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LOCAL STABILITY OF THE ADDITIVE EQUATION

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Abstract: In this paper, the local stability problems of the additive functional equation will be investigated for large classes of restricted domains. Moreover, some asymptotic properties of additive mappings will be studied.

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1. Introduction

The starting point of studying the stability of functional equations seems to be the famous talk of S. M. Ulam [13] in 1940, in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms:

> *Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. For any given $\varepsilon > 0$, does there exist a $\delta > 0$ such that for any mapping $h : G_1 \to G_2$ satisfying the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?*

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The case of approximately additive mappings was solved by D. H. Hyers [3] under the assumption that G_1 and G_2 are Banach spaces. Later, the result of Hyers was further generalized by Th. M. Rassias [9]. It should be remarked that we can find in the books [4] a lot of references concerning the stability of functional equations (see [1, 5, 6]).

In [11, 12], F. Skof investigated the Hyers-Ulam stability of the additive functional equation for many cases of restricted domains in $\mathbb{R}$ (cf. [7]). Later, L. Losonczi [8] proved the local stability of the additive equation for more general cases and applied the result to the proof of stability of the Hosszú's functional equation.

In [11], F. Skof showed that for a given ε-additive mapping $f : [0, a)^N \to F$ (F is a Banach space), there exists an additive mapping $A : \mathbb{R}^N \to F$, close to f. In [7], Z. Kominek has the result that for any ε-additive mapping $f : D \to F$, with $D \subset \mathbb{R}^N$ a bounded set containing the origin in its interior with the property $\frac{1}{2}D \subset D$, there exists an additive mapping $A : \mathbb{R}^N \to F$ which is close to f.

In this paper, we prove that for any arbitrarily small open cone $C \subset E$ (E is a normed space of any (possibly infinite) dimension), which may or may not contain the origin but is close to it, if $f : C \to F$ is an approximately additive mapping, then there exists a unique additive mapping $A : E \to F$ which is close to f. We remark that the papers [7, 11] do not obtain the uniqueness of such additive mappings. Here we show that such additive mapping $A : E \to F$ close to f is unique. Moreover, some asymptotic properties of additive mappings will be studied.

Throughout this paper, by a normed space we mean a real (or complex) normed vector space and by a cone we will mean

$$C = \tilde{C} \cap B(0, r),$$

where $\tilde{C}$ is a set in a vector space with the property

$$\tilde{C} = \{tx : x \in \tilde{C},\, t > 0\},$$

and $B(0, r) = \{x \in E : \|x\| < r\}$, for some $0 < r \leq \infty$ (in some literature, $\tilde{C}$ might be called a cone.)

Let $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of all positive integers and all nonnegative integers, respectively.

2. Local Stability for Bounded Domains

In this section, we will prove a general stability theorem for the additive functional equation for a large class of mappings defined on bounded domains.

Theorem 1. *Let E be a normed space, let F be a Banach space, and $C = \tilde{C} \cap B(0,r)$, an open cone where $B(0,r) = \{x \in E : \|x\| < r\}$ with $0 < r \le \infty$ and $\tilde{C}$ is an open set with the property $\tilde{C} = \{tx : x \in \tilde{C},\, t > 0\}$. Suppose that there is a mapping $\varphi : C \times C \to [0,\infty)$ such that*

$$\Phi(x,y) = \sum_{m=1}^{\infty} 2^m \varphi(2^{-m}x, 2^{-m}y) < \infty\,, \tag{1}$$

for any $x, y \in C$. If a mapping $f : E \to F$ satisfies the following inequality

$$\|f(x+y) - f(x) - f(y)\| \le \varphi(x,y)\,, \tag{2}$$

for all $x, y \in C$, then there exists a unique additive mapping $A : E \to F$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2}\Phi(x,x)\,, \tag{3}$$

for all $x \in C$.

Proof. In view of the structure of C, for any $x \in \tilde{C}$, there is a smallest integer $n(x) \in \mathbb{N}_0$ such that $2^{-n(x)}x \in C$. Then for any integers l and m with $l > m \ge n(x)$, it follows from (2) that

$$\begin{aligned}
\|f(2^{-m}x) - 2f(2^{-(m+1)}x)\| &\le \varphi(2^{-(m+1)}x, 2^{-(m+1)}x),\\
\|2f(2^{-(m+1)}x) - 2^2 f(2^{-(m+2)}x)\| &\le 2\varphi(2^{-(m+2)}x, 2^{-(m+2)}x),\\
\cdots \quad &\quad \cdots\\
\|2^{l-m-1} f(2^{-l+1}x) - 2^{l-m} f(2^{-l}x)\| &\le 2^{l-m-1}\varphi(2^{-l}x, 2^{-l}x).
\end{aligned}$$

Adding up the left-hand sides and the right-hand sides respectively, we have

$$\|f(2^{-m}x) - 2^{l-m} f(2^{-l}x)\| \le \sum_{k=1}^{l-m} 2^{k-1}\varphi(2^{-(m+k)}x, 2^{-(m+k)}x).$$

In other words,

$$\|2^m f(2^{-m}x) - 2^l f(2^{-l}x)\| \le \frac{1}{2}\sum_{j=m+1}^{\infty} 2^j \varphi(2^{-j}x, 2^{-j}x)\,, \tag{4}$$

for any $x \in \tilde{C}$ and $l > m \geq n(x)$.

Therefore, we see by (1) that $\{2^k f(2^{-k}x)\}$ is a Cauchy sequence for each $x \in \tilde{C}$. Since F is complete, we can define a mapping $A : \tilde{C} \to F$ by

$$A(x) = \lim_{k\to\infty} 2^k f(2^{-k}x).$$

Then, we see that since for any $x \in C$ we can take $n(x) = 0$, we can put $m = 0$ in (4) to get

$$\|f(x) - 2^l f(2^{-l}x)\| \leq \frac{1}{2}\sum_{j=1}^{\infty} 2^j \varphi(2^{-j}x, 2^{-j}x)\,,$$

for any $l \in \mathbb{N}$, i.e., the inequality (3) is satisfied for any $x \in C$.

We assert that if $x + y \in \tilde{C}$ with $x, y \in \tilde{C}$, then $A(x + y) = A(x) + A(y)$. To see this, choose $m_0 \in \mathbb{N}_0$ such that $2^{-m_0}x, 2^{-m_0}y, 2^{-m_0}(x + y) \in C$. Then for any $m \geq m_0$, the inequality (2) yields

$$\big\|2^m f(2^{-m}(x+y)) - 2^m f(2^{-m}x) - 2^m f(2^{-m}y)\big\| \leq 2^m \varphi\big(2^{-m}x, 2^{-m}y\big)\,,$$

and taking $m \to \infty$, and considering (1), we have

$$A(x + y) = A(x) + A(y)\,, \tag{5}$$

for all $x, y, x + y \in \tilde{C}$.

When $0 \in \tilde{C}$, then $0 \in C$. Hence by (1) we get $\varphi(0, 0) = 0$ and further by (2) we obtain $f(0) = 0$. Therefore

$$A(0) = \lim_{k\to\infty} 2^m f(2^{-m}0) = 0 \tag{6}$$

for $0 \in \tilde{C}$. When 0 is not in $\tilde{C}$, and when for some $x \in \tilde{C}$, $-x$ is also in $\tilde{C}$, then choose $m \in \mathbb{N}_0$ such that $2^{-m}x \in C$ and we get by (2)

$$\big\|2^m f(2^{-m}0) - 2^m f(2^{-m}x) - 2^m f(-2^{-m}x)\big\| \leq 2^m \varphi\big(2^{-m}x, -2^{-m}x\big)\,.$$

Taking $m \to \infty$, we find that since the limits $A(x)$ and $A(-x)$ exist, $f(0)$ must be 0 and we have $A(x) + A(-x) = 0$. Therefore, it holds that

$$A(x) + A(-x) = 0\,, \tag{7}$$

for any $x \in \tilde{C}$ with $-x \in \tilde{C}$. Let us define $A(0) = 0$ if $0 \notin \tilde{C}$. Then by (5), (6) and (7) all together we get

$$A(x + y) = A(x) + A(y)\,, \tag{8}$$

for all $x, y \in \tilde{C} \cup \{0\}$ with $x + y \in \tilde{C} \cup \{0\}$.

Choose a nonzero vector $p \in \tilde{C}$ and let $B(p, r) = \{x \in E : \|x - p\| < r\} \subset \tilde{C}$ for small $r > 0$ and $S = \{tx : x \in B(p, r),\ t > 0\} \subset \tilde{C}$. Here, we note that $\tilde{C}$ is an open set. Then, S is in $\tilde{C}$ and we show that if $u, v \in S$, then $u + v \in S$. If $u, v \in S$, there are some $t, s > 0$ such that $\|tu - p\| < r$ and $\|sv - p\| < r$. This means that $\|u - \frac{1}{t}p\| < \frac{r}{t}$ and $\|v - \frac{1}{s}p\| < \frac{r}{s}$, which again implies that $\|(u + v) - \frac{t+s}{ts}p\| < \frac{t+s}{ts}r$, or $\|\frac{ts}{t+s}(u + v) - p\| < r$, i.e., $u + v \in S$. We state this as

$$u + v \in S \cup \{0\} \quad \text{if} \quad u, v \in S \cup \{0\}, \tag{9}$$

and because $S \subset \tilde{C}$, by (8) and (9) we have

$$A(u + v) = A(u) + A(v)\,, \tag{10}$$

for all $u, v \in S \cup \{0\}$.

We note that any vector $x \in E$ can be written as

$$x = u - v \quad \text{for some} \quad u, v \in S \cup \{0\},$$

since for any $x \in E$, we see that $tp + x = t(p + \frac{1}{t}x) \in S \cup \{0\}$ if $t > \frac{\|x\|}{r}$.

Let us define a mapping $A' : E \to F$ by

$$A'(x) = A(u) - A(v)\,, \tag{11}$$

for each $x = u - v \in E$ with $u, v \in S \cup \{0\}$. It is well defined because if $x = u - v = u' - v' \in E$ with $u, v, u', v' \in S \cup \{0\}$, then using (9) and (10), we have

$$A(u + v') = A(u' + v)\,, \quad \text{or} \quad A(u) - A(v) = A(u') - A(v').$$

We show now $A'(x) = A(x)$ for $x \in \tilde{C}$. For $x \in \tilde{C}$, let $x = u - v$ with $u, v \in S \cup \{0\}$. Then $u = x + v$ and all $x, v, u = x + v$ are in $\tilde{C} \cup \{0\}$ and by (8), we get

$$A(u) = A(x) + A(v)\,,$$

or by (11),

$$A(x) = A(u) - A(v) = A'(x)\,, \tag{12}$$

for any $x \in \tilde{C}$.

To see that $A' : E \to F$ is additive, suppose $x, y \in E$ are given, where $x = u - v$ and $y = u' - v'$ with $u, v, u', v' \in S \cup \{0\}$. Then by using (9), (10) and (11),

$$\begin{aligned} A'(x) + A'(y) &= A(u) - A(v) + A(u') - A(v') \\ &= A(u + u') - A(v + v') \ = \ A'(x + y). \end{aligned}$$

From now on, let $A(x)$ stand for $A'(x)$ for simplicity. It is clear by (12) that this extended mapping $A : E \to F$ has the property (3) for any $x \in C$.

We now show that $A(x)$ is the unique additive mapping with such property. Suppose that $A_1 : E \to F$ is another additive mapping with the same property (3). First, for any $x \in S$ there is a nonnegative integer $n(x) \in \mathbb{N}_0$ such that $m \geq n(x)$ implies $2^{-m}x \in C$. Thus, it follows from (3) that

$$\begin{aligned} \|A(x) - A_1(x)\| &= 2^m \|A(2^{-m}x) - A_1(2^{-m}x)\| \\ &\leq 2^m \|A(2^{-m}x) - f(2^{-m}x)\| \\ &\quad +2^m \|f(2^{-m}x) - A_1(2^{-m}x)\| \\ &\leq 2^m \cdot \frac{1}{2}\Phi(2^{-m}x, 2^{-m}x) \cdot 2 \\ &= \sum_{j=m+1}^{\infty} 2^j \varphi(2^{-j}x, 2^{-j}x) \\ &\to 0 \text{ as } m \to \infty. \end{aligned}$$

Therefore, $A(x) = A_1(x)$ for any $x \in S \cup \{0\}$. Now $A(x)$ and $A_1(x)$ are both additive mappings which agree on $S \cup \{0\}$ and therefore $A(x) = A_1(x)$ for any $x \in E$, since for $x = u - v \in E$ with $u, v \in S \cup \{0\}$, we have

$$A(x) = A(u) - A(v) = A_1(u) - A_1(v) = A_1(x),$$

as required. □

Theorem 1 covers very general cases for the stability of additive functional equation. For example, when we put $\tilde{C} = E$, $r = \infty$ and $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for some $\theta \geq 0$ and $p > 1$, Theorem 1 reduces to the Gajda's Theorem (see [2]):

Corollary 2. *Let E and F be the same spaces given in Theorem 1. If a mapping $f : E \to F$ satisfies the inequality*

$$\|f(x + y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p), \tag{13}$$

for all $x, y \in E$ and for some $\theta \geq 0$ and $p > 1$, then there exists a unique additive mapping $A : E \to F$ such that

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2^p - 2}\|x\|^p,$$

for each $x \in E$.

Assuming that f is 2-homogeneous, we can apply Theorem 1 to a verification of an asymptotic property of additive mappings:

Corollary 3. *Let E and F be the same spaces given in Theorem 1. A 2-homogeneous mapping $f : E \to F$ is additive if and only if f satisfies the asymptotic condition*

$$\|f(x+y) - f(x) - f(y)\| = O(\|x\|^p + \|y\|^p)\,,$$

as $\|x\| + \|y\| \to 0$, where $p > 1$ is a constant.

Proof. Using the asymptotic condition, we can choose a sufficiently large integer n and a constant $\theta \geq 0$ such that the inequality (13) is satisfied for all $x, y \in E$ with $\|x\| + \|y\| < 1/n$. If we put $\tilde{C} = E$ and $r = 1/(2n)$, then the inequality (13) is satisfied for all $x, y \in C = B(0, 1/(2n))$. According to Theorem 1, there exists a unique additive mapping $A : E \to F$ that satisfies the inequality (3) for all $x \in B(0, 1/(2n))$.

Let $x \in E$ be given. We can choose a nonnegative integer $n(x)$ such that $m \geq n(x)$ implies $2^{-m}x \in B(0, 1/(2n))$. Hence, the 2-homogeneity of f and (3) yield

$$\begin{aligned} \|f(x) - A(x)\| &= 2^m \|f(2^{-m}x) - A(2^{-m}x)\| \\ &\leq \theta \|x\|^p \sum_{j=m+1}^{\infty} 2^{(1-p)j} \\ &\to 0 \ \text{ as } \ m \to \infty, \end{aligned}$$

which implies that f itself is an additive mapping. The reverse statement is trivial. □

3. Local Stability for Unbounded Domains

In the previous section, we proved a local stability theorem concerning the additive equation on bounded domains. We will now investigate a local stability problem of the additive functional equation for the mappings defined on unbounded domains.

Theorem 4. *Let E be a normed space, let F be a Banach space, and let $\tilde{C}$ be an open set with the property $\tilde{C} = \{tx : x \in \tilde{C},\, t > 0\}$. Let $B_R = \{x \in$*

$E : \|x\| > R\}$, where $0 \le R < \infty$, and $C = \tilde{C} \cap B_R$ and suppose that there is a mapping $\psi : C \times C \to [0, \infty)$ such that

$$\Psi(x, y) = \sum_{m=0}^{\infty} 2^{-m} \psi(2^m x, 2^m y) < \infty\,, \tag{14}$$

for any $x, y \in C$. If a mapping $f : E \to F$ satisfies the following inequality

$$\|f(x + y) - f(x) - f(y)\| \le \psi(x, y)\,, \tag{15}$$

for all $x, y \in C$, then there exists a unique additive mapping $A : E \to F$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2}\Psi(x, x)\,,$$

for all $x \in C$.

Proof. Analogously to the proof of Theorem 1, we will first define a mapping $A : \tilde{C} \to F$ by using (15). For any $x \in \tilde{C} \setminus \{0\}$, there is a smallest integer $n(x) \in \mathbb{N}_0$ such that $2^{n(x)} x \in C$. Then for any integers $l > m \ge n(x)$, it follows from (15) that

$$\begin{aligned}
\|2^{-1} f(2^{m+1}x) - f(2^m x)\| &\le 2^{-1}\psi(2^m x, 2^m x), \\
\|2^{-2} f(2^{m+2}x) - 2^{-1} f(2^{m+1}x)\| &\le 2^{-2}\psi(2^{m+1} x, 2^{m+1} x), \\
\cdots \quad &\quad \cdots \\
\|2^{-(l-m)} f(2^l x) - 2^{-(l-m-1)} f(2^{l-1}x)\| &\le 2^{-(l-m)}\psi(2^{l-1} x, 2^{l-1} x).
\end{aligned}$$

Adding up the left-hand sides and the right-hand sides respectively, we have

$$\|2^{m-l} f(2^l x) - f(2^m x)\| \le \sum_{k=1}^{l-m} 2^{-k}\psi(2^{m+k-1}x, 2^{m+k-1}x).$$

In other words,

$$\|2^{-l} f(2^l x) - 2^{-m} f(2^m x)\| \le \frac{1}{2}\sum_{j=m}^{\infty} 2^{-j}\psi(2^j x, 2^j x)\,,$$

for any $x \in \tilde{C}\setminus\{0\}$ and $l > m \ge n(x)$.

Therefore, we see by (14) that $\{2^{-k} f(2^k x)\}$ is a Cauchy sequence for $x \in \tilde{C}\setminus\{0\}$. We now define a mapping $A : \tilde{C} \to F$ by

$$A(x) = \begin{cases} \lim\limits_{k\to\infty} 2^{-k} f(2^k x) & \text{for } x \in \tilde{C}\setminus\{0\}, \\ 0 & \text{for } x = 0 \text{ and when } 0 \in \tilde{C}. \end{cases}$$

To accomplish our proof, we need follow the proof of Theorem 1 and only change the signs of the exponents of 2. □

If we set $\tilde{C} = E$, $R = 0$ and $\psi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for some $\theta \geq 0$ and $p < 1$ in Theorem 4, we obtain the Rassias' theorem (see [9, 10]):

Corollary 5. *Let E and F be a normed space and a Banach space, respectively. If a mapping $f : E \to F$ satisfies the inequality* (13) *for all $x, y \in E\backslash\{0\}$ and for some $\theta \geq 0$ and $p < 1$, then there exists a unique additive mapping $A : E \to F$ such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p\,,$$

for any $x \in E\backslash\{0\}$.

Analogously to the proof of Corollary 3, we can use Theorem 4 to prove the following corollary:

Corollary 6. *Let E and F be a normed space and a Banach space, respectively. Assume that $f : E \to F$ is a 2-homogeneous mapping. Then, f is an additive mapping if and only if*

$$\|f(x + y) - f(x) - f(y)\| = O(\|x\|^p + \|y\|^p)\,,$$

as $\|x\| + \|y\| \to \infty$, where $p < 1$ is a constant.

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