilon} + \int_0^{\infty} \big(x+\frac{1}{2}\big)^{-1-\epsilon} dx = 2^{1+\epsilon} + \frac{2^\epsilon}{\epsilon}. \tag{3.3}$$

If there exists a positive parameter $\lambda(\leq \min\{p,q\})$ such that the constant factor in (3.1) is not the best possible, then there exists a positive constant $K < B\big(\frac{\lambda}{p}, \frac{\lambda}{q}\big)$, such that (3.1) is valid if we replace $B\big(\frac{\lambda}{p}, \frac{\lambda}{q}\big)$ by K. In particular, we have

$$I = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\overline{a}_m \overline{b}_n}{(m+n+1)^\lambda} < K$$

$$\times \Big\{ \sum_{n=0}^{\infty} \big(n+\frac{1}{2}\big)^{(1-\lambda)(p-1)} \overline{a}_n^p \Big\}^{1/p} \Big\{ \sum_{n=0}^{\infty} \big(n+\frac{1}{2}\big)^{(1-\lambda)(q-1)} \overline{b}_n^q \Big\}^{1/q}. \tag{3.4}$$

By (2.9) and (3.3), we find

$$2^\epsilon \int_0^{\infty} \frac{1}{(1+u)^\lambda} u^{\frac{\lambda}{p}-1-\frac{\epsilon}{q}} du - \epsilon\big(\frac{\lambda}{p} - \frac{\epsilon}{q}\big)^{-2} < K(\epsilon 2^{1+\epsilon} + 2^\epsilon).$$

Setting $\epsilon \to 0^+$ in the above inequality, we conclude that $B\big(\frac{\lambda}{p}, \frac{\lambda}{q}\big) \leq K$. This is a contradiction. Hence, the constant factor in (3.1) is the best possible. The theorem is proved. □

Theorem 3.2. *If* $a_n \geq 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $0 < \lambda \leq \min\{p, q\}$, *and*

$$0 < \sum_{n=0}^{\infty} \big(n + \frac{1}{2}\big)^{(1-\lambda)(p-1)} a_n^p < \infty,$$

then we have

$$\sum_{n=0}^{\infty} \big(n + \frac{1}{2}\big)^{\lambda-1} \big[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+1)^{\lambda}}\big]^p < \big[B\big(\frac{\lambda}{p}, \frac{\lambda}{q}\big)\big]^p \sum_{n=0}^{\infty} \big(n + \frac{1}{2}\big)^{(1-\lambda)(p-1)} a_n^p, \quad (3.5)$$

where the constant factor $\big[B\big(\frac{\lambda}{p}, \frac{\lambda}{q}\big)\big]^p$ *is the best possible. Inequality (3.5) is equivalent to (3.1). In particular, if* $1 < \lambda = p \leq q$, *we have*

$$\sum_{n=0}^{\infty} \big(n + \frac{1}{2}\big)^{p-1} \big[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+1)^{p}}\big]^p < \big(\frac{1}{p-1}\big)^p \sum_{n=0}^{\infty} \big(n + \frac{1}{2}\big)^{-(1-p)^2} a_n^p. \quad (3.6)$$

Proof. Since $0 < \sum_{n=0}^{\infty} \big(n + \frac{1}{2}\big)^{(1-\lambda)(p-1)} a_n^p < \infty$, there exists $k_0 \in N_0$ shuch for any $k \geq k_0$, $0 < \sum_{n=0}^{k} \big(n + \frac{1}{2}\big)^{(1-\lambda)(p-1)} a_n^p < \infty$. We set

$$b_n(k) = \big(n + \frac{1}{2}\big)^{\lambda-1} \big[\sum_{m=0}^{k} \frac{a_m}{(m+n+1)^{\lambda}}\big]^{p-1},$$

and use (3.1) to obtain

$$0 < \sum_{n=0}^{k} \big(n + \frac{1}{2}\big)^{(1-\lambda)(q-1)} b_n^q(k) = \sum_{n=0}^{k} \big(n + \frac{1}{2}\big)^{\lambda-1} \big[\sum_{m=0}^{k} \frac{a_m}{(m+n+1)^{\lambda}}\big]^p$$
$$= \sum_{n=0}^{k} \sum_{m=0}^{k} \frac{a_m b_n(k)}{(m+n+1)^{\lambda}} < B\big(\frac{\lambda}{p}, \frac{\lambda}{q}\big) \big\{\sum_{n=0}^{k} \big(n + \frac{1}{2}\big)^{(1-\lambda)(p-1)} a_n^p\big\}^{1/p}$$
$$\times \big\{\sum_{n=0}^{k} \big(n + \frac{1}{2}\big)^{(1-\lambda)(q-1)} b_n^q(k)\big\}^{1/q}. \quad (3.7)$$

Hence, we find

$$\Big[\sum_{n=0}^{k}\big(n+\frac{1}{2}\big)^{(1-\lambda)(q-1)}b_n^q(k)\Big]^{1/p} < B\big(\frac{\lambda}{p},\frac{\lambda}{q}\big)\Big\{\sum_{n=0}^{k}\big(n+\frac{1}{2}\big)^{(1-\lambda)(p-1)}a_n^p\Big\}^{1/p}. \quad (3.8)$$

It follows that $0<\sum_{n=0}^{\infty}\big(n+\frac{1}{2}\big)^{(1-\lambda)(q-1)}b_n^q(\infty)<\infty$. Hence (3.7) is valid as $k\to\infty$ by (3.1). So is (3.8). Thus, inequality (3.5) holds. We have proved that (3.1) implies (3.5). We need show that (3.5) implies (3.1).

By Holder's inequality,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}} = \sum_{n=0}^{\infty}\Big[\big(n+\frac{1}{2}\big)^{(\lambda-1)/p}\sum_{m=0}^{\infty}\frac{a_m}{(m+n+1)^{\lambda}}\Big]\Big[\big(n+\frac{1}{2}\big)^{(1-\lambda)/p}b_n\Big]$$

$$\leq \Big\{\sum_{n=0}^{\infty}\big(n+\frac{1}{2}\big)^{\lambda-1}\Big[\sum_{m=0}^{\infty}\frac{a_m}{(m+n+1)^{\lambda}}\Big]^p\Big\}^{1/p} \times\Big\{\sum_{n=0}^{\infty}\big(n+\frac{1}{2}\big)^{(1-\lambda)(q-1)}b_n^q\Big\}^{1/q}. \quad (3.9)$$

By (3.5), we have (3.1). It follows that (3.1) and (3.5) are equivalent.

If the constant in (3.5) is not the best possible, then by (3.9), we may get a contradiction that the constant in (3.1) is not the best possible. The theorem is proved. □

Remarks. For $\lambda=1$, inequality (3.1) reduces to (1.1), and (3.5) reduces to (1.2). Hence (3.1) is an extension of (1.1) but other than (1.3), and (3.5) is an extension of (1.2). Inequality (3.5) is independent of q. Since (3.1) and (3.5) are equivalent inequalities with the best constant factors, we give some new results.

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