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# ON THE RITZ METHOD AND ITS GENERALIZATION FOR ILL-POSED EQUATIONS WITH NON-SELFADJOINT OPERATORS

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**Abstract:** Lavrent'ev's method or simplified regularization for an ill-posed operator equation $Tx = y$ is usually applied when $T$ is a positive selfadjoint operator. We modify this procedure to suit for a general bounded operator $T$ by considering $|T|$ which results in an *apparent Ritz method*, and also consider its generalization to obtain better convergence rates using a priori and a posteriori parameter choice strategies.

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## 1. Introduction

Suppose $H$ is a complex infinite dimensional Hilbert space and $T : H \to H$ is a bounded linear operator. For solving equations of the first kind

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$$Tx = y \tag{1.1}$$

one uses regularization methods to convert the (possibly) ill-posed problem into a well-posed problem. The well known Tikhonov regularization is based on the problem of finding the minimizer $x_\alpha^T$ of the quadratic functional

$$Q_\alpha(x) := \|Tx - y\|^2 + \alpha \|x\|^2 \ , \quad x \in H,$$

for $\alpha > 0$. Then $x_\alpha^T$ is given by

$$x_\alpha^T = (T^*T + \alpha I)^{-1} T^* y \ .$$

If $T$ is a positive selfadjoint operator, then one can determine the minimizer $x_\alpha^R$ of the Ritz functional

$$R_\alpha(x) := (Tx, x) - 2\mathrm{Re}(y, x) + \alpha(x, x) \ , \quad x \in H,$$

for $\alpha > 0$, which leads to the regularization principle which is widely known as simplified or Lavrent'ev's regularization. In this case,

$$x_\alpha^R = (\alpha I + T)^{-1} y \ . \tag{1.2}$$

Simplified regularization has been considered by many authors (see, for example, Groetsch and Guacaneme [3], Schock [5], Schock [6], George and Nair [2]). It has already been proved (see Schock [5]) that, in case the operator is positive and selfadjoint, then simplified regularization is better than Tikhonov regularization in terms of speed of convergence and condition numbers involved in the setting of finite dimensional approximations.

We may observe that the normal form of the equation (1.1), namely,

$$T^*Tx = T^*y \tag{1.3}$$

fits into a form to consider the simplified regularization, with the positive selfadjoint operator $T^*T$ in place of $T$, and $T^*y$ in place of $y$. Another situation is to consider (1.1) in the form

$$TT^*u = y \, , \tag{1.4}$$

by assuming that the solution $x$ that we are looking for belongs to the range of $T^*$ so that it is of the form $x = T^*u$. The above formulations does not seem to yield any better result than those available for Tikhonov regularization. In the setting of (1.3), the simplified regularization is nothing but Tikhonov regularization, and the setting of (1.4) is not suitable for less smooth solutions. Also, these formulations are not, in anyway, simpler than Tikhonov regularization in case the operator $T$ is positive and selfadjoint.

In the case that $T$ is not selfadjoint or at least not positive definite, the Ritz functional may not have a minimizer in $H$. Our idea, in this paper, is to consider a regularization procedure for a general bounded linear operator $T$ so that, in case of a positive selfadjoint $T$, it reduces to the simplified regularization. Our approach enables us to make use of the analysis available for simplified regularization to be applicable for a general bounded linear operator, as well. We shall also consider more general versions of such procedure. For choosing the regularization parameter, we employ an Arcangeli's-type discrepancy principle, as well as a priori procedures.

## 2. The Apparent Ritz Method

In order to extend the Ritz method to case of a non-selfadjoint bounded operator $T : H \to H$ on a Hilbert space $H$, we shall make use of the *polar decomposition* of $T$, namely,

$$T = U\,|T|\,, \tag{2.1}$$

where $U$ is a partial isometry, and $|T| = (T^*T)^{1/2}$ (cf. Convey [1]). Since $U$ is a partial isometry, its restriction to $N(U)^\perp$ is an isometry. Hence, it follows that $U^*U$ and $UU^*$ are orthogonal projections. In many cases the partial isometry $U$ of $T$ may be known, for example, $T$ is a compact operator and if its singular value decomposition is known.

Using the representation (2.1), we can re-write the equation (1.1) in the form

$$|T|\,x = U^*y\,, \tag{2.2}$$

and we obtain an apparent Ritz formulation if we compute

$$x_\alpha^R = (\alpha I + |T|)^{-1}U^*y. \tag{2.3}$$

We shall call the above procedure as *appa-Ritz method*, and the solution $x_\alpha^R$ as *appa-Ritz regularized solution* of (1.1). Note that if $T$ is positive definite and selfadjoint, then we recover (1.2) from (2.3).

**Example 1.** Let $H = L^2[0,1]$ and $\varphi : [0,1] \to \mathbb{C}$ be a continuous function. Then the multiplication operator $M_\varphi : H \to H$, defined by

$$M_\varphi f = \varphi \cdot f$$

is not an isomorphism, if $\varphi$ has a zero $t_0 \in [0,1]$. It can be seen that if $\varphi$ is real-valued, then $M_\varphi$ is selfadjoint. Also,

$$M_\varphi^* f = \bar{\varphi} \cdot f \,, \quad |M_\varphi|\, f = |\varphi| \cdot f.$$

The polar decomposition of $M_\varphi$ is given by

$$M_\varphi = U \cdot |M_\varphi|$$

with $Uf = \operatorname{sgn} \varphi \cdot f$, where

$$(\operatorname{sgn} \psi)(t) = \begin{cases} \dfrac{\psi(t)}{|\psi(t)|}, & \psi(t) \neq 0\,, \\ 0, & \psi(t) = 0\,, \end{cases}$$

is the signum function. In this case,

$$x_\alpha^R = \frac{\operatorname{sgn} \bar{\varphi}}{\alpha + |\varphi|} \cdot y$$

is the appa-Ritz regularization of the equation $M_\varphi x = y$.

**Example 2.** Suppose $T$ is a compact normal operator on a Hilbert space, and let

$$Tx = \sum \gamma_j (x, u_j) u_j$$

be its spectral representation, where $\gamma_j$ are complex eigenvalues of $T$ and $(u_j)$ is an orthonormal sequence of eigenvectors. Then the appa-Ritz regularized solution of (1.1) is given by

$$x_\alpha^R = \sum \frac{\operatorname{sgn} \bar{\gamma}_j}{\alpha + |\gamma_j|}\, (y, u_j)\, u_j.$$

### 2.1. Convergence and Convergence Rates

Suppose $y \in R(T) + R(T)^\perp$ and $\hat{x}$ is the minimal norm LRN-solution of the equation (1.1), i.e., $\hat{x} = T^\dagger y$, where $T^\dagger$ is the Moore-Penrose inverse of $T$. In the due course we shall make use of the following.

**Lemma 2.1.** *For $0 < \eta \leq s$, $\mu \geq 0$,*

$$\sup_{\lambda > 0} \frac{\alpha^\mu \lambda^\eta}{\alpha + \lambda^s} \leq \alpha^{\mu - 1 + \eta/s} \quad \forall \alpha > 0.$$

*Proof.* Clearly, if $\eta = s$, then $\alpha\lambda^\eta/(\alpha+\lambda^s) \le \alpha$. Now, let $\eta < s$. Then taking $a = s/\eta$ and $b > 0$ such that $a + b = ab$, we have

$$\alpha + \lambda^s \ge \frac{\alpha}{b} + \frac{\lambda^s}{a} \ge \alpha^{1/b}\lambda^{s/a} \quad \forall \lambda > 0.$$

Hence

$$\frac{\alpha^\mu \lambda^\eta}{\alpha+\lambda^s} \le \alpha^{\mu-1/b}\lambda^{\eta-s/a} = \alpha^{\mu-1/b} = \alpha^{\mu-1+\eta/s} \quad \forall \lambda > 0.$$

This completes the proof. □

Now, we show the convergence of $x_\alpha^R$ to $\hat{x}$, and also obtain estimates for the error $\hat{x} - x_\alpha^R$ under some smoothness assumptions on $\hat{x}$.

**Theorem 2.2.** *Suppose $y \in R(T)$. Then*

$$\hat{x} - x_\alpha^R \to 0 \quad \textit{as} \quad \alpha \to 0.$$

*If $\hat{x} \in R(|T|^\nu)$ for some $\nu \in (0,1]$, then*

$$\|\hat{x} - x_\alpha^R\| = O\left(\alpha^\nu\right).$$

*Proof.* From the equations

$$|T|\hat{x} = U^*y, \qquad (\alpha I + |T|)x_\alpha^R = U^*y,$$

it follows that

$$\hat{x} - x_\alpha^R = \alpha(\alpha I + |T|)^{-1}\hat{x}.$$

Let $S_\alpha := \alpha(\alpha I + |T|)^{-1}$ for $\alpha > 0$. Then

$$\hat{x} - x_\alpha^R = S_\alpha \hat{x}.$$

Note that $\|S_\alpha\| \le 1$ for all $\alpha > 0$, and $S_\alpha u \to 0$ for every $u \in R(|T|)$. Since $R(|T|)$ is dense in the Hilbert space $N(T)^\perp$, and since $\hat{x} \in N(T)^\perp$, it follows that

$$\hat{x} - x_\alpha^R = \alpha(\alpha I + |T|)^{-1}\hat{x} = S_\alpha\hat{x} \to 0 \quad \text{as} \quad \alpha \to 0.$$

Now, suppose $\hat{x} = |T|^\nu \hat{u}$ for some $\hat{u} \in H$ and for some $\nu \in (0,1]$. Then we have

$$\hat{x} - x_\alpha^R = \alpha(\alpha I + |T|)^{-1}\hat{x} = \alpha(\alpha I + |T|)^{-1}|T|^\nu\hat{u}.$$

Hence, by spectral results and Lemma 2.1,

$$\|\hat{x} - x_\alpha^R\| \le \|\hat{u}\| \sup_{0<\lambda\le\|T\|} \frac{\alpha\lambda^\nu}{\alpha+\lambda} = O(\alpha^\nu).$$

Thus, the proof is completed. □

### 2.2. Regularization with Inexact Data

Suppose the data $y$ is not available exactly, but only an approximation of it, namely $\tilde{y}$, is available. In this case, instead of $x_\alpha^R$ in (2.3), we consider

$$\tilde{x}_\alpha^R = (\alpha I + |T|)^{-1} U^* \tilde{y}. \tag{2.4}$$

We assume that

$$\|y - \tilde{y}\| \leq \delta\,, \tag{2.5}$$

for some known error level $\delta > 0$.

**Theorem 2.3.** *Suppose $y \in R(T)$ and $\hat{x} := T^\dagger y \in R(|T|^\nu)$ for some $\nu \in (0,1]$, and $\tilde{y} \in Y$ satisfies (2.5). Then*

$$\|\hat{x} - \tilde{x}_\alpha^R\| = O\left(\alpha^\nu + \frac{\delta}{\alpha}\right).$$

*In particular, if $\alpha = c\delta^{1/(\nu+1)}$ for some $c > 0$, then*

$$\|\hat{x} - \tilde{x}_\alpha^R\| = O\left(\delta^{\nu/(\nu+1)}\right).$$

*Proof.* Clearly, from (2.3) and (2.4), we have

$$x_\alpha^R - \tilde{x}_\alpha^R = (\alpha I + |T|)^{-1} U^* (y - \tilde{y})\,,$$

so that

$$\|x_\alpha^R - \tilde{x}_\alpha^R\| \leq \|(\alpha I + |T|)^{-1}\|\,\|U^*\|\,\|y - \tilde{y}\| = O\left(\frac{\delta}{\alpha}\right).$$

This, together with the estimate obtained in Theorem 2.2 for $\|\hat{x} - x_\alpha^R\|$, will give the required error estimate. □

For $\nu > 0$, $\rho > 0$ let

$$M_{\nu,\rho} := \{|T|^\nu v : \|v\| \leq \rho\}.$$

Then, it is known that the *best possible maximal error estimate* corresponding to the above *source set* $M_{\nu,\rho}$ is $O\left(\delta^{\nu/(\nu+1)}\right)$, i.e.,

$$\inf_R \{\|x - R\tilde{y}\| : x \in M_{\nu,\rho},\, \|\,|T|x - U^*\tilde{y}\| \leq \delta\} = O\left(\delta^{\nu/(\nu+1)}\right),$$

where the infimum is taken over all maps $R : H \to H$ (see, for example, Vainikko [7]). Thus, Theorem 2.3 gives an a priori choice of the parameter $\alpha$, which leads to the optimal estimate when $0 < \nu \leq 1$.

### 2.3. An a Posteriori Parameter Choice Strategy

In Theorem 2.3, we obtained the optimal rate for the appa-Ritz method under an a priori choice of the regularization parameter $\alpha$. The difficulty with the above choice of the *regularization parameter* $\alpha$ is that it depends on the *smoothness* $\nu$ of the unknown solution $\hat{x}$. So, a parameter choice strategy, which is independent of the knowledge on $\hat{x}$ is desirable.

For simplified regularization for positive selfadjoint operator $T$ , Arcangeli's discrepancy principle was considered by Groetsch and Guacaneme [3] for choosing the regularization parameter $\alpha$. Although convergence analysis of this procedure was carried out in that paper, no attempt was done to obtain error estimates. Later, George and Nair [2] considered a general form of Arcangeli's discrepancy principle and obtained error estimate. Such generalized Arcangeli's discrepancy principle for a general bounded operator was first considered by Schock [4].

In order to choose the regularization parameter in the setting of appa-Ritz method, we use a discrepancy principle as in George and Nair [2], that is, $\alpha$ is to be chosen such that the equation

$$\|U^*\tilde{y} - |T|\tilde{x}_\alpha^R\| = \frac{\delta^p}{\alpha^q}, \quad p > 0,\ q > 0 \tag{2.6}$$

is satisfied. In view of the following proposition, the above discrepancy principle can also be written as

$$\|Q\tilde{y} - T\tilde{x}_\alpha^R\| = \frac{\delta^p}{\alpha^q}, \quad p > 0,\ q > 0,$$

where $Q$ is the orthogonal projection $UU^*$.

**Proposition 2.4.** *Let $Q$ be the orthogonal projection $UU^*$. Then*

$$\|U^*v - |T|x\| = \|Q\tilde{y} - Tx\| \quad \forall x, v \in H.$$

*Proof.* We observe that $T^* = |T|U^*$ so that $N(U^*) \subseteq N(T^*)$. Also, we have

$$R(Q) = \mathrm{cl}R(UU^*) = \mathrm{cl}R(U) = N(U^*)^\perp.$$

Hence,

$$R(T) \subseteq N(T^*)^\perp \subseteq N(U^*)^\perp = R(Q).$$

Therefore, for every $x, v \in H$, we have

$$\begin{aligned} \|U^*v - |T|x\|^2 &= \langle U^*v - |T|x, U^*v - |T|x\rangle \\ &= \langle U^*v, U^*v\rangle - \langle U^*v, |T|x\rangle - \langle |T|x, U^*v\rangle \\ &\quad + \langle |T|x, |T|x\rangle \\ &= \langle UU^*v, v\rangle - \langle v, U|T|x\rangle - \langle U|T|x, v\rangle + \langle |T|^2x, x\rangle \\ &= \langle Qv, v\rangle - \langle v, Tx\rangle - \langle Tx, v\rangle + \langle Tx, Tx\rangle \\ &= \|Q\tilde{y} - Tx\|. \end{aligned}$$

This completes the proof. □

Using similar arguments as in George and Nair [2], we obtain the following result.

**Theorem 2.5.** *Suppose $y \in R(T)$ and $\alpha$ is chosen according to (2.6). Then*

$$\tilde{x}_\alpha^R \to \hat{x} \quad \text{as} \quad \delta \to 0.$$

*If $\hat{x} := T^\dagger y \in R(|T|^\nu)$ for some $\nu \in (0, 1]$, and if $p < q + 1$, then*

$$\|\hat{x} - \tilde{x}_\alpha^R\| = O(\delta^t)\,,$$

*where*

$$t = \min\Big\{\frac{p\nu}{q+1},\ 1 - \frac{p}{q+1}\Big\}.$$

*In particular,*

$$\|\hat{x} - \tilde{x}_\alpha^R\| = O(\delta^{p\nu/(q+1)}) \quad \text{whenever} \quad 0 < \nu \leq \frac{q+1}{p} - 1.$$

Observe that for $\nu = \nu_0 := (q + 1 - p)/p$, we obtain the optimal rate $O(\delta^{\nu_0/(\nu_0+1)})$.

### 2.4. Condition Numbers

In the actual process of solving the ill-posed problem in an infinite dimensional Hilbert space, it is necessary to discretize or to use some projection methods to bring it to a finite dimensional problem. Here, we describe a projection method by the use of orthoprojections. In the case of ill-posed problems these linear systems are ill-conditioned. We will compare the usual Tikhonov regularization with the appa-Ritz regularization for a compact operator $T$ on $H$.

Let $P$ be an orthogonal projection onto a finite dimensional subspace $E$ of $H$. We compare the finite dimensional equations

$$\begin{aligned}(PT^*TP+\alpha I)\,\tilde{x}_\alpha^T &= PT^*y,\\ (P\,|T|\,P+\alpha I)\,\tilde{x}_\alpha^R &= PU^*y.\end{aligned}$$

Since $T=U\,|T|$ and $T^*=|T|\,U^*$, we can re-write the above equations in the form

$$\begin{aligned}\left(P\,|T|^2\,P+\alpha I\right)\tilde{x}_\alpha^T &= P\,|T|\,U^*y,\\ (P\,|T|\,P+\alpha I)\,\tilde{x}_\alpha^R &= PU^*y,\end{aligned}$$

respectively. Since $\tilde{x}_\alpha^R\in E$, they take the form

$$\begin{aligned}\left(\tilde{T}\tilde{T}^*+\alpha I\right)\tilde{x}_\alpha^T &= \tilde{T}U^*y,\\ \left(\tilde{T}+\alpha I\right)\tilde{x}_\alpha^R &= PU^*y,\end{aligned}$$

respectively, where $\tilde{T}=P\,|T|$.

Therefore we have to compare the condition numbers of the matrices

$$A=\left(\tilde{T}\tilde{T}^*+\alpha I\right)^{-1}\tilde{T},\qquad B=\left(\tilde{T}+\alpha I\right)^{-1}$$

We denote by $(\lambda_k)$ the monotone decreasing sequence of eigenvalues of $|T|$, and by $(u_k)$, the corresponding orthonormal eigenvectors. Then it is known (see e.g. Schock [6]) that the optimal condition numbers are obtained if $P$ is the orthogonal projection onto the space $E$, spanned by the first $n$ eigenvectors $u_1,u_2,\ldots,u_n$. Without loss of generality we will assume $\lambda_1=1$. Then the condition numbers - as the quotient of the largest and the smallest eigenvalue - associated with $A$ and $B$ are

$$\kappa_n(A)=\begin{cases}(\lambda_n^2+\alpha)/[\lambda_n(1+\alpha)] & \text{if } \alpha\ge\lambda_{\mathrm{n}}\,,\\ \lambda_n(1+\alpha)/(\lambda_n^2+\alpha) & \text{if } \alpha\le\lambda_{\mathrm{n}}\,,\end{cases}$$

and

$$\kappa_n(B)=\frac{1+\alpha}{\lambda_n+\alpha},$$

respectively.

We see that for large $n$, hence for small $\lambda_n$, and for a fixed $\alpha$, the condition numbers $\kappa_n(A)$ diverge to infinity, but the $\kappa_n(B)$ converge to $(1+\alpha)/\alpha$. On the other hand, it is remarkable that for $\alpha = \lambda_n$ we obtain $\kappa_n(A) = 1$, while $\lim_{n\to\infty} \kappa_n(B) = \infty$ for any choice of $\alpha = \alpha_n$, tending to zero.

Mildly ill-posed problems have singular values tending to zero slowly. In this case we assume $\alpha < \lambda_n$. Hence,

$$\kappa_n(A) > \frac{\lambda_n}{\alpha}, \quad \text{while} \quad \kappa_n(B) > \frac{1+\alpha}{\alpha}.$$

But, for severely ill-posed problems with rapidly decreasing singular values we may assume $\alpha > \lambda_n$ and we are in the same situation as for fixed $\alpha$. Hence $\kappa_n(A)$ may tend to infinity.

## 3. Generalization of Appa-Ritz Method

Recall that the appa-Ritz method gives a rate atmost $O(\delta^{1/2})$, which is attained for $\hat{x} \in R(|T|)$. Now, we consider a generalized form of appa-Ritz method which can give better estimate than $O(\delta^{1/2})$ under higher smoothness assumptions on $\hat{x}$. For this, we observe from (2.2) that the minimal norm solution $\hat{x}$ of (1.1) satisfies the equation

$$|T|^s \hat{x} = |T|^{s-1} U^* y\,, \tag{3.1}$$

for any $s > 1$, as well. This motivates us to consider the regularized equations as

$$(\alpha I + |T|^s) x_{\alpha,s} = |T|^{s-1} U^* y, \tag{3.2}$$

$$(\alpha I + |T|^s) \tilde{x}_{\alpha,s} = |T|^{s-1} U^* \tilde{y}, \tag{3.3}$$

respectively for $s > 1$. We assume that

$$y \in R(T) \quad \text{and} \quad \|y - \tilde{y}\| \le \delta.$$

### 3.1. Convergence and Error Estimate

From equations (3.1), (3.2) and (3.3), we have

$$\hat{x} - x_{\alpha,s} = \alpha(\alpha I + |T|^s)^{-1} \hat{x}, \tag{3.4}$$

$$x_{\alpha,s} - \tilde{x}_{\alpha,s} = [\alpha I + |T|^s]^{-1} |T|^{s-1} U^* (y - \tilde{y}). \tag{3.5}$$

Let

$$S_{\alpha,s} := \alpha[\alpha I + |T|^2]^{-1}.$$

Then $\|S_{\alpha,s}\| \leq 1$, and for every $v \in R(|T|^s)$, $\|S_{\alpha,s}v\| \to 0$ as $\alpha \to 0$. Since $R(|T|^s)$ is dense in $N(T)^\perp$, it follows that

$$\|\hat{x} - x_{\alpha,s}\| = \|S_{\alpha,s}v\| \to 0 \quad \text{as} \quad \alpha \to 0.$$

Next, we obtain estimates for the error $\|\hat{x} - x_{\alpha,s}\|$ under certain smoothness assumptions on $\hat{x}$.

**Theorem 3.1.** *Suppose $x_{\alpha,s}$ and $\tilde{x}_{\alpha,s}$ are as in (3.4) and (3.5) respectively. Assume that $\hat{x} \in R(|T|^\nu)$, where $0 < \nu \leq s$, and $\hat{x} = |T|^\nu \hat{u}$ for some $\hat{u} \in H$. Then*

$$\begin{aligned} \|\hat{x} - x_{\alpha,s}\| &\leq \|\hat{u}\|\alpha^{\nu/s}, \\ \|x_{\alpha,s} - \tilde{x}_{\alpha,s}\| &\leq \alpha^{-1/s}\delta. \end{aligned}$$

*In particular, if $\alpha = c\delta^{s/(\nu+1)}$ for some $c > 0$, then*

$$\|\hat{x} - \tilde{x}_{\alpha,s}\| = O(\delta^{\nu/(\nu+1)}).$$

*Proof.* If $\hat{x} = |T|^\nu \hat{u}$, then, from (3.4), we have

$$\hat{x} - x_{\alpha,s} = \alpha[\alpha I + |T|^s]^{-1}\hat{x} = \alpha[\alpha I + |T|^s]^{-1}|T|^\nu \hat{u}.$$

Hence,

$$\|\hat{x} - x_{\alpha,s}\| \leq \|\hat{u}\| \sup_{\lambda > 0} \frac{\alpha\lambda^\nu}{\alpha + \lambda^s}.$$

Also, from equation (3.5), we have

$$\|x_{\alpha,s} - \tilde{x}_{\alpha,s}\| \leq \|[\alpha I + |T|^s]^{-1}|T|^{s-1}U^*(y - \tilde{y})\| \leq \delta \sup_{\lambda > 0} \frac{\lambda^{s-1}}{\alpha + \lambda^s}.$$

Therefore, by Lemma 2.1, we obtain the inequalities

$$\|\hat{x} - x_{\alpha,s}\| \leq \|\hat{u}\|\alpha^{\nu/s}, \qquad \|x_{\alpha,s} - \tilde{x}_{\alpha,s}\| \leq \alpha^{-1/s}\delta.$$

The last part of the theorem is obvious. □

Note that, if $\nu = s$, then we obtain the optimal rate $O(\delta^{s/(s+1)})$.

### 3.2. Discrepancy Principle

As in Section 2, we consider the discrepancy principle

$$\|U^*\tilde{y} - |T|\tilde{x}_{\alpha,s}\| = \frac{\delta^p}{\alpha^q}, \quad p > 0,\ q > 0, \tag{3.6}$$

for choosing the regularization parameter $\alpha$.

**Proposition 3.2.** *Suppose $\alpha$ is chosen according to the discrepancy principle (3.6). Then*

$$\alpha = O(\delta^{p/(q+1)}).$$

*If $\hat{x} \in R(|T|^\nu)$, $0 < \nu \le s$, then*

$$\frac{\delta^p}{\alpha^q} = O(\delta + \alpha^{\nu/s}).$$

*Proof.* Note that

$$\begin{aligned}
\|U^*\tilde{y}\| - \frac{\delta^p}{\alpha^q} &= \|U^*\tilde{y}\| - \|U^*\tilde{y} - |T|\tilde{x}_{\alpha,s}\| \le \||T|\tilde{x}_{\alpha,s}\| \\
&= \frac{1}{\alpha}\||T|\tilde{x}_{\alpha,s}\| \\
&= \frac{1}{\alpha}\||T|^s U^*\tilde{y} - |T|^{s+1}\tilde{x}_{\alpha,s}\| \\
&\le \frac{1}{\alpha}\|T\|^s\ \|U^*\tilde{y} - |T|\tilde{x}_{\alpha,s}\| \\
&\le \frac{\|T\|^s\delta^p}{\alpha^{q+1}}.
\end{aligned}$$

From this it follows that $\alpha = O(\delta^{p/(q+1)})$. For obtaining the estimate for $\delta^p/\alpha^q$, first we observe that

$$\begin{aligned}
U^*\tilde{y} - |T|\tilde{x}_{\alpha,s} = U^*\tilde{y} - |T|(\alpha I + |T|^s)^{-1}|T|^{s-1}U^*\tilde{y} \\
= \alpha(\alpha I + |T|^s)^{-1}U^*\tilde{y}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{\delta^p}{\alpha^q} &= \|U^*\tilde{y} - |T|\tilde{x}_{\alpha,s}\| \\
&= \|\alpha(\alpha I + |T|^s)^{-1}U^*\tilde{y}\| \\
&\le \|\alpha(\alpha I + |T|^s)^{-1}U^*(\tilde{y} - y)\| + \|\alpha(\alpha I + |T|^s)^{-1}U^*y\| \\
&\le \delta + \|\alpha(\alpha I + |T|^s)^{-1}U^*y\|.
\end{aligned}$$

Now let $\hat{x} = |T|^\nu \hat{u}$ for some $\hat{u} \in H$. Then

$$\begin{aligned}\|\alpha(\alpha I + |T|^s)^{-1} U^* y\| &= \|\alpha(\alpha I + |T|^s)^{-1}|T|\hat{x}\| \\ &= \|\alpha(\alpha I + |T|^s)^{-1}|T|^{\nu+1}\hat{u}\|,\end{aligned}$$

and by using Lemma 2.1, we have

$$\|\alpha(\alpha I + |T|^s)^{-1}|T|^{\nu+1}\hat{u}\| \le \||T|\hat{u}\| \sup_{\lambda>0} \frac{\alpha\lambda^\nu}{\alpha+\lambda^s} \le \||T|\hat{u}\|\alpha^{\nu/s}.$$

Thus, $\delta^p/\alpha^q \le \delta + \||T|\hat{u}\|\alpha^{\nu/s}$, and the proof is complete. □

**Theorem 3.3.** *Suppose $\alpha$ is chosen according to the discrepancy principle (3.6). Assume that $p > 0$ and $q > 0$ are such that $p \le \min\{qs,\ q+1\}$, and $\hat{x} \in R(|T|^\nu)$ for $0 < \nu \le s$ . Then*

$$\ell := 1 - \frac{p\nu}{(q+1)s}\left[1 + \frac{1}{q}\left(1 - \frac{\nu}{s}\right)\right] > 0,$$

*and*

$$\|\hat{x} - \tilde{x}_{\alpha,s}\| = O(\delta^t),$$

*where*

$$t = \min\left\{\frac{p\nu}{(q+1)s}, \quad \ell\right\}.$$

*Proof.* From Theorem 3.1, we have

$$\|\hat{x} - x_{\alpha,s}\| \le \|\hat{u}\|\alpha^{\nu/s}, \qquad \|x_{\alpha,s} - \tilde{x}_{\alpha,s}\| \le \alpha^{-1/s}\delta.$$

Hence,

$$\|\hat{x} - \tilde{x}_{\alpha,s}\| = O\left(\alpha^{\nu/s} + \alpha^{-1/s}\delta\right).$$

By Proposition 3.2, we have $\alpha^{\nu/s} = O\left(\delta^{\frac{p\nu}{(q+1)s}}\right)$, and

$$\begin{aligned}\alpha^{-1/s}\delta &= \delta^{1-p/qs}\left(\frac{\delta^p}{\alpha^q}\right)^{1/qs} \\ &= O\left(\delta^{1-p/qs}(\delta + \alpha^{\nu/s})^{1/qs}\right) \\ &= O\left(\left[\delta^{qs-p}(\delta + \alpha^{\nu/s})\right]^{1/qs}\right) \\ &= O\left(\left[\delta^{qs-p+1} + \delta^{qs-p+p\nu/(q+1)s})\right]^{1/qs}\right).\end{aligned}$$

Since $p\nu/(q+1)s \leq p/(q+1) \leq 1$, it follows that $\alpha^{-1/s}\delta = O(\delta^{\ell})$, where

$$\ell := 1 - \frac{p}{qs} + \frac{p\nu}{(q+1)qs^2} = 1 - \frac{p\nu}{(q+1)s}\left[1 + \frac{1}{q}\left(1 - \frac{\nu}{s}\right)\right].$$

Clearly the condition $p \leq qs$ implies that $\ell \geq 0$. This completes that proof. □

From the above theorem, it follows that

$$t = \frac{p\nu}{(q+1)s} \iff \frac{p}{(q+1)s} \leq \frac{1}{\nu + 1 + \frac{1}{q}\left(1 - \frac{\nu}{s}\right)}.$$

In particular, we have the following.

**Corollary 3.4.** *Suppose $\alpha$ is chosen according to the discrepancy principle (3.6). Assume that $p > 0$ and $q > 0$ are such that*

$$p \leq qs, \quad \text{and} \quad p/(q+1) \leq s/(s+1).$$

*If $\hat{x} \in R(|T|^s)$, then*

$$\|\hat{x} - \tilde{x}_{\alpha,s}\| = O(\delta^t), \qquad t = \frac{p}{q+1}.$$

If we take $p, q$ such that

$$\frac{p}{q+1} = \frac{s}{s+1} \quad \text{and} \quad q \geq \frac{1}{s},$$

then by the above corollary, we obtain the optimal order $O(\delta^{s/(s+1)})$ whenever $\hat{x} \in R(|T|^s)$. In particular, $p = 1$ and $q = 1/s$ yield the above optimal order.

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