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DECAY ESTIMATES BY MOMENTS AND MASSES OF INITIAL DATA FOR LINEAR DAMPED WAVE EQUATIONS

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Abstract: We present new decay estimates of solutions to the Cauchy problem of an equation $v_{tt} - v_{xx} + v_t = 0$, which has a moment type of weighted initial data
$[v_0, v_1] \in (H^1(\mathbf{R}) \cap L^{1,2(k+1)}(\mathbf{R})) \times (L^2(\mathbf{R}) \cap L^{1,2(k+1)}(\mathbf{R}))$
(for definition of $L^{1,\gamma}(\mathbf{R})$, see below) with $k \in \mathbf{N}$.

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1. Introduction

We are concerned with the following Cauchy problem in $\mathbf{R}$:

$$V_{tt}(t,x) - V_{xx}(t,x) + V_t(t,x) = 0, \quad (t,x) \in (0,+\infty) \times \mathbf{R}, \tag{1.1}$$

$$V(0,x) = V_0(x), \quad V_t(0,x) = V_1(x), \quad x \in \mathbf{R}. \tag{1.2}$$

Notations: Throughout this paper, $\|\cdot\|_q$ and $\|\cdot\|_{H^l}$ stand for the usual $L^q(\mathbf{R})$-norm and $H^l(\mathbf{R})$-norm, respectively. For simplicity of notations, in particular, we write $\|\cdot\|$ instead of $\|\cdot\|_2$. Furthermore, for $0 < T \leq +\infty$ we set

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$$X_1(0,T) = C([0,T); H^1(\mathbf{R})) \cap C^1([0,T); L^2(\mathbf{R})).$$

On the other hand, let $\gamma \in [0,+\infty)$. Then weighted function spaces $L^{1,\gamma}(\mathbf{R})$ are defined as follows: $u \in L^{1,\gamma}(\mathbf{R})$ iff $u \in L^1(\mathbf{R})$ and

$$\|u\|_{1,\gamma} = \int_{\mathbf{R}} (1+|x|)^\gamma |u(x)|dx < +\infty.$$

In this connection we see that the continuous imbedding $L^{1,\gamma}(\mathbf{R}) \hookrightarrow L^{1,\mu}(\mathbf{R})$ holds good for $0 \leq \mu \leq \gamma$.

In this article we shall show the following moment type decay estimates.

Theorem 1.1. *Let $k \in \mathbf{N}$. If $[V_0, V_1] \in (H^1(\mathbf{R}) \cap L^{1,2(k+1)}(\mathbf{R})) \times (L^2(\mathbf{R}) \cap L^{1,2(k+1)}(\mathbf{R}))$, then there exists a unique solution $V \in X_1(0,+\infty)$ to the problem* (1.1)-(1.2) *satisfying*

$$\|V(t,\cdot)\|^2 \leq Ce^{-\mu t}(\|V_0\| + \|V_1\|)^2$$

$$+C(\|V_0\|_{1,2(k+1)} + \|V_1\|_{1,2(k+1)})^2(1+t)^{-(4k+5)/2}$$

$$+C(\|V_0\|_{1,2k+1} + \|V_1\|_{1,2k+1})^2(1+t)^{-(4k+3)/2}$$

$$+C_k\sum_{l=0}^{k}(\frac{1}{(2l)!})^2\{(\int_{\mathbf{R}} x^{2l}V_0(x)dx)^2 + (\int_{\mathbf{R}} x^{2l}V_1(x)dx)^2\}(1+t)^{-(4l+1)/2}$$

$$+C_k\sum_{l=1}^{k}(\frac{1}{(2l-1)!})^2(\int_{\mathbf{R}} x^{2l-1}V_0(x)dx)^2(1+t)^{-(4l-1)/2}$$

$$+C_k\sum_{l=1}^{k}(\frac{1}{(2l-1)!})^2(\int_{\mathbf{R}} x^{2l-1}V_1(x)dx)^2(1+t)^{-(4l-1)/2},$$

with some constants $C_k > 0$ and $\mu > 0$.

Remark 1.1. In a sense the result in Theorem 1.1 generalizes that of Ikehata [6]. Although we can also derive the corresponding L^∞ moment type decay estimates, we shall leave it to the reader's exercise, and for this we can refer the reader to Matsumura [11]. On the other hand, Milani et al [12] derived some decay estimates of L^1-norm of a solution to the equation (1.1), which has V_{xx} replaced by ΔV in $\mathbf{R}^N$ in the case of weighted L^1-initial data.

Furthermore, for the energy of the equation (1.1) we can derive the following decay estimates.

Theorem 1.2 *Let $k \in \mathbf{N}$. If $[V_0, V_1] \in (H^1(\mathbf{R}) \cap L^{1,2(k+1)}(\mathbf{R})) \times (L^2(\mathbf{R}) \cap L^{1,2(k+1)}(\mathbf{R}))$, then there exists a unique solution $V \in X_1(0,+\infty)$ to the problem (1.1)-(1.2) satisfying:*

(i)
$$\|V_x(t,\cdot)\|^2 \le Ce^{-\mu t}(\|V_0\|_{H^1}^2 + \|V_1\|^2)$$
$$+C_k(1+t)^{-(7+4k)/2}(\|V_0\|_{1,2(k+1)}^2 + \|V_1\|_{1,2(k+1)}^2)$$
$$+C_k(1+t)^{-(5+4k)/2}(\|V_0\|_{1,2k+1}^2 + \|V_1\|_{1,2k+1}^2)$$
$$+C_k\sum_{l=0}^{k}(\frac{1}{(2l)!})^2\{(\int_{\mathbf{R}} x^{2l}V_0(x)dx)^2 + (\int_{\mathbf{R}} x^{2l}V_1(x)dx)^2\}(1+t)^{-(3+4l)/2}$$
$$+C_k\sum_{l=1}^{k}(\frac{1}{(2l-1)!})^2(\int_{\mathbf{R}} x^{2l-1}V_0(x)dx)^2(1+t)^{-(1+4l)/2}$$
$$+C_k\sum_{l=1}^{k}(\frac{1}{(2l-1)!})^2(\int_{\mathbf{R}} x^{2l-1}V_1(x)dx)^2(1+t)^{-(1+4l)/2},$$

(ii)
$$\|V_t(t,\cdot)\|^2 \le Ce^{-\mu t}(\|V_0\|_{H^1}^2 + \|V_1\|^2)$$
$$+C_k(1+t)^{-(9+4k)/2}(\|V_0\|_{1,2(k+1)}^2 + \|V_1\|_{1,2(k+1)}^2)$$
$$+C_k(1+t)^{-(7+4k)/2}(\|V_0\|_{1,2k+1}^2 + \|V_1\|_{1,2k+1}^2)$$
$$+C_k\sum_{l=0}^{k}(\frac{1}{(2l)!})^2\{(\int_{\mathbf{R}} x^{2l}V_0(x)dx)^2 + (\int_{\mathbf{R}} x^{2l}V_1(x)dx)^2\}(1+t)^{-(5+4l)/2}$$
$$+C_k\sum_{l=1}^{k}(\frac{1}{(2l-1)!})^2(\int_{\mathbf{R}} x^{2l-1}V_0(x)dx)^2(1+t)^{-(3+4l)/2}$$
$$+C_k\sum_{l=1}^{k}(\frac{1}{(2l-1)!})^2(\int_{\mathbf{R}} x^{2l-1}V_1(x)dx)^2(1+t)^{-(3+4l)/2}\,,$$

with some constants $C_k > 0$ and $\mu > 0$.

These decay estimates of a moment type have already been established in Duoandixoetxea et al [1] (at least) in the linear heat equation case in $\mathbf{R}^N$. Quite recently, in Prado et al [15] they have derived asymptotic expansions of a solution to the equation

$$u_t - \Delta u_t - \Delta u + (\vec{b} \cdot \nabla u) = \nabla \cdot F(u)\,,$$

in terms of moments and masses of initial data.

On the other hand, it is well-known in Han et al [2], Ikehata [3], Ikehata et al [7], Karch [8], Marcati et al [10] and Nishihara [13, 14] that the asymptotic profiles of solutions to the problem (1.1)-(1.2) are equal to those of the corresponding heat equations, which have an appropriate type of initial data. So, a question naturally arises whether the decay estimates of a moment type hold or not also to the present problem (1.1)-(1.2). At least in the 1-dimensional case we can obtain the decay estimates of a moment type as in Theorem 1.1. Of course we may deal with higher dimensional case. The restriction to the 1-dimensional case is due to just the technical one.

The main idea of this paper originally seems to be in the works from Vainberg [18], Schonbek [16], Shibata [17] and so on. Based on these ideas Ikehata [6] has recently derived new optimal decay estimates to the linear equation (1.1) with V_{xx} replaced by ΔV in $\mathbf{R}^N$, and applied it to the critical exponent problem of the equation (1.1) with a nonlinear blowup term $|V|^p$ in the 1-dimensional exterior domain. As it has already been stated in Remark 1.1, the purpose of this paper is to generalize the decay estimates of [6, Theorems 1.3 and 1.4] for the linear equation (1.1) at least in the one dimensional case. We believe that our results will be useful in the case when we study (for example) lower bounds of decay rates for solutions of the equation (1.1).

For closely related results to the damped linear wave equations we refer the reader to Ikehata [4, 5], Kawashima et al [9] and Matsumura [11].

2. Proof of Theorems 1.1 and 1.2

In this section, we shall prove Theorems 1.1 and 1.2. The proof will be done based on the previous work by Matsumura [11] except for new estimates of the low frequency part through the Fourier transform. Throughout this section $C > 0$ is a generic constant, whose value may change from line to line.

In the following proof, (if necessary) we may assume $[V_0, V_1] \in C_0^\infty(\mathbf{R}) \times C_0^\infty(\mathbf{R})$ so that the corresponding solution $V(t,x)$ to (1.1)-(1.2) are also sufficiently smooth functions with the compact support in $\mathbf{R}$, because the solution $V(t,x)$ is given as the limit of smooth solutions $V_n(t,x)$ with the initial data $V_n(0,x) = V_{0,n} \in C_0^\infty(\mathbf{R})$ and $\frac{\partial}{\partial t}V_n(0,x) = V_{1,n} \in C_0^\infty(\mathbf{R})$ such that $V_{0,n} \to V_0$ in $H^1(\mathbf{R}) \cap L^{1,\gamma}(\mathbf{R})$ and $V_{1,n} \to V_1$ in $L^2(\mathbf{R}) \cap L^{1,\gamma}(\mathbf{R})$ $(0 \leq \gamma < +\infty)$. Then the solution $V(t,x)$ to (1.1)-(1.2) can be considered as the compactly supported smooth solution $V(t,x)$ to the Cauchy problem in $\mathbf{R}$:

$$V_{tt}(t,x) - V_{xx}(t,x) + V_t(t,x) = 0, \quad (t,x) \in (0,\infty) \times \mathbf{R}, \tag{2.1}$$

$$V(0,x) = V_0(x), \quad V_t(0,x) = V_1(x), \quad x \in \mathbf{R}. \tag{2.2}$$

First of all, we can represent the solution of (2.1)-(2.2) as follows (see Matsumura [11, Lemma 1.1]):

$$V(t,x) = K_1 * V_1 + K_2 * V_0.$$

Let $R_i(t,\xi)$ be the Fourier transform of $K_i(t,x)$ $(i = 1,2)$. Here, by

$$\mathcal{F}(f)(\xi) = \int_{\mathbf{R}} e^{-\sqrt{-1}x\xi} f(x)dx,$$

we define a Fourier transform of a function $f \in L^1(R)$.

Then R_i satisfies

$$\begin{aligned} \frac{d^2}{dt^2}R_i + \frac{d}{dt}R_i + |\xi|^2 R_i = 0, \\ R_1(0,\xi) = 0, \\ \frac{d}{dt}R_1(0,\xi) = 1, \\ R_2(0,\xi) = 1, \\ \frac{d}{dt}R_2(0,\xi) = 0. \end{aligned} \tag{2.3}$$

We can solve (2.3) exactly, so that

$$R_1(t,\xi) = \begin{cases} \frac{2e^{-(1/2)t}}{\sqrt{1-4|\xi|^2}} \sinh(\frac{\sqrt{1-4|\xi|^2}}{2}t), & \text{for } |\xi| \le 1/2\,, \\ \frac{2e^{-(1/2)t}}{\sqrt{4|\xi|^2-1}} \sin(\frac{\sqrt{4|\xi|^2-1}}{2}t), & \text{for } |\xi| > 1/2\,, \end{cases}$$

$$R_2(t,\xi) = R_1(t,\xi) + R_3(t,\xi),$$

$$R_3(t,\xi) = \begin{cases} e^{-(1/2)t} \cosh(\frac{\sqrt{1-4|\xi|^2}}{2}t), & \text{for } |\xi| \le 1/2 \\ e^{-(1/2)t} \cos(\frac{\sqrt{4|\xi|^2-1}}{2}t), & \text{for } |\xi| > 1/2. \end{cases}$$

Thus, it follows that

$$\begin{aligned} \|V(t,\cdot)\| &\le C(\|K_1(t,\cdot) * V_1\| + \|K_2(t,\cdot) * V_0\|) \\ &\le C(\|\mathcal{F}(K_1(t,\cdot))(\xi)\mathcal{F}(V_1)\| + \|\mathcal{F}(K_2(t,\cdot))(\xi)\mathcal{F}(V_0)\|) \\ &\le C(\|R_1(t,\cdot)\mathcal{F}(V_1)\| + \|R_2(t,\cdot)\mathcal{F}(V_0)\|) \end{aligned}$$

$$\le C(\|R_1(t,\cdot)\mathcal{F}(V_1)\| + \|R_1(t,\cdot)\mathcal{F}(V_0)\| + \|R_3(t,\cdot)\mathcal{F}(V_0)\|). \tag{2.4}$$

So, it suffices to derive the decay estimates of the quantities $\|R_i(t,\cdot)\mathcal{F}(f)\|$ $(i = 1, 3)$ for the function $f \in L^{1,\gamma}(\mathbf{R}) \cap L^2(\mathbf{R})$ or $\in L^{1,\gamma}(\mathbf{R}) \cap H^1(\mathbf{R})$.

Now we see that for $\delta > 0$ small,

$$\|R_1(t,\cdot)\mathcal{F}(f)\|^2 = \int_{\mathbf{R}} |R_1(t,\xi)|^2|\mathcal{F}(f)(\xi)|^2 d\xi$$

$$= \int_{|\xi|\ge 1} + \int_{1/2\le|\xi|\le 1} + \int_{\delta\le|\xi|\le 1/2} + \int_{|\xi|\le\delta}$$

$$= I_1 + I_2 + I_3 + I_4.$$

From Matsumura [11, Lemma 1.1], for the high frequency parts we can have

$$I_1 \le Ce^{-t}\|f\|^2, \quad I_2 \le Cte^{-t}\|f\|^2, \quad I_3 \le Cte^{-(1-\sqrt{1-4\delta^2})t}\|f\|^2. \tag{2.5}$$

Therefore, the following lemma is crucial in our argument.

Lemma 2.1. *Let $k \in \mathbf{N}$. If $f \in L^{1,2(k+1)}(\mathbf{R}) \cap L^2(\mathbf{R})$, then it holds that*

$$I_4 = \int_{|\xi|\le\delta} |R_1(t,\xi)|^2|\mathcal{F}(f)(\xi)|^2 d\xi$$

$$\le C_k\{(1+t)^{-(4k+5)/2}\|f\|^2_{1,2(k+1)} + (1+t)^{-(4k+3)/2}\|f\|^2_{1,2k+1}\}$$

$$+C_k\sum_{l=0}^{k}(\frac{1}{(2l)!})^2|\int_R x^{2l}f(x)dx|^2(1+t)^{-(4l+1)/2}$$

$$+C_k\sum_{l=1}^{k}(\frac{1}{(2l-1)!})^2|\int_R x^{2l-1}f(x)dx|^2(1+t)^{-(4l-1)/2},$$

where $\delta \in (0, 1/2)$ is an arbitrarily fixed number.

In order to prove Lemma 2.1 we shall prepare several tools on real analysis. The first one is elementary.

Lemma 2.2. *For each $k \in \mathbf{N}$, it holds that*

$$M_k = \sup_{\theta\ne 0}\frac{|\cos\theta - (\sum_{l=0}^{k}(-1)^l\frac{\theta^{2l}}{(2l)!})|}{|\theta|^{2(k+1)}} < +\infty,$$

$$K_k = \sup_{\theta \neq 0} \frac{|\sin\theta - (\sum_{l=1}^{k}(-1)^{l-1}\frac{\theta^{2l-1}}{(2l-1)!})|}{|\theta|^{2k+1}} < +\infty.$$

The next lemma is crucial in our argument, which is a generalization of [6, Lemma 3.1].

Lemma 2.3. *For each $k \in \mathbf{N}$ and each $f \in L^{1,2(k+1)(\mathbf{R})}$, it holds that*

$$|\mathcal{F}(f)(\xi)| \leq M_k|\xi|^{2(k+1)}\|f\|_{1,2(k+1)} + K_k|\xi|^{2k+1}\|f\|_{1,2k+1}$$

$$+\sum_{l=0}^{k}|\int_{\mathbf{R}} x^{2l}f(x)dx|\frac{|\xi|^{2l}}{(2l)!} + \sum_{l=1}^{k}|\int_{\mathbf{R}} x^{2l-1}f(x)dx|\frac{|\xi|^{2l-1}}{(2l-1)!}.$$

Proof. We first obtain the identity:

$$\mathcal{F}(f)(\xi) = \int_{\mathbf{R}} (\cos(x\xi) - \sqrt{-1}\sin(x\xi))f(x)dx$$

$$= \int_{\mathbf{R}} \{\cos(x\xi) - (\sum_{l=0}^{k}(-1)^l\frac{(x\xi)^{2l}}{(2l)!})\}f(x)dx$$

$$-\sqrt{-1}\int_{\mathbf{R}} \{\sin(x\xi) - (\sum_{l=1}^{k}(-1)^{l-1}\frac{(x\xi)^{2l-1}}{(2l-1)!})\}f(x)dx$$

$$+\int_{\mathbf{R}}(\sum_{l=0}^{k}(-1)^l\frac{(x\xi)^{2l}}{(2l)!})f(x)dx - \sqrt{-1}\int_{\mathbf{R}}(\sum_{l=1}^{k}(-1)^{l-1}\frac{(x\xi)^{2l-1}}{(2l-1)!})f(x)dx.$$

Thus, we see

$$|\mathcal{F}(f)(\xi)| \leq \int_{\mathbf{R}} |\cos(x\xi) - (\sum_{l=0}^{k}(-1)^l\frac{(x\xi)^{2l}}{(2l)!})||f(x)|dx$$

$$+\int_{\mathbf{R}} |\sin(x\xi) - (\sum_{l=1}^{k}(-1)^{l-1}\frac{(x\xi)^{2l-1}}{(2l-1)!})||f(x)|dx$$

$$+|\sum_{l=0}^{k}\frac{(-1)^l\xi^{2l}}{(2l)!}\int_{\mathbf{R}} x^{2l}f(x)dx|$$

$$+|\sum_{l=1}^{k}\frac{(-1)^{l-1}\xi^{2l-1}}{(2l-1)!}\int_{\mathbf{R}}x^{2l-1}f(x)dx|. \tag{2.6}$$

Here, for $|\xi|>0$ with the aid of Lemma 2.2 we have

$$I_{1,\epsilon}=\int_{|x|\ge\epsilon}|\cos(x\xi)-(\sum_{l=0}^{k}(-1)^l\frac{(x\xi)^{2l}}{(2l)!})||f(x)|dx$$

$$=\int_{|x|\ge\epsilon}|x|^{2(k+1)}|\xi|^{2(k+1)}\frac{|\cos(x\xi)-(\sum_{l=0}^{k}(-1)^l\frac{(x\xi)^{2l}}{(2l)!})|}{|x\xi|^{2(k+1)}}|f(x)|dx$$

$$\le M_k|\xi|^{2(k+1)}\|f\|_{1,2(k+1)}. \tag{2.7}$$

Similarly we obtain

$$I_{2,\epsilon}=\int_{|x|\ge\epsilon}|\sin(x\xi)-(\sum_{l=1}^{k}(-1)^{l-1}\frac{(x\xi)^{2l-1}}{(2l-1)!})||f(x)|dx$$

$$\le K_k|\xi|^{2k+1}\|f\|_{1,2k+1}. \tag{2.8}$$

Letting $\epsilon\downarrow 0$ in (2.7) and (2.8), we can deduce the desired inequality:

$$|\mathcal{F}(f)(\xi)|\le M_k|\xi|^{2(k+1)}\|f\|_{1,2(k+1)}+K_k|\xi|^{2k+1}\|f\|_{1,2k+1}$$

$$+\sum_{l=0}^{k}\frac{|\xi|^{2l}}{(2l)!}|\int_{\mathbf{R}}x^{2l}f(x)dx|+\sum_{l=1}^{k}\frac{|\xi|^{2l-1}}{(2l-1)!}|\int_{\mathbf{R}}x^{2l-1}f(x)dx|. \qquad \square$$

Remark 2.1. In the case when $k=0$ (formally) in Lemma 2.3, its statement should be replaced by the following estimate:

$$|\mathcal{F}(f)(\xi)|\le M_0|\xi|^2\|f\|_{1,2}+K_0|\xi|\|f\|_{1,1}+|\int_{\mathbf{R}}f(x)dx|,$$

for $f\in L^{1,2}(\mathbf{R})$.

Proof of Lemma 2.1. We first note that $-1+\sqrt{1-4|\xi|^2}\le -2|\xi|^2$ for $|\xi|\le\delta<1/2$ implies

$$|R_1(t,\xi)|^2\le Ce^{-2t|\xi|^2}.$$

Thus, it follows from Lemma 2.3 that

$$\int_{|x|\le\delta}|R_1(t,\xi)|^2|\mathcal{F}(f)(\xi)|^2d\xi$$

$$\le C_k \int_{|\xi|\le\delta} e^{-2t|\xi|^2}|\xi|^{4(k+1)}d\xi\|f\|^2_{1,2(k+1)}$$

$$+C_k \int_{|\xi|\le\delta} e^{-2t|\xi|^2}|\xi|^{4k+2}d\xi\|f\|^2_{1,2k+1}$$

$$+C_k \sum_{l=0}^{k}\int_{|\xi|\le\delta} e^{-2t|\xi|^2}|\xi|^{4l}d\xi\cdot P_l^2$$

$$+C_k \sum_{l=1}^{k}\int_{|\xi|\le\delta} e^{-2t|\xi|^2}|\xi|^{4l-2}d\xi\cdot Q_l^2, \tag{2.9}$$

where

$$P_l = \frac{1}{(2l)!}|\int_{\mathbf{R}} x^{2l}f(x)dx|,$$

$$Q_l = \frac{1}{(2l-1)!}|\int_{\mathbf{R}} x^{2l-1}f(x)dx|.$$

The well-known inequality:

$$\int_0^\delta e^{-2t|\xi|^2}|\xi|^m d|\xi| \le C(1+t)^{-(m+1)/2}$$

implies the desired estimates. □

With regard to the estimate for another quantity $\|R_3(t,\cdot)\mathcal{F}(f)\|$, we can have the following one for the core part of R_3.

Lemma 2.4. *Let $k\in\mathbf{N}$. If $f\in L^{1,2(k+1)}(\mathbf{R})\cap L^2(\mathbf{R})$, then it holds that*

$$K_4 = \int_{|\xi|\le\delta}|R_3(t,\xi)|^2|\mathcal{F}(f)(\xi)|^2 d\xi$$

$$\le C_k\{(1+t)^{-(4k+5)/2}\|f\|^2_{1,2(k+1)} + (1+t)^{-(4k+3)/2}\|f\|^2_{1,2k+1}\}$$

$$+C_k\sum_{l=0}^{k}(\frac{1}{(2l)!})^2|\int_R x^{2l}f(x)dx|^2(1+t)^{-(4l+1)/2}$$

$$+C_k\sum_{l=1}^{k}(\frac{1}{(2l-1)!})^2|\int_R x^{2l-1}f(x)dx|^2(1+t)^{-(4l-1)/2},$$

where $\delta\in(0,1/2)$ is an arbitrarily fixed number.

Proof. Note that the following relation holds:

$$\{e^{-t/2}\cosh(\frac{\sqrt{1-4|\xi|^2}}{2}t)\}^2 = \frac{1}{4}(e^{-\frac{1-\sqrt{1-4|\xi|^2}}{2}t} + e^{-\frac{1+\sqrt{1-4|\xi|^2}}{2}t})^2$$
$$\leq e^{-(1-\sqrt{1-4|\xi|^2})t}.$$

Since we also have $-1+\sqrt{1-4|\xi|^2} \leq -2|\xi|^2$ for $|\xi| \leq \delta < 1/2$, we find that

$$|R_3(t,\xi)|^2 \leq e^{-2t|\xi|^2}, \tag{2.10}$$

so that

$$K_4 \leq \int_{|\xi|\leq\delta} e^{-2t|\xi|^2}|\mathcal{F}(f)(\xi)|^2 d\xi.$$

The desired estimate follows from Lemma 2.3 and the same argument as in the proof of Lemma 2.1. □

Proof of Theorem 1.1. The estimate for the high frequency part of $\|R_3(t,\cdot)\mathcal{F}(f)\|$ is done similarly to (2.5), and we obtain

$$\int_{|\xi|\geq\delta} |R_3(t,\xi)|^2|\mathcal{F}(f)(\xi)|^2 d\xi \leq Ce^{-\mu t}\|f\|^2, \tag{2.11}$$

where $\mu > 0$ depends only on $\delta > 0$. Thus, Theorem 1.1 is a direct consequence of Lemmas 2.1 and 2.4, (2.4), (2.5) and (2.11). □

Next let us prove Theorem 1.2. To begin with, we shall show (ii) in Theorem 1.2.

Now similarly to the proof of Theorem 1.1 we first obtain

$$\|V_t(t,\cdot)\| \leq C(\|\frac{\partial}{\partial t}R_1(t,\cdot)\mathcal{F}(V_1)\| + \|\frac{\partial}{\partial t}R_2(t,\cdot)\mathcal{F}(V_0)\|)$$
$$\leq C(\|\frac{\partial}{\partial t}R_1(t,\cdot)\mathcal{F}(V_1)\| + \|\frac{\partial}{\partial t}R_1(t,\cdot)\mathcal{F}(V_0)\|)$$
$$+C\|\frac{\partial}{\partial t}R_3(t,\cdot)\mathcal{F}(V_0)\|. \tag{2.12}$$

Thus, it also suffices to derive the decay estimates of quantities $\|\frac{\partial}{\partial t}R_i(t,\cdot)\mathcal{F}(f)\|$ $(i=1,3)$ for functions $f \in L^{1,2(k+1)}(\mathbf{R})\cap L^2(\mathbf{R})$ or $\in L^{1,2(k+1)}(\mathbf{R})\cap H^1(\mathbf{R})$.

Indeed, we first obtain for $\delta > 0$ small

$$\|\frac{\partial}{\partial t}R_1(t,\cdot)\mathcal{F}(f)\|^2 = \int_{\mathbf{R}} |\frac{\partial}{\partial t}R_1(t,\xi)|^2|\mathcal{F}(f)(\xi)|^2 d\xi$$

$$= \int_{|\xi|\geq 1} + \int_{1/2\leq|\xi|\leq 1} + \int_{\delta\leq|\xi|\leq 1/2} + \int_{|\xi|\leq\delta}$$
$$= J_1 + J_2 + J_3 + J_4.$$

From Matsumura [11, Lemma 1.1], for the high frequency parts we can have

$$J_1 \leq Ce^{-t}\|f\|^2, \quad J_2 \leq C(1+t)^2 e^{-t}\|f\|^2,$$
$$J_3 \leq Ce^{-(1-\sqrt{1-4\delta^2})t}\|f\|^2. \tag{2.13}$$

On the other hand, we also devide J_4 into two parts as follows.

$$J_4 = \int_{-\delta}^{\delta} |\mathcal{F}(f)(\xi)|^2 \{e^{-t/2}\cosh(\frac{\sqrt{1-4|\xi|^2}t}{2})$$
$$-\frac{e^{-t/2}}{\sqrt{1-4|\xi|^2}}\sinh(\frac{\sqrt{1-4|\xi|^2}t}{2})\}^2 d\xi$$
$$= \frac{1}{4}\int_{|\xi|\leq\delta} \frac{|\mathcal{F}(f)(\xi)|^2}{1-4|\xi|^2}|\sqrt{1-4|\xi|^2}e^{-t(1-\sqrt{1-4|\xi|^2})/2} - e^{-t(1-\sqrt{1-4|\xi|^2})/2}$$
$$+\sqrt{1-4|\xi|^2}e^{-t(1+\sqrt{1-4|\xi|^2})/2} + e^{-t(1+\sqrt{1-4|\xi|^2})/2}|^2 d\xi$$
$$\leq C\int_{-\delta}^{\delta} \frac{|\mathcal{F}(f)(\xi)|^2}{1-4|\xi|^2}(1+\sqrt{1-4|\xi|^2})^2 e^{-(1+\sqrt{1-4|\xi|^2})t} d\xi$$
$$+C\int_{-\delta}^{\delta} \frac{|\mathcal{F}(f)(\xi)|^2}{1-4|\xi|^2}(1-\sqrt{1-4|\xi|^2})^2 e^{-(1-\sqrt{1-4|\xi|^2})t} d\xi$$
$$= J_{4,1} + J_{4,2}, \tag{2.14}$$

where $C > 0$ is a generous constant. It is easy to derive

$$J_{4,1} \leq Ce^{-t}\|f\|^2. \tag{2.15}$$

So, it suffices to derive the desirable estimate for $J_{4,2}$. Indeed, by the fact

$$-4|\xi|^2 \leq -1 + \sqrt{1-4|\xi|^2} \leq -2|\xi|^2$$

for $|\xi| \leq \delta < 1/2$ we see

$$J_{4,2} \leq \frac{C}{1-4\delta^2}\int_{|\xi|\leq\delta} e^{-2t\xi^2}|\xi|^4|\mathcal{F}(f)(\xi)|^2 d\xi.$$

Therefore, it follows from Lemma 2.3 that

$$J_{4,2} \le \frac{C}{1-4\delta^2}\int_{|\xi|\le\delta} e^{-2t\xi^2}|\xi|^4\{M_k|\xi|^{2(k+1)}\|f\|_{1,2(k+1)}$$

$$+K_k|\xi|^{2k+1}\|f\|_{1,2k+1} + \sum_{l=0}^{k} P_l|\xi|^{2l} + \sum_{l=1}^{k} Q_l|\xi|^{2l-1}\}^2 d\xi$$

$$\le C_{\delta,k}\int_{|\xi|\le\delta} e^{-2t\xi^2}|\xi|^{4+4(k+1)}d\xi\|f\|^2_{1,2(k+1)}$$

$$+C_{\delta,k}\int_{|\xi|\le\delta} e^{-2t\xi^2}|\xi|^{4+4k+2}d\xi\|f\|^2_{1,2k+1}$$

$$+C_{\delta,k}\sum_{l=0}^{k} P_l^2(\int_{|\xi|\le\delta}|\xi|^{4+4l}e^{-2t|\xi|^2}d\xi)$$

$$+C_{\delta,k}\sum_{l=1}^{k} Q_l^2(\int_{|\xi|\le\delta}|\xi|^{2+4l}e^{-2t|\xi|^2}d\xi),$$

so that we obtain

$$J_{4,2} \le C_{\delta,k}\{(1+t)^{-(9+4k)/2}\|f\|^2_{1,2(k+1)} + (1+t)^{-(7+4k)/2}\|f\|^2_{1,2k+1}$$

$$+\sum_{l=0}^{k} P_l^2(1+t)^{-(5+4l)/2} + \sum_{l=1}^{k} Q_l^2(1+t)^{-(3+4l)/2}\}. \tag{2.16}$$

(2.13), (2.14), (2.15) and (2.16) imply the following lemma.

Lemma 2.5. *Let $k \in \mathbf{N}$ and $f \in L^2(\mathbf{R}) \cap L^{1,2(k+1)}(\mathbf{R})$. Then it holds that*

$$\|\frac{\partial}{\partial t}R_1(t,\cdot)\mathcal{F}(f)\|^2 \le Ce^{-\mu t}\|f\|^2 + C_k(1+t)^{-(9+4k)/2}\|f\|^2_{1,2(k+1)}$$

$$+C_k(1+t)^{-(7+4k)/2}\|f\|^2_{1,2k+1} + \sum_{l=0}^{k}(\frac{1}{(2l)!}\int_{\mathbf{R}} x^{2l}f(x)dx)^2(1+t)^{-(5+4l)/2}$$

$$+\sum_{l=1}^{k}(\frac{1}{(2l-1)!}\int_{\mathbf{R}} x^{2l-1}f(x)dx)^2(1+t)^{-(3+4l)/2},$$

with some constants $C_k > 0$ and $\mu > 0$.

Proof of (ii) of Theorem 2.2. Similarly to Lemma 2.5 we can also obtain the desired estimate for $\|\frac{\partial}{\partial t}R_3(t,\xi)\mathcal{F}(f)\|$ with ease. Roughly speaking, the following relations hold approximately for $|\xi| \le \delta$:

$$\|\frac{\partial}{\partial t}R_3(t,\xi)\mathcal{F}(f)\|^2$$

$$\approx \int_{|\xi|\le\delta} |R_3(t,\xi)|^2|\mathcal{F}(f)(\xi)|^2 d\xi$$

$$+\int_{|\xi|\le\delta}(1-4|\xi|^2)e^{-t}(\sinh(\frac{\sqrt{1-4|\xi|^2}}{2}t))^2|\mathcal{F}(f)(\xi)|^2 d\xi.$$

Thus, from Lemma 2.4 (see also (2.11)) it suffices to obtain the estimate for

$$\int_{|\xi|\le\delta} e^{-t}(\sinh(\frac{\sqrt{1-4|\xi|^2}}{2}t))^2|\mathcal{F}(f)(\xi)|^2 d\xi.$$

But, the estimate for this part is done in the same way as in Lemma 2.1.

Next, let $f \in H^1(\mathbf{R})$. Then for the high frequency part of $\|\frac{\partial}{\partial t}R_3(t,\xi)\mathcal{F}(f)\|$ we also obtain

$$\int_{|\xi|\ge 1} |\frac{\partial}{\partial t}R_3(t,\xi)|^2|\mathcal{F}(f)(\xi)|^2 d\xi$$

$$= \int_{|\xi|\ge 1} |\frac{1}{2}e^{-t/2}\cos(\frac{t\sqrt{4|\xi|^2-1}}{2})$$

$$+e^{-t/2}\sin(\frac{t\sqrt{4|\xi|^2-1}}{2})\frac{\sqrt{4|\xi|^2-1}}{2}|^2|\mathcal{F}(f)(\xi)|^2 d\xi$$

$$\le C\int_{|\xi|\ge 1} e^{-t}|\cos(\frac{t\sqrt{4|\xi|^2-1}}{2})|^2|\mathcal{F}(f)(\xi)|^2 d\xi$$

$$+C\int_{|\xi|\ge 1} e^{-t}|\sin(\frac{t\sqrt{4|\xi|^2-1}}{2})|^2(4|\xi|^2-1)|\mathcal{F}(f)(\xi)|^2 d\xi$$

$$\le Ce^{-t}\|f\|^2 + Ce^{-t}\|f\|^2_{H^1}.$$

The other part for $\delta \le |\xi| \le 1$ can be handled similarly.

Combining these arguments together with Lemma 2.5 we have the desired estimate (ii). □

Finally, let us give the outline of proof for (i) in Theorem 1.2. The assumption on the initial data $V_0 \in H^1(\mathbf{R})$ is used in the proof of the high frequency parts of $\|\frac{\partial}{\partial x}(K_2(t,\cdot) * f)\|$ (see [11]). Note that the estimate of a quantity

$$\|\frac{\partial}{\partial x}(K_i(t,\cdot) * f)\| \approx \|(\sqrt{-1}\xi)R_i(t,\xi)\mathcal{F}(f)\|$$

is almost the same as that of $\|R_i(t,\xi)\mathcal{F}(f)\|$ except for that of the high frequency part $|\xi| \geq 1$, and it is easy to derive the estimate for the high frequency part $(i = 1, 2)$.

First of all, note that the following relation holds approximately.

$$\|V_x(t,\cdot)\| \approx \|(K_1(t,\cdot) * V_1)'\| + \|(K_2(t,\cdot) * V_0)'\|$$

$$\approx \|(\sqrt{-1}\xi)R_1(t,\cdot)\mathcal{F}(V_1)\| + \|(\sqrt{-1}\xi)R_3(t,\cdot)\mathcal{F}(V_0)\|. \qquad (2.17)$$

In the case of the low frequency part $|\xi| \leq \delta$ of (2.17), however, we have

$$\|(\sqrt{-1}\xi)R_1(t,\cdot)\mathcal{F}(V_1)\| + \|(\sqrt{-1}\xi)R_3(t,\cdot)\mathcal{F}(V_0)\|$$

$$\approx \|R_1(t,\cdot)\mathcal{F}(V_1)\| + \|R_3(t,\cdot)\mathcal{F}(V_0)\|,$$

and this parts are calculated similarly to Lemmas 2.1 and 2.4 with a slight modification. In fact,

$$\int_{|\xi|\leq\delta} |\xi|^2 |R_1(t,\xi)|^2 |\mathcal{F}(f)(\xi)|^2 d\xi$$

$$\leq C_{\delta,k}\int_{|\xi|\leq\delta} |\xi|^{6+4k} e^{-2t|\xi|^2} d\xi \|f\|^2_{1,2(k+1)}$$

$$+C_{\delta,k}\int_{|\xi|\leq\delta} |\xi|^{4+4k} e^{-2t|\xi|^2} d\xi \|f\|^2_{1,2k+1}$$

$$+C_{\delta,k}\sum_{l=0}^{k} P_l^2 \int_{|\xi|\leq\delta} |\xi|^{2+4l} e^{-2t|\xi|^2} d\xi + C_{\delta,k}\sum_{l=1}^{k} Q_l^2 \int_{|\xi|\leq\delta} |\xi|^{4l} e^{-2t|\xi|^2} d\xi$$

$$\leq C_{\delta,k}(1+t)^{-(7+4k)/2}\|f\|^2_{1,2(k+1)} + C_{\delta,k}(1+t)^{-(5+4k)/2}\|f\|^2_{1,2k+1}$$

$$+C_{\delta,k}\sum_{l=0}^{k} P_l^2 (1+t)^{-(3+4l)/2} + C_{\delta,k}\sum_{l=0}^{k} Q_l^2 (1+t)^{-(1+4l)/2}.$$

On the other hand, we have the following decay estimates for the high frequency part $|\xi| \geq 1$ of (2.17). Indeed, we obtain

$$\int_{|\xi|\geq 1} |\xi|^2 |R_1(t,\xi)|^2 |\mathcal{F}(V_1)(\xi)|^2 d\xi$$

$$\leq 4\int_{|\xi|\geq 1} |\xi|^2 \frac{e^{-t}}{4|\xi|^2-1} |\sin(\frac{\sqrt{4|\xi|^2-1}}{2}t)|^2 |\mathcal{F}(V_1)(\xi)|^2 d\xi$$

$$\leq C\int_{|\xi|\geq 1} e^{-t} |\mathcal{F}(V_1)(\xi)|^2 d\xi \leq Ce^{-t}\|\mathcal{F}(V_1)\|^2 \leq C\|V_1\|^2 e^{-t},$$

and

$$\int_{|\xi|\geq 1} |\xi|^2 |R_3(t,\xi)|^2 |\mathcal{F}(V_0)(\xi)|^2 d\xi$$

$$= \int_{|\xi|\geq 1} |\xi|^2 e^{-t} \cos^2(\frac{\sqrt{4|\xi|^2-1}}{2}t) |\mathcal{F}(V_0)(\xi)|^2 d\xi$$

$$\leq C\int_{|\xi|\geq 1} e^{-t} |\sqrt{-1}\xi|^2 |\mathcal{F}(V_0)(\xi)|^2 d\xi \leq C\|V_0\|^2_{H^1} e^{-t}.$$

The other part for $\delta \leq |\xi| \leq 1$ can be handled similarly, too. Summing up these arguments we have

Lemma 2.6. *Let $k \in \mathbf{N}$ and $f \in H^1(\mathbf{R}) \cap L^{1,2(k+1)}(\mathbf{R})$. Then it holds that*

$$\|V_x(t,\cdot)\|^2 \leq Ce^{-\mu t}(\|V_0\|^2_{H^1} + \|V_1\|^2)$$

$$+C_k(1+t)^{-(7+4k)/2}(\|V_0\|^2_{1,2(k+1)} + \|V_1\|^2_{1,2(k+1)})$$

$$+C_k(1+t)^{-(5+4k)/2}(\|V_0\|^2_{1,2k+1} + \|V_1\|^2_{1,2k+1})$$

$$+C_k\sum_{l=0}^{k}(\frac{1}{(2l)!}\int_{\mathbf{R}} x^{2l}V_0(x)dx)^2(1+t)^{-(3+4l)/2}$$

$$+C_k\sum_{l=0}^{k}(\frac{1}{(2l)!}\int_{\mathbf{R}} x^{2l}V_1(x)dx)^2(1+t)^{-(3+4l)/2}$$

$$+C_k\sum_{l=1}^{k}(\frac{1}{(2l-1)!}\int_{\mathbf{R}} x^{2l-1}V_0(x)dx)^2(1+t)^{-(1+4l)/2}$$

$$+C_k\sum_{l=1}^{k}(\frac{1}{(2l-1)!}\int_{\mathbf{R}} x^{2l-1}V_1(x)dx)^2(1+t)^{-(1+4l)/2}$$

with some constants $C_k > 0$ and $\mu > 0$.

Proof of (i) of Theorem 1.2. This part is a direct consequence of Lemma 2.6. □

Remark 2.2. In (for example) Theorem 1.1, we can not deal with the case $k = 0$. But we can obtain similar decay estimates even in the case when $k = 0$ because of Remark 2.1: If $[V_0, V_1] \in (H^1(\mathbf{R}) \cap L^{1,2}(\mathbf{R})) \times (L^2(\mathbf{R}) \cap L^{1,2}(\mathbf{R}))$, then there exists a unique solution $V \in X_1(0, +\infty)$ to the problem (1.1)-(1.2) satisfying

$$\|V(t,\cdot)\|^2 \le Ce^{-\mu t}(\|V_0\| + \|V_1\|)^2 + C(\|V_0\|_{1,2} + \|V_1\|_{1,2})^2(1+t)^{-5/2}$$

$$+C(\|V_0\|_{1,1} + \|V_1\|_{1,1})^2(1+t)^{-3/2}$$

$$+C\{(\int_{\mathbf{R}} V_0(x)dx)^2 + (\int_{\mathbf{R}} V_1(x)dx)^2\}(1+t)^{-1/2}$$

with some constants $C > 0$ and $\mu > 0$.

This result precisely expresses that of [6, Theorem 1.3] with $N = 1$.

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