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VERSIONS OF QUANTUM KDV

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1. Introduction - Abstract

Various forms of "quantum" KdV type equations are derived in various contexts. The hierarchy picture for qKdV is well developed of course (cf. [1, 2, 5, 6, 7, 8, 9, 10, 25, 26, 27, 33, 41, 42]) and is sketched below. There are many other ways of deriving "quantum" KdV equations via differential calculi, Moyal deformation, quantum spaces, Maurer-Cartan equations, Virasoro algebras, etc. Many of these equations are not however equivalent to qKdV and yet they are obtained by extending known derivations of KdV to various contexts, some with quantum group theoretic significance. Hence all such quantum KdV equations seem to be worthy of consideration and represent candidates for describing some kind of "quantum" phenomena.

2. QKDV

One defines $D_q f(x) = [f(qx) - f(x)]/(q-1)x$ with $Df(x) = f(qx)$. Set (**A1**) $L = D_q + a_0 + \sum_1^\infty a_i D_q^{-i}$ and define qKP via (**A2**) $\partial_j L = [L_+^j, L]$ (cf. [1, 2, 5, 6, 8, 9, 25, 26, 27, 41, 42]). For qKdV (cf. also [10]) one

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writes $L^2 = D_q^2 + (q-1)xuD_q + u$ with $u = D_q\partial_1 \log[\tau_q(x,t)D\tau_q(x,t)]$ and (**A3**) $L = D_q + s_0 + s_q D_q^{-1} + \cdots$ with $s_0 = (q-1)x\partial_1\partial_q \log\tau_q$ (here $\tau_q = \tau(c(x)+t)$, $c(x) = \left(\frac{(1-q)^n x^n}{n(1-q^n)}\right)$, $t = (t_1, t_2, \cdots)$, and τ is an ordinary KdV tau function). In [5, 10] we exhibited the qKdV hierarchy equation in the form

$$\partial_t u = [L_+^3, L^2]_0 = [D_q^3 + w_2 D_q^2 + w_1 D_q + w_0, D_q^2 + u_1 D_q + u]_0 \tag{2.1}$$

$$= (D_q^3 u) + w_2(D_q^2 u) + w_1(D_q u) - [(D_q^2 w_0) + u_1(D_q w_0)]\,,$$

where $u_1 = (q-1)xu$ and $w_i = w_i(u)$. However,

$$w_2 = D^2 s_0 + u_1 = D^2 s_0 + Ds_0 + s_0, \tag{2.2}$$

$$w_1 = (q+1)(DD_q s_0) + \tau^2 s_1 + [(Ds_0) + s_0](Ds_0) + u,$$

$$w_0 = D_q^2 s_0 + (q+1)(DD_q s_1) + u_1 D_q s_0 + u_1(Ds_1) + us_0 + D^2 s_2\,,$$

and the determination of the s_i requires infinite series calculations based e.g. on formulas like

$$s_1 + Ds_1 = u - D_q s_0 - s_0^2 = f \Rightarrow s_1(x) = \sum_0^\infty (-1)^n f(q^n x)\,. \tag{2.3}$$

Similarly (**A4**) $Ds_2 + s_2 = -D_q s_1 - s_0 s_1 - s_1 D^{-1} s_0$, etc. This seems to suggest that an explicit form for a qKdV equation of this type with all coefficients specified in terms of u or s_0 would involve an infinite series in powers of D.

3. Candidate Structures

We recall first the derivation of KP from a differential calculus in [8, 11, 19, 20] (fundamentals of QG theory will be assumed as we proceed (cf. [8, 34, 36]).

3.1. Differential Calculi

The examples here are somewhat ad hoc and experimental.

Example 3.1. Consider a calculus based on **(A)** $dt^2 = dx^2 = dxdt + dtdx = 0$ **(B)** $[dt, t] = [dx, t] = [dt, x] = 0$ and $[dx, x] = \eta dt$. Assumingt the Leibnitz rule $d(fg) = (df)g + f(dg)$ for functions and $d^2 = 0$ one obtains (**A5**) $df = f_x dx + (f_t + (1/2)\eta f_{xx})dt$. For a connection $A = wdt + udx$ the zero curvature condition $F = dA + A^2 = 0$ leads to (**A6**) $(u_t - w_x + (\eta/2)u_{xx} + \eta uu_x = 0$, which for $w_x = 0$ is a form of Burger's equation.

Example 3.2. Next consider **(A)** $[dt, t] = [dx, t] = [dt, x] = [dy, t] = [dt, y] = [dy, y] = 0$ with **(B)** $[dx, x] = 2bdy$ and $[dx, y] = [dy, x] = 3adt$. Further **(C)** $dt^2 = dy^2 = dtdx + dxdt = dydt + dtdy = dydx + dxdy = 0$. Then (**A7**) $df = f_x dx + (f_y + bf_{xx})dy + (f_t + 3af_{xy} + abf_{xxx}dt)$. For $A = vdx + wdt + udy$ one finds that $dA + A^2 = F = 0$ implies

$$\begin{aligned} u_x &= v_y + bv_{xx} + 2bvv_x\,, \\ w_x &= 3av_{xy} + abv_{xxx} + 3auv_x + 3av(v_y + bv_{xx}), \end{aligned} \tag{3.1}$$

$$w_x + bw_{xx} = u_t + 3au_{xy} + abu_{xxx} + 3auu_x - v[2bw_x - 3a(u_y + bu_{xx})]\,.$$

Taking e.g. $w_x = (3a/2b)u_y + (3a/2)u_{xx}$ in the last equation to decouple one arrives at (**A8**) $\partial_x(u_t - (ab/2)u_{xxx} + 3auu_x) = (3a/2b)u_{yy}$, for suitable a, b this is KP. If the equation is independent of y we obtain a version of KdV.

It is surprisingly difficult to convert these examples into meaningful q-calculus equations and in that spirit for guidance we were motivated to develop many formulas concerning qKP, qKdV, etc. We recall first the first order differential calculus (FODC) Γ_+ from [34] on a quantum plane or Manin plane (cf. also [8, 44]), this is based on $xp = qpx$ with

$$\begin{aligned} &dx^2 = dp^2 = 0, \quad && dxdp = -q^{-1}dpdx, \\ &xdx = q^2 dxx, \quad && xdp = qdpx + (q^2-1)dxp, \end{aligned} \tag{3.2}$$

$$pdx = qdxp,\ pdp = q^2 dpp,\ \partial_x\partial_p = q^{-1}\partial_p\partial_x,$$
$$\partial_x x = 1 + q^2 x\partial_x + (q^2-1)p\partial_p,\ \partial_x p = qp\partial_x,$$
$$\partial_p x = qx\partial_p,\ \partial_p p = 1 + q^2 p\partial_p,\ \partial_x dx = q^{-2}dx\partial_x,\ \partial_x dp = q^{-1}dp\partial_x,$$
$$\partial_p dx = q^{-1}dx\partial_p,\ \partial_p dp = q^{-2}p\partial_p + (q^2-1)dx\partial_x\,.$$

In this FODC the partial derivatives ∂_i of Γ_+ act on $\mathfrak{A}$ = formal power series with x, p ordering, via ($\partial_p x^n = q^n x^n \partial_p$ and $\partial_x p^n = q^n p^n \partial_x$)

$$\begin{aligned} \partial_x(f(x)h(p)) = (D^x_{q^2}f(x))h(p),\ \partial_p(f(x)h(p)) & \\ = (T_q f(x))(D^p_{q^2}h(p))\,, & \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\partial_x(x^n) = D_{q^2}x^n = [[n]]_{q^2}x^{n-1}, \qquad [[n]]_{q^2} = \frac{q^{2n}-1}{q^2-1}, \\ &\partial_p p^n = [[n]]_{q^2}p^{n-1}\,. \end{aligned} \tag{3.4}$$

Example 3.3. Consider the q-plane situation

$$(dx)^2 = (dt)^2 = 0, \quad xt - qtx = 0, \\ dxdt = -q^{-1}dtdx, \quad xdx = q^2dxx, \tag{3.5}$$

$$xdt = qdtx + (q^2-1)dxt, \; tdx = qdxt, \; tdt = q^2dtt\,.$$

Then

$$df = D_t^{-1}\partial_x f dx + \partial_t f dt \tag{3.6}$$

(with $\partial_x \sim D_{q^2}$ as in Γ_+, etc.). Then $A = udx + wdt$ yields

$$dA + A^2 = 0 \rightsquigarrow -q\partial_t u + D_t^{-1}\partial_x w + uD_x^{-2}D_t^{-2}w \tag{3.7}$$

$$-\frac{wqt}{x}[D_x^{-1}D_t^{-1}w - D_x^{-3}D_t^{-1}w] - qwD_x^{-1}D_t^{-2}u = 0\,.$$

For $q \to q^{-1}$ evidently $\partial_x f = [f(q^2x) - f(x)]/(q^2-1)x \to [f(q^{-2}x) - f(x)]/(q^{-2}-1)x$ and we write this latter as $\hat{\partial}_x$. This gives

$$-q^{-1}\hat{\partial}_t u + D_t\hat{\partial}_x w + uD_x^2D_t^2 w - q^{-1}wD_xD_t^2u \\ - \frac{wt}{qx}[D_xD_tw - D_x^3D_tw] = 0 \tag{3.8}$$

Note that as $q \to 1$ (3.7) becomes (**A9**) $-\partial_t u + \partial_x w + w - wu = 0$. Taking $w = u_x$ one has then (**A10**) $-u_t + u_{xx} + u_x - uu_x = 0 \rightsquigarrow u_t + uu_x + u_{xx} - u_x = 0$, which is a kind of perturbed Burger's equation with perturbation $-u_x$.

Example 3.4. Consider the generalized q-plane with an algebra generated by x, y, x^{-1}, y^{-1}, where $xy = qyx$, $xdx = qdxx$, $ydx = q^{-1}dxy$, $xdy = qdyx$, and $ydy = q^{-1}dyy$. Also from $qdyx = dxy$ we have $qdydx = dxdy$ and a little calculation yields (**A11**) $dx^n = [(1-q^{-n})/(1-q^{-1})]x^{n-1}dx$ with $dy^m = [(1-q^m)/(1-q)]y^{m-1}dy$. Working from $f = \sum a_{nm}x^ny^m$ one obtains then (note $dxy^m = q^my^mdx$)

$$df = D_yD^x_{q^{-1}}fdx + D^y_qfdy\,. \tag{3.9}$$

Set then $A = wdy + udx$ with $dA = D_yD^x_{q^{-1}}wdxdy + D^y_qudydx$ and, noting that $dyx^n = q^{-n}x^ndy$, $dyy^m = q^my^mdy$, and $dxy^m = q^my^mdx$ with $dxx^n = q^{-n}x^ndx$ one gets $dyw = D_x^{-1}D_ywdy$ and $dxu = D_x^{-1}D_yudx$ leading to $A^2 = wD_x^{-1}D_yudydx + uD_x^{-1}D_ywdxdy$, and

$$dA + A^2 = 0 = qD_yD^x_{q^{-1}}w + D^y_qu + wD_x^{-1}D_yu + quD_x^{-1}D_yw\,. \tag{3.10}$$

Setting then e.g. $qw = D_y^{-1} D_{q^{-1}}^x u$ one gets

$$D_q^y u + (D_{q^{-1}}^x)^2 u + q^{-1}(D_x^{-1} D_y u)(D_y^{-1} D_{q^{-1}}^x u) + u D_x^{-1} D_{q^{-1}}^x u = 0\,. \qquad (3.11)$$

For $q \to 1$ we have (**A12**) $u_y + u_{xxx} + 2uu_x = 0$ so this appears to be an exact q-form of Burger's equation.

Example 3.5. We will try now a somewhat different approach. First we take q-derivatives only in x, as in the case of qKP for example and we know from Example 3.2 that **(A7)** leads to interesting consequences so begin with an assumption ($f_y = \partial f/\partial y$, etc.)

$$df = D_q^x f dx + (f_y + b(D_q^x)^2 f)dy + (f_t + 3a\partial_y D_q^x f + ab(D_q^x)^3 f)dt\,. \qquad (3.12)$$

Then we can determine what elementary commutation relations between the variables are consistent with (3.12). This is rather ad hoc but we stipulate x, y, t ordering and then there are relations

$$\begin{aligned} dxx &= qxdx + b[2]_q dy, & dyx &= q^2 xdy + a[3]_q dt, \\ dtx &= q^3 xdt, & dxy &= ydx + 3adt\,, \end{aligned} \qquad (3.13)$$

along with

$$[dt, y] = [dy, y] = [dy, t] = [dx, t] = [dt, t] = [dx, y] = 0\,, \qquad (3.14)$$

which are determined by (3.12). The underlying structure for x, y, t is not visible from (3.12) but (3.13) - (3.14) do lead to (3.12) and whatever zero curvature equations subsequently arise (see [5] for details). Returning now to (3.12) it remains to check now the zero curvature equation arising in the spirit of Example 3.2 (some extra factors and terms will arise via noncommutativity). Thus, assume first $dx^2 = dy^2 = dt^2 = 0$ and take $A = vdx + wdt + udy$, so

$$\begin{aligned} dA = (v_y + b(D_q^x)^2 v)dydx + (v_t + 3a\partial_y D_q^x v \\ + ab(D_q^x)^3 v)dtdx + D_q^x w dxdt + (w_y + b(D_q^x)^2 w)dydt \\ + D_q^x u dxdy + (u_t + 3a\partial_y D_q^x u + ab(D_q^x)^3 u)dtdy\,. \end{aligned} \qquad (3.15)$$

For A^2 one puts tentatively (♣) $dxdt + dtdx = 0 = dxdy + dydx = dydt + dtdy$ and some calculation (as in [5]) yields $dA + A^2 = 0$, which compares with Example 3.2 as follows (there are a number of cancellations of the form $uv - vu$, which do not cancel in the q-situation due to factors of D_x)

- $dxdy$: $-2bvu_x \sim vD_x u - uD_x^2 v - b[2]_q v D_x D_q^x v$.

- $dtdx$: $3auv_x + 3avv_y + 3abvv_{xx} \sim wD_x^3v - vD_xw + a[3]_q uD_x^2D_q^x v + 3avD_x\partial_y v + ab[3]_q vD_x(D_q^x)^2 v.$
- $dtdy$: $3avu_y + 3auu_x + 3abvu_{xx} - 2bvw_x \sim wD_x^2u - uD_x^3w + 3avD_x\partial_y u + a[3]_q uD_x^2D_q^x u + ab[3]_q vD_x(D_q^x)^2u - b[2]_q vD_xD_q^x w.$

These are all close in the same manner, with distortions due to factors of D_x and q-numbers. One expects now to be able to produce a qKP type equation directly from these formulas and some manipulation as in Example 3.2. Note that there does not seem to be any explicit formula in the literature for qKP, it is defined via a hierarchy and calculations involving specific equations are difficult, even in the Hirota forms (cf. [9]). We compare now the KP type equations derivable from Examples 3.2 and 3.5. First

$$\begin{aligned} &v_y - u_x + bv_{xx} + 2vvv_x = 0 \\ &\quad \sim v_y - D_q^x u + b(D_q^x)^2 v + [2]_q bvD_xD_q^x v + uD_x^2 v - vD_x u = 0\,, \end{aligned} \tag{3.16}$$

$$\begin{aligned} &v_t + 3av_{xy} + abv_{xxx} - w_x + 3auv_x + 3avv_y + 3abvv_{xx} = 0 \\ &\quad \sim v_t + 3a\partial_y D_q^x v + ab(D_q^x)^3 v - D_q^x w + [3]_q auD_x^2D_q^x v \\ &\qquad + 3avD_x\partial_y v + [3]_q abvD_x(D_q^x)^2 v = 0\,, \end{aligned} \tag{3.17}$$

$$\begin{aligned} &u_t + 3au_{xy} + abu_{xxx} - bw_{xx} - w_y + 3avu_y + 3auu_x + 3abvu_{xx} - 2bvw_x = 0 \\ &\quad \sim u_t + 3a\partial_y\partial_q^x u + ab(D_q^x)^3 u - b(D_q^x)^2 w \\ &\qquad - w_y + wD_x^3 u - uD_x^2 w + 3avD_x\partial_y u + [3]_q auD_x^2D_q^x u \\ &\qquad + [3]_q abvD_x(D_q^x)^2 u - b[2]_q vD_q^x w = 0\,. \end{aligned} \tag{3.18}$$

The similarities are obvious. Now look at the derivation of KP in Example 3.2. One takes $w_x = (3a/2b)u_y + (3a/2)u_{xx}$ to decouple in (3.18) which reduces (3.18) to (♦) $u_t + 3au_{xy} + abu_{xxx} - bw_{xx} - w_y = 0$. Thus, setting (**A13**) $w = \partial^{-1}\left(\frac{3a}{2b}u_y\right) + \frac{3a}{2}u_x$ leads to a KP type equation and repeating such a procedure for the q-situation (3.16)-(3.18) leads to a q version of KP in the form

$$\begin{aligned} &u_t + ab(D_q^x)^3 u - \frac{[3]_q ab}{[2]_q} D_q^x D_x (D_q^x)^2 u + [3]_q auD_x^2 D_q^x u \\ &\qquad - \frac{3a}{[2]_q b}(D_q^x)^{-1} D_x u_{yy} = A(u,w) + B(u)\,, \end{aligned} \tag{3.19}$$

where

$$w = \frac{3a}{[2]_q b}(D_q^x)^{-1}D_x\partial_y u + \frac{[3]_q a}{[2]_q q}D_x D_q^x u, \tag{3.20}$$

$$A(u,w) = wD_x^3 u - uD_x^2 w,$$

$$B(u) = \frac{[3]_q a}{[2]_q q}D_x\partial_y D_q^x u + \frac{3a}{[2]_q}D_q^x D_x\partial_y u - 3a\partial_y D_q^x u\,.$$

Then $A \to 0$ and $B \to 0$ as $q \to 1$ and the equation (3.20) goes to the standard KP form. Since this equation was derived from exactly the same procedures as the classical KP equation (via zero curvature considerations) one expects it to be a good candidate for a qKP equation. We see no immediate connection to qKP however. For y independent u we obtain

$$u_t + ab(D_q^x)^3 u - \frac{[3]_q ab}{[2]_q}D_q^x D_x (D_q^x)^2 u + [3]_q a u D_x^2 D_q^x u = \frac{[3]_q a}{[2]_q b}\left(D_x D_q^x u D_x^3 u - u D_x^3 D_q^x u\right) \tag{3.21}$$

which is a quantum KdV equation.

3.2. Geometry

We indicate two "geometrical" contexts in a classical vein and subsequently give "quantum" versions of these.

Example 3.6. One can devise a procedure directly from [16]. Thus, look at $SL(2,\mathbf{R})$ with matrices (**A14**) $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc = 1$. The right invariant Maurer-Cartan (MC) form is

$$\omega = dXX^{-1} = \begin{pmatrix} \omega_1^1 & \omega_1^2 \\ \omega_2^1 & \omega_2^2 \end{pmatrix}, \tag{3.22}$$

where $\omega_1^1 + \omega_2^2 = 0$. The structure equation of $SL(2,\mathbf{R})$ or MC equation is (**A15**) $d\omega = \omega \wedge \omega$ or explicitly (**A16**) $d\omega_1^1 = \omega_1^2 \wedge \omega_2^1$, $d\omega_1^2 = 2\omega_1^1 \wedge \omega_1^2$, $d\omega_2^1 = 2\omega_2^1 \wedge \omega_1^1$. Now let U be a neighborhood in the (x,t) plane and consider a smooth map $f : U \to SL(2,\mathbf{R})$. The pullback of the MC form can be written as (**A17**) $\omega_1^1 \sim \eta dx + Adt$, $\omega_1^2 \sim Qdx + Bdt$, $\omega_2^1 \sim rdx + Cdt$ with coefficient functions of x,t. The equations **(A16)** become

1. $-\eta_t + A_x - QC + rB = 0.$

2. $-Q_t + B_x - 2\eta B + 2QA = 0$.

3. $-r_t + C_x - 2rA + 2\eta C = 0$.

Take $r = 1$ with η independent of (x, t) and set $Q = u(x, t)$. Then from (1) and (3) one gets (**A18**) $A = \eta C + \frac{1}{2}C_x$, $B = uC - \eta C_x - \frac{1}{2}C_{xx}$. Putting this in the (2) above yields $u_t = K(u)$, where (**A19**) $K(u) = u_x C + 2uC_x + 2\eta^2 C_x - \frac{1}{2}C_{xxx}$. In the special case $C = \eta^2 - (1/2)u$ one gets the KdV equation (**A20**) $u_t = \frac{1}{4}u_{xxx} - \frac{3}{2}uu_x$.

Example 3.7. Following [3], let $Vec(S^1)$ denote the Lie algebra of smooth vector fields on S^1 and then the Virasoro algebra is $Vir = Vec(S^1) \oplus \mathbf{R} = \mathfrak{W} \oplus \mathbf{R}$ with (note the minus sign convention involving $f'g - fg'$)

$$[(f(x)\partial_x, a), (g(x)\partial_x, b)] = \left((f'g - fg')\partial_x, \int_{S^1} f'g''dx\right) \quad (3.23)$$

($\mathfrak{W} \sim$ Witt algebra). Here $\int_{S^1} f'g''dx$ is called the Gelfand-Fuks cocycle, where a cocycle on a Lie algebra $\mathfrak{g}$ is a bilinear skew symmetric form $c(\cdot, \cdot)$ satisfying (**A21**) $\sum c([f, g], h) = 0$, where the sum is over cyclic permutations of f, g, h. This means that $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbf{R}$ (central extension) with commutator $[(f, a), (g, b)] = ([f, g], c(f, g))$ satisfies the Jacobi identity of a Lie algebra. Next one defines the Virasoro group as the set of pairs $(\phi(x), a) \in Diff(S^1) \oplus \mathbf{R}$ with multiplication law

$$(\phi(x), a) \circ (\psi(x), b) = \left(\phi(\psi(x)), a + b + \int_{S^1} \log(\phi \circ \psi(x))' d \log \psi'(x)\right). \quad (3.24)$$

This can be equipped with a right invariant Riemannian metric via an energy like quadratic form on Vir ($\sim$) tangent space at the group identity) of the form

$$H(f(x)\partial_x, a) = \frac{1}{2}\left(\int_{S^1} f^2(x)dx + a^2\right). \quad (3.25)$$

Then the KdV equation on the circle is the evolution equation (**A22**) $\partial_t u + uu' + u''' = 0$, where $' \sim \partial_x$. More precisely the Euler equation corresponding to geodesic flow is a 1-parameter family of KdV equations. To see how this arises consider (**A3**) $Vir^* = \{(u(x)dx^2, c),\ u$ smooth on S^1 and $c \in \mathbf{R}\}$. Then

$$< (v(x)\partial_x, a), (u(x)dx^2, c) >= \int_{S^1} v(x)u(x)dx + ac. \quad (3.26)$$

The coadjoint action of $(f\partial_x, a) \in Vir$ on $(udx^2, c) \in Vir^*$ is ($ad_v^* : \mathfrak{g}^* \to \mathfrak{g}^*,\ ad_v^* w(u) = w(ad_v u)$)

$$ad^*_{(f\partial_x,a)}(udx^2, c) = (2f'u + fu' + cf''')dx^2, 0)\,, \tag{3.27}$$

which arises from the identity

$$< [(f\partial_x, a), (g\partial_x, b)], (udx^2, c) > \\ =< (g\partial_x, b), ad^*_{(f\partial_x,a)}(udx^2, c) > . \tag{3.28}$$

Note here from (3.23) and (3.26)

$$< \left((f'g - fg')\partial_x, \int_{S^1} f'g''dx \right), (udx^2, c) > \\ = \int_{S^1}(f'g - fg')udx + c\int_{S^1} f'g''dx\,, \tag{3.29}$$

while from (3.26) and (3.27)

$$< (g\partial_x, b), ad^*_{f\partial_x,a)}(udx^2, c) > \\ =< (g\partial_x, b), ((2f'u + fu' + cf''')dx^2, 0) > . \tag{3.30}$$

Now for S^1 there are no boundary terms in integration (for single valued functions) so, integrating by parts,

$$\int_{S^1} g(2f'u + fu' + cf''')dx = \int_{S^1}[u(gf' - fg') + cf'g'']dx \tag{3.31}$$

in agreement with (3.29). Now a function H on $\mathfrak{g} = Vir$ determines a tautological inertia operator $A : Vir \to Vir^* : (u\partial_x, c) \to (udx^2, c)$ and hence a quadratic Hamiltonian on Vir^* via

$$H(udx^2, c) = \frac{1}{2}\left(\int_{S^1} udx^2 + c^2\right) = \frac{1}{2} < (u\partial_x, c), (udx^2, c) > \\ = \frac{1}{2} < (u\partial_x, c), A(u\partial_x, c) > . \tag{3.32}$$

Following [34] the corresponding Euler equation is $\dot{m} = -ad^*_{A^{-1}m}m$ ($m \in \hat{\mathfrak{g}}$), which here takes the form

$$\partial_t(udx^2, c) = -ad^*_{A^{-1}(udx^2,c)}(udx^2, c)\,, \tag{3.33}$$

which becomes via (3.27) with (**A24**) $(f\partial_x, a) = A^{-1}(udx^2, c) = (u\partial_x, c)$

$$\partial_t u = -2u'u - uu' - cu''' = -3uu' - cu''' \,, \tag{3.34}$$

where c is independent of time. One notes also that without the central extension of $Vec(S^1)$ we arrive in the same manner at a nonviscous Burger's equation (**A25**) $\partial_t u = -3uu'$.

4. Q-Virasoro

For q-Virasoro we go directly to [35] and refer to the bibliography in the references given above in the introduction for other approaches. Thus work on S^1 with $(q \neq 0, \pm 1)$

$$\partial_q z = \frac{q^m z^m - q^{-m} z^m}{(q - q^{-1} z} = z^{m-1}[m],\ [m] = \frac{q^m - q^{-m}}{q - q^{-1}} \,. \tag{4.1}$$

We adapt the formalism of [35] as follows. Let $D_n = -z^{n+1}\partial$ with $\partial : z^m \to q^m[m]z^{m-1}$ so $\partial \sim \partial_q \tau$, where $\tau f(z) = f(qz)$. Generally we will think of $z = e^{i\theta} \in S^1$ so $(1/2\pi i)\int_{S^1} z^n dz = (1/2\pi)\int z^{n+1} d\theta = \delta_{(-1,0)}$, which will be written as (**A26**) $\int z^n = \delta_{(-1,0)}$. Write also (**A27**) $\ell_n \sim z^{n+1}\partial = -D_n = z^{n+1}\partial_q \tau$. It is known that q-brackets are needed now where

$$[\ell_m, \ell_n]_q = q^{m-n}\ell_m \ell_n - q^{n-m}\ell_n \ell_m = [m-n]\ell_{m+n} \,. \tag{4.2}$$

For a central term in a putative Vir_q one wants (cf. [3, 14, 35]) a formula (**A28**) $c[m+1][m][m-1]\delta_{m+m,0}$ (see Remark 3.1 for an optimal term). First, we want to formulate the q-bracket in terms of vector fields as follows (the central term will be added later in a somewhat ad hoc manner). This can be done as a direct calculation (note $\partial_q f = (\tau f)\partial_q + (\partial_q f)\tau^{-1}$)

$$\begin{aligned}
[z^n\partial, z^m\partial]_q &\sim q^{n-m} z^n \partial(z^m \partial) - q^{m-n} z^m \partial(z^n \partial) \\
&= q^{n-m} z^n q^m \partial_q z^m \tau\partial - q^{m-n} z^m q^n \partial_q z^n \tau \partial \\
&= q^{n-m} z^n q^m (q^m z^m \partial_q + [m]_q z^{m-1}\tau^{-1})\tau\partial \\
&\quad - q^{m-n} z^m q^n (q^n z^n \partial_q + [n]_q z^{n-1}\tau^{-1})\tau\partial \\
&= (q^n[m] - q^m[n]) z^{m+n-1}\partial = [n-m] z^{m+n-1}\partial \,.
\end{aligned} \tag{4.3}$$

Let now $v \sim \sum a_n z^n$ and $w \sim \sum b_m z^m$, then we define a bracket in $Vec(S^1)$ via

$$
\begin{aligned}
[v\partial, w\partial]_q &\sim (\tau v)\partial(\tau^{-1}w)\partial - (\tau w)\partial(\tau^{-1}v)\partial \\
&= (\tau v)\partial_q \tau(\tau^{-1}w)\partial - (\tau w)\partial_q \tau(\tau^{-1}v)\partial \\
&= (\tau v)(\partial_q w \tau\partial) - (\tau w)(\partial_q v\tau\partial) \\
&= \sum a_n q^n z^n (\sum b_m (q^m z^m \partial_q + [m]z^{m-1}\tau^{-1}))\tau\partial \\
&- \sum b_m z^m q^m (\sum a_n (q^n z^n \partial_q + [n]z^{n-1}\tau^{-1}))\tau\partial \\
&= \sum a_n b_m (q^n[m] - q^m[n]) z^{m+n-1})\partial \\
&= \sum a_n b_m [n-m] z^{m+n-1}\partial\,. \qquad (4.4)
\end{aligned}
$$

Proposition 4.1. From (4.4) we have a correspondence

$$
\begin{aligned}
v'w - vw' &\sim -[v\partial_x, w\partial_x] \sim -[v\partial, w\partial]_q \\
&= -\{(\tau v)(\partial_q w) - (\tau w)(\partial_q v)\}\tau\,. \qquad (4.5)
\end{aligned}
$$

Remark 4.1. In [35] one defines the q-analogue of the enveloping algebra of the Witt algebra $\mathfrak{W}$ as the associative algebra $\mathfrak{U}_q(\mathfrak{W})$ having generators ℓ_m ($m \in \mathbf{Z}$) and relations (4.2). The q-deformed Virasoro algebra is defined as the associative algebra $\mathfrak{U}_q(Vir)$ having generators ℓ_m ($m \in \mathbf{Z}$) and relations ($q \neq$ root of unity)

$$
\begin{aligned}
q^{m-n}\ell_m\ell_n - q^{n-m}\ell_n\ell_m \\
= [m-n]\ell_{m+n} + \delta_{m+n,0}\frac{[m+1][m][m-1]}{[2][3] < m >}\hat{c}\,, \qquad (4.6)
\end{aligned}
$$

where $< m >= q^m + q^{-m}$ and $\hat{c}\ell_m = q^{2m}\ell_m\hat{c}$ (thus, $\hat{c}$ is an operator, which we examine below and we refer to [14, 35] for the central term). Then $\mathfrak{U}_q(Vir) \sim Vir_q$ is a $\mathbf{Z}$ graded algebra with $\deg(\ell_m) = m$ and $\deg(\hat{c}) = 0$. One also introduces in [35] a larger algebra $\mathfrak{U}(V_q)$ = associative algebra generated by $J^{\pm 1}$, $\hat{c}$, d_m ($m \in \mathbf{Z}$) with relations

$$
JJ^{-1} = J^{-1}J = 1,\ Jd_mJ^{-1} = q^m d_m,\ \hat{c}J = J\hat{c},\ \hat{c}d_m = q^m d_m\hat{c}, \qquad (4.7)
$$

$$
q^m d_m d_n J - q^n d_n d_m J = [m-n]d_{m+n} + \delta_{m+n,0}\frac{[m+1][m][m-1]}{[2][3] < m >}\hat{c}
$$

The subalgebra of $\mathfrak{U}(V_q)$ generated by $\ell'_m = d_m J$ and $\hat{c}' = \hat{c}J$ ($m \in \mathbf{Z}$) is the same as $\mathfrak{U}_q(Vir)$. We will treat (4.6) as displaying the natural (preferred) central term for our purposes.

4.1. Calculations

Now we mimic the framework of Example 3.7 and it is interesting to note that an ordinary integral $\int_{S^1}$ will suffice. One does not need a Jackson type integral in order to deal with integration by parts. Thus, we observe that

$$\int_{S^1} f = \int \sum f_n z^n = f_{-1},\ \int \partial_q f = \frac{1}{q-q^{-1}} \int \frac{f(qz) - f(q^{-1}z)}{z}$$
$$= \frac{1}{q-q^{-1}} \int \sum f_n z^{n-1}(q^n - q^{-n}) = \frac{1}{q-q^{-1}}(f_0 - f_0) = 0\,. \quad (4.8)$$

Since $\partial_q(fh) = (\tau f)(\partial_q h) + (\partial_q f)(\tau^{-1}h)$ we have an integration by parts formula

$$\int (\tau f)(\partial_q h) = -\int (\partial_q f)(\tau^{-1}h) \Rightarrow \int f\partial_q(\tau h) = -\int \partial_q(\tau^{-1}f)h\,. \quad (4.9)$$

This can be written as (recall $\partial \sim \partial_q \tau$) (**A29**) $\int f\partial h = -\int h\hat{\partial} f$ for $\hat{\partial} = \partial_q \tau^{-1}$. Now we think of $\mathfrak{U}_q(Vir)$ with elements $(f\partial, a)$ as in (2.1) with

$$[(f\partial, a), (g\partial, b)] = (-[f\partial, g\partial]_q \partial, \int (\tau\partial^3 f)(\tau g)\hat{c})\,. \quad (4.10)$$

The central term is defined tentatively via (**A30**) $\int(\tau\partial^3 f)(\tau g)\hat{c} = \psi(f\partial, g\partial)$ where one has (**A31**) $\psi(f\partial, g\partial) = q^{-1}\int g\partial^3 f\hat{c}$ since

$$\int \tau a \tau b = \int \sum a_n b_m q^{n+m} z^{n+m} = \sum a_n b_{-n-1} q^{-1} = q^{-1} \int ab\,. \quad (4.11)$$

We will want to put the central operator $\hat{c}$ into the integral **(A30)** or **(A31)**, acting on f, and will see below that $\hat{c} \sim \tau^2$ for example and $\tau^2 F(z) = F(q^2 z)\tau^2$ so it eventually automatically passes to the right in our qKdV type equations. Hence for the moment think of $\hat{c} = \tau^2$ put into (5.2) or **(A30)** via e.g. $\tau\partial^3\tau^{-2}\hat{c}f \equiv \tau\partial^3 f$ and ignored at the end except when exhibiting formulas like (4.6) on generators (see also remarks below).

Now duality as in (2.4) will be expressed here via (**A32**), $<(v\partial, a), (u, c)>= \int v\tau^{-1}u + ac$ and hence from (4.5)

$$<[(f\partial, a), (g\partial, b)], (u, c)>= \int -[f\partial, g\partial]_q \tau^{-1} u + c\psi(f\partial, g\partial) \quad (4.12)$$
$$= \int (\tau g)(\partial_q f)u - (\tau f)(\partial_q g)u + c\int (\tau\partial^3 f)(\tau g)\hat{c}$$

This puts us in the framework of (2.7), (2.9), etc. Finally note (**A33**), $\partial_q(g(\tau^2 f)\tau u) = (\partial_q g)(\tau f)u + (\tau g)\partial_q((\tau^2 f)\tau u)$ so (5.4) becomes

$$\int (\tau g)[(\partial_q f)u + \partial_q((\tau^2 f)\tau u) + c\tau\partial^3 f] \tag{4.13}$$

Theorem 4.1. *In the spirit of Example 3.7* (4.13) *leads to a tentative qKdV type equation for $f = u$ (note $\partial_q\tau = q\tau\partial_q$)*

$$u_t = -c\tau\partial^3 u - (\partial_q u)u - \partial_q((\tau^2 u)\tau u) = -c\tau\partial^3 u - (1+q\tau)^2 u\partial_q u\,, \tag{4.14}$$

where $\partial^3 u = \partial_q\tau\partial_q\tau\partial_q\tau u$ (see remarks and results below especially where the operator $\hat{c}$ is put into the equation more meaningfully and the central term follows (4.6) *acting on generators).*

Remark 4.2. $\psi(f\partial, g\partial)$ appears to be a perfectly satisfactory central term even though it does not seem to be a cocycle on two counts and thus the theorem uses a reduced structure for its derivation (e.g. there will not be a Jacobi identity with brackets (5.2)). First, note via (**A31**) $\psi(f\partial, g\partial) = q^{-1}\int g\partial^3 f = q^{-1}\hat{\psi}(f\partial, g\partial)$ and an elementary calculation gives (**A34**) $\int g\partial^3 f = -\int f\hat{\partial}^3 g$ with $\hat{\partial}$ as in **(A29)**. However, the antisymmetry condition $\psi(f\partial, g\partial) = -\psi(g\partial, f\partial)$ does not hold since for $f = \sum f_{n+1}z^{n+1}$ and $g = \sum g_{m+1}z^{m+1}$

$$\int f\partial^3 g = \sum f_{n+1}g_{-n+1}[-n+1][-n][-n-1]q^{-3n}\,. \tag{4.15}$$

We note, however, that $\int(\partial^3 z^{n+1})z^{m+1} = [n+1][n][n-1]\int z^{m+n-1} = [n+1][n][n-1]\delta_{m+n,0}$ as in **(A28)** but this falls short of (4.6) by an n dependent term $< n >^{-1}$. The remaining cocycle condition can be written as

$$\hat{\psi}(f\partial, [g\partial, h\partial]_q\partial) + \hat{\psi}(h\partial, [f\partial, g\partial]_q\partial) + \hat{\psi}(g\partial, [h\partial, f\partial]_q\partial) = 0\,, \tag{4.16}$$

and one can write out the terms directly to see that cancellation does not occur. One can salvage a bit here by taking as central term

$$\begin{aligned}\tilde{\psi}(f\partial, g\partial) &= \int(\tau\partial^3\tau^{-3}f)(\tau g) = q^{-1}\int(\partial^3\tau^{-3}f)g \\ &= q^{-4}\int\sum f_{n+1}g_{m+1}[n+1][n][n-1]x^{n+m-1} \\ &= q^{-4}\sum f_{n+1}g_{-n+1}[n+1][n][n-1]\,. \end{aligned} \tag{4.17}$$

For this we have

$$\tilde{\psi}(g\partial, f\partial) = q^{-4}\sum f_{n+1}g_{-n+1}[-n+1][-n][-n-1] \\ = -\tilde{\psi}(f\partial, g\partial)\,, \quad (4.18)$$

so antisymmetry is realized. However a little calculation shows that the cocycle condition (4.16) again does not hold. In any event, using $\tilde{\psi}$, one gets a tentative qKdV equation

$$u_t = -c\tau\partial^3\tau^{-3}u - (1+q\tau)^2 u\partial_q u \quad (4.19)$$

analogous to (4.14).

Remark 4.3. Regarding structure on $Vec(S^1)\oplus\mathbf{R}$ one does not have a central extension (as with $Vir = Vec(S^1)\oplus\mathbf{R}$ in the classical case), but there is an antisymmetric bracket (cf. (5.2))

$$[(g\partial, b), (f\partial, a)] = (-[g\partial, f\partial]_q, \tilde{\psi}(g\partial, f\partial)) = -[(f\partial, a), (g\partial, b)] \\ = -(-[f\partial, g\partial]_q, \tilde{\psi}(f\partial, g\partial)) = -([g\partial, f\partial]_q, \tilde{\psi}(f\partial, g\partial))\,. \quad (4.20)$$

However, without a cocycle we do not have a Jacobi identity. Nevertheless, a dual structure can be defined as in **(A32)** and manipulated as in Example 3.7 (with no need to refer to quadratic differentials, etc.). Further the structure imposed by $\tilde{\psi}$ does give (reinserting $\hat{c}$ on generators as discussed above after (5.3))

$$\tilde{\psi}(\ell_n, \ell_m) = \tilde{\psi}(z^{n+1}\partial, z^{m+1}\partial) = q^{-4}[n+1][n][n-1]\delta_{m+n,0}\hat{c}\,, \quad (4.21)$$

as stipulated in **(A28)** so we are speaking of $Vec_q(S^1)\oplus\mathbf{R}$ defined on generators without specifying (or needing) any additional structure. Note also that the definition **(A32)** of a $(Vec(S^1)\oplus\mathbf{R})^*$ (vector space dual) related to **(A23)** is straightforward as is the generalization of ad^* in (3.30). No algebraic structure is used or needed in tracing the development from Example 3.7 although the significance of the resulting equations seems dependent on the form of central term and would presumably be enhanced in the presence of a "q-cocycle" of some sort.

Remark 4.4. Now from [35] we know that $\mathfrak{U}_q(\mathfrak{W})$ is an associative algebra with generators ℓ_m and q-bracket as in (4.2). The central term in [35] is (**A35**) $([m+1][m][m-1]/[2][3] < m >)\delta_{m+n,0}c$, where $< m >= q^m + q^{-m}$. In our situation we have in (4.21) a term $q^{-4}[n+1][n][n-1]\hat{c} \sim -q^{-4}[m+1][m][m-1]\hat{c}$ and to bring this into line with Remark 4.1 one takes first

$\hat{c} = \tau^2$ where (**A36**) $\tau^2\ell_m = \tau^2(z^{m+1}\partial_q\tau) = q^{2m}z^{m+1}\partial_q\tau\tau^2 = q^{2m}\ell_m\tau^2$ (note $\partial_q\tau = q\tau\partial_q$ and $\partial_q\tau^{-1} = q^{-1}\tau^{-1}\partial_q$). Then, to get a factor $1/<m>$ note simply (**A37**) $(\tau+\tau^{-1})z^m =<m> z^m$ and consequently

$$(\tau+\tau^{-1})^{-1}z^m =<m>^{-1} z^m\,. \tag{4.22}$$

So, let us use $\hat{c}\sim\tau^2$ to get $\hat{c}\ell_m = q^{2m}\ell_m\hat{c}$ as indicated in Remark 3.1. Then we will work $(\tau+\tau^{-1})^{-1}$ into the calculation to obtain $<m>^{-1}$. Finally, pick a $c = c'q^4/[2][3]$ for arbitrary c'. Following up as remarked after (5.3) we note that the presence of an operator $\hat{c}\sim\tau^2$ would complicate our putative qKdV equations such as (4.14) or (4.19) if it is left hanging on the end. On the other hand, we do not want to leave it out of the calculation so we move it to the left and let it work on z^{n+1} for example or on f. Thus, we set (**A38**) $\hat{c}(z^{n+1}) = q^{2n+2}z^{n+1}$ (omitting the trailing $\hat{c}$, except when needed) and this is illustrated below. Thus, consider a generic situation and define now a new ψ via ($\hat{c}\sim\tau^2,\ \partial=\partial_q\tau$)

$$\psi(\ell_n,\ell_m) = q^6\int(\tau z^{m+1})\tau(\partial^2\hat{c}(\tau+\tau^{-1})^{-1}\partial\tau^{-5}z^{n+1}) \tag{4.23}$$

$$= q^5\int z^{m+1}(\partial^2\hat{c}(\tau+\tau^{-1})^{-1}\partial_q\tau^{-4}z^{n+1})$$

$$= q\int q^{-2n}z^{m+1}(\partial^2 z^n)\frac{[n+1]}{<n>}$$

$$= q\int q^{-2n}z^{m+1}q^n\partial z^{n-1}\frac{[n+1][n]}{<n>} = \frac{[n+1][n][n-1]}{<n>}\delta_{m+n,0}\hat{c}\,,$$

where $\hat{c}=\tau^2$ is restored at the right in the last equation (since $\tau F(x) = F(qx)\tau$) so one recovers (4.6) up to a factor of [2][3] (which can be recovered by writing $\hat{c} = c'\tau^2$ for a suitable constant c', independent of n). Applying this more generally, we write for $\hat{c} = (q^{-6}/[2][3])c'\tau^2$ (c' arbitrary)

$$\psi(f\partial, g\partial) = \int(\tau g)\tau(\partial^2\hat{c}(\tau+\tau^{-1})^{-1}\partial\tau^{-5}f)\,. \tag{4.24}$$

This means that a possibly "canonical" qKdV type equation can be obtained in the form

$$u_t = -c''\tau\partial^2\tau^2(\tau+\tau^{-1})^{-1}\partial\tau^{-5}u - (1+q\tau)^2u\partial_q u\,, \tag{4.25}$$

where $\partial = \partial_q\tau$. It would be possible to vary the third derivative term in various ways by repositioning of τ^2 and $(\tau+\tau^{-1})^{-1}$ and we have chosen this way for convenience. One very interesting feature of this equation involves the $(\tau + \tau^{-1})^{-1}$ term. This serves to bring the equation into line with the structure of the classical qKdV hierarchy equation suggested implicitly in [5], since expansion of $(\tau+\tau^{-1})^{-1}$ would involve an infinite number of terms. The use of q-brackets and formulas like (4.6) seem to bypass the quest for an embedding of $Vec_q(S^1)$ into qPSDO involving the logarithmic cocycle of [33]. It would be interesting to write out explicitly a form for qKdV using the embedding of [33].

Remark 4.5. One checks now that ψ as defined in (4.24) is antisymmetric but the cocycle condition (4.16) still does not hold. An alternative rendering of (4.24) - (4.25) could be taken as

$$\psi(f\partial, g\partial) = \int (\tau g)(\tau\tau^{-5}\partial^2(\tau+\tau^{-1})^{-1}\partial(\tau^2 c' f)\,, \tag{4.26}$$

from which we obtain (**A39**) $\psi(\ell_n, \ell_m) = c'q^6([n+1][n][n-1]/ < n >)\delta_{m+n,0}\hat{c}$ so one chooses $c' = c''q^{-6}/[2][3]$, etc. The corresponding qKdV equation would be formally

$$u_t = -c''\tau^{-4}\partial^2(\tau+\tau^{-1})^{-1}\partial\tau^2 u - (1+q\tau)^2 u\partial_q u \tag{4.27}$$

$$= -(1+q\tau)^2 u\partial_q u - c''\tau^{-4}\partial_q\tau\partial_q\tau\sum_0^\infty(-1)^n\tau^{2n+3}\partial_q\tau u\,.$$

A similar formula with an infinite series would apply for (4.25). Still another formula is obtained via $\Gamma = (q^{-1}\tau + q\tau^{-1}) = q\tau^{-1}(1+q^{-2}\tau^2)$ with $\Gamma^{-1} = (\tau/q)(1+q^{-2}\tau^2)^{-1}$ so $\Gamma z^{n+1} =< n > z^{n+1}$. Then consider $(\hat{c}\sim\tau^2)$

$$\psi(f\partial, g\partial) = \int (\tau g)\tau(\tau^{-5}\partial^3 c'\hat{c}\Gamma^{-1} f)\,. \tag{4.28}$$

It follows that $\psi(\ell_n, \ell_m) = (c'q^6[n+1][n][n-1]/ < n >)\delta_{m+n,0}\hat{c}$ and

$$u_t = -(1+q\tau)^2 u\partial_q u - c''\tau^{-4}\partial_q\tau\partial_q\tau\partial_q\sum_0^\infty(-1)^n q^{-2n-1}\tau^{2n+4}u\,. \tag{4.29}$$

Note, also that

$$[m-n][m+n-p] < p > +[n-p][n+p-m] < m > + [p-m][p+m-n] < n >= 0\,, \tag{4.30}$$

and this is related to a quantum Jacobi condition for the $Witt_q \sim \mathfrak{W}_q$ algebra based on $\Gamma(\ell_p) =< p > \ell_p$. Thus, writing $\sigma_m = ([m+1][m][m-1]/ < m > [2][3])\delta_{m+n,0}\hat{c}$, we have

$$[[\ell_m, \ell_n]_q, \Gamma(\ell_p)]_q = [m-n][\ell_{m+n}, < p > \ell_p]_q + \sigma_m \delta_{m+n,0}[\hat{c}, \Gamma(\ell_p)]_q \,. \quad (4.31)$$

However ($\deg(\hat{c}) = 0$) $[\hat{c}, \ell_p] = \hat{c}\ell_p q^{-p} - \ell_p \hat{c} q^p = \ell_p(q^p - q^p)\hat{c} = 0$ so since $\hat{c}\ell_p = q^{2p}\ell_p\hat{c}$

$$[[\ell_m, \ell_n]_q, \Gamma(\ell_p)]_q = [m-n][m+n-p] < p > \ell_{m+n+p} + \sigma_{m+n}[m-n] < p > \delta_{m+n+p,0}\hat{c} \,. \quad (4.32)$$

Hence (via (4.30))

$$[[\ell_m, \ell_n]_q, \Gamma(\ell_p)]_q + [[\ell_n, \ell_p]_q \Gamma(\ell_m)]_q + [[\ell_p, \ell_m]_q, \Gamma(\ell_n)]_q = \{\sigma_{m+n}[m-n] < p > + \sigma_{n+p}[n-p] < m > + \sigma_{p+m}[p-m] < n >\}\sigma_{m+n+p}\hat{c} \,. \quad (4.33)$$

The right side comes only from the Virasoro central term so we get zero for the $\mathfrak{W}_q$ algebra. It is stated in [35] that Vir_q is the universal quantum central extension of $\mathfrak{W}_q$ but this is not clear since the right side of (4.33) does not vanish.

5. Maurer-Cartan

To develop the idea of Maurer-Cartan (MC) formulas in a quantum group context one recalls that generally there is no exponential map for q-group situations but we can work with duality between $Fun(G)$ and $\mathfrak{U}^*(\mathfrak{g})$ = dual of the enveloping algebra $\mathfrak{U}(\mathfrak{g})$ in the following manner (cf. [4, 8, 13, 34]). For $X \in \mathfrak{g}$ write

$$(\rho(X)\phi)(g) = X|_g\phi = \frac{d}{dt}\phi(e^{tX}g)|_{t=0} \quad (5.1)$$

which enables a duality $< \cdot, \cdot >$: $Fun(G) \times \mathfrak{U}(\mathfrak{g}) \to \mathbf{C}$ to be defined via (**A40**) $< \phi, a >= (\rho(a)\phi)(e) \in \mathbf{C}$ for $a \in \mathfrak{U}(\mathfrak{g})$ and $\phi \in Fun(G)$. This defines an embedding of $Fun(G)$ in $\mathfrak{U}(\mathfrak{g})^* = \mathfrak{U}^*(\mathfrak{g})$ (i.e. $\phi \in \mathfrak{U}^*(\mathfrak{g})$). Then one tries to preserve the duality under suitable quantization of $Fun(G)$ and $\mathfrak{U}(\mathfrak{g})$. We note also that $Fun(G)$ is a Hopf algebra via (here $Fun(G) \otimes Fun(G)$ is identified with $Fun(G \times G)$ which should be refined via $\hat{\otimes}$)

$$m(\phi \otimes \psi)(g) = (\phi\psi)(g) = \phi(g)\psi(g),\ \Delta(\phi)(g_1 \otimes g_2) = \phi(g_1 g_2), \tag{5.2}$$

$$\eta(\alpha) = \alpha u,\ u(g) = 1\ \forall g \in G,\ u \in Fun(G),\ \alpha \in \mathbf{C},$$

$$\epsilon(\phi) = \phi(e),\ S(\phi)(g) = \phi(g^{-1})$$

(cf. [8, 13]). Similarly $\mathfrak{U}(\mathfrak{g})$ is a Hopf algebra (where maps on $\mathfrak{g}$ are extended uniquely to $\mathfrak{U}(\mathfrak{g})$ and m is the associative multiplication in $\mathfrak{U}(\mathfrak{g})$)

$$\Delta(X) = X \otimes I + I \otimes X\ (I = id \in \mathfrak{U}(\mathfrak{g})),\ \eta(\alpha) = \alpha I\ (\alpha \in \mathbf{C}), \tag{5.3}$$

$$\epsilon(I) = 1,\ \epsilon(X) = 0\ (\forall X \neq I\ in\ \mathfrak{U}(\mathfrak{g})),\ S(X) = -X\ (X \in \mathfrak{g})$$

Then $Fun(G)$ and $\mathfrak{U}(\mathfrak{g})$ are dual where the dual to a Hopf algebra $(A, m, \Delta, \eta, \epsilon, S)$ is defined as $(A^*, m^*, \Delta^*, \eta^*, \epsilon^*, S^*)$ (here A^* is a dual space to A)

$$< m^*(\phi \otimes \psi), X >=< \phi \otimes \psi, \Delta(X) >, \tag{5.4}$$

$$< \Delta^*(\phi), X \otimes Y >=< \phi, XY >,$$

$$< \eta^*(\alpha), X >= \alpha\epsilon(X),$$

$$\epsilon^*(\phi) =< \phi, 1 >,\ < S^*(\phi), X >=< \phi, S(X) > .$$

Now, the idea of quantization via e.g. Moyal brackets (cf. [4, 13]) involves deforming a Poisson structure on a manifold M via a Poisson bracket $\{\ ,\ \} : Fun(M) \otimes Fun(M) \to Fun(M)$ and generally

- Quantization is a transition from commutative to noncommutative (or for dual objects from cocommutative to noncocommutative) Hopf algebras.

- The relation between a Lie group and a Lie algebra survive after quantization in the form of duality of the corresponding Hopf algebras.

The whole procedure is spelled out in a lovely way in [13], the connections to Moyal quantization are indicated and the emergence of q arises naturally in the context of r and R matrices, etc. There is no room for detail here but we sketch a bit.

Example 5.1. We consider e.g. $\mathfrak{g} = \mathfrak{s}\ell_2$ and write down some of the facts. First (**A41**) $r = (1/2)[J_0 \otimes J_0 + 2J_+ \otimes J_-]$ and one defines a map $\zeta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ via (**A42**) $\zeta(Y) = [r, 1 \otimes Y + Y \otimes 1]$ where $r = r^{\mu\nu} X_\mu \otimes X_\nu$. Using $[J_0, J_\pm] = \pm 2J_\pm$ with $[J_+, J_-] = J_0$ one obtains (**A43**) $\zeta(J_0) = 0$ and $\zeta(J_\pm) = (J_0 \otimes J_\pm - J_\pm \otimes J_0)$. Now, the comultiplication Δ_χ (for a Moyal deformation) has the general form (**A44**) $\Delta_\chi = \sum_0^\infty (\chi^n/n!)\Delta_n$ and the standard conditions $\Delta(ab) = \Delta(a)\Delta(b)$ and $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$ then give the recursive relations (**A45**) $\sum_0^n \begin{pmatrix} n \\ k \end{pmatrix} (\Delta_k \otimes 1 - 1 \otimes \Delta_k)\Delta_{n-k} = 0$. One gets two first approximations $\Delta_0 = \Delta$ and Δ_1 from standard rules leading to (**A46**) $\Delta_1(H) = 0$ and $\Delta_1(X^\pm) = (1/2)(H \otimes X^\pm - X^\pm \otimes H)$ (here $J_\pm \to X^\pm$ and $J_0 \to H$ in the deformed algebra). Then by induction there results

$$\Delta_n(H) = 0,\ \Delta_n(X^\pm) = \left(X^\pm \otimes \left(-\frac{H}{2} \right)^n + \left(\frac{H}{2} \right)^n \otimes X^\pm \right), \tag{5.5}$$

where H^n is defined by the multiplication m_χ and χ is the Moyal deformation parameter. This yields (**A47**) $\Delta_\chi(H) = H \otimes 1 + 1 \otimes H,\ \Delta_\chi(X^\pm) = X^\pm \otimes q^{-H/2} + q^{H/2} \otimes X^\pm$, where $q = \exp(\chi)$ and this shows how a q-deformation arises from general Moyal type theory. Now the various Hopf algebra structure maps are also deformed and one obtains

$$[H, X^\pm] = \pm 2X^\pm,\ [X^+, X^-] = [H]_q,\ \Delta_\chi(H) = H \otimes 1 + 1 \otimes H, \tag{5.6}$$

$$S_\chi(H) = -H,$$

$$S_\chi(X^\pm) = -q^{\mp 1} X^\pm,\ \Delta_\chi X^\pm = X^\pm \otimes q^{-H/2} + q^{H/2} \otimes X^\pm,$$

$$\epsilon_\chi(X^\pm) = \epsilon_\chi(H) = 0,\ \epsilon_\chi(1) = 1,\ \eta(\alpha) = \alpha 1$$

(recall $[x]_q = (q^x - q^{-x})/(q - q^{-1})$).

A corresponding technique for any simple Lie algebra is indicated in [13] (this goes back to Drinfeld and Jimbo). Generally to quantize the algebra of functions on a group $Fun(G)$ one can use duality. Thus, $Fun(G)$ is isomorphic to the algebra of matrix elements of finite dimensional representations of the UEA $\mathfrak{U}(\mathfrak{g})$ (generalized Peter-Weyl Theorem) and an algebra $Fun_q(G)$ of matrix elements for the QUEA $\mathfrak{U}_q(\mathfrak{g})$ is called an algebra of functions on a quantum group G_q. A quantum group G_q is then interpreted as a spectrum (set of representations) of $Fun_q(G)$. Explicitly let $\rho : A \to End(V)$ be a finite dimensional representation of a complex algebra A. then the matrix elements $T_{ij}(a)$ are linear functionals on A defined via $\rho : a \to (T_{ij}(a))$. If A is a Hopf algebra then the T_{ij} inherit a Hopf algebra structure, e.g.

(**A48**) $m(T_{ij}(a), T_{k\ell}(a)) = (T_{ij} \otimes T_{k\ell})\Delta(a)$. We think here of $A \sim \mathfrak{U}(\mathfrak{g})$. Then $Fun_q(G) \sim$ quantization of the Poisson algebra $Fun(G)$ with bracket

$$\{T_{ij}(Y), T_{k\ell}(Z)\} = [(T \otimes T)(-ir_0), T(Y)) \otimes T(Z)]_{ij,k\ell}, \tag{5.7}$$

$$-ir_0 = \frac{1}{2} \sum_{\alpha \in \Delta_+} (X_{-\alpha} \otimes X_\alpha - X_\alpha \otimes X_{-\alpha}\,,$$

where $X, Y, Z \in \mathfrak{g}$ and $\Delta_+ \sim$ positive roots (e.g. G is simple). The resulting $Fun_q(G)$ is then a quantization of $Fun(G)$. We recall now the following definition.

Definition 5.1. Let $(A, m, \Delta, \eta, \epsilon, S)$ be a Hopf algebra and $\mathfrak{R}$ an invertible element of $A \otimes A$. Then the pair $(A, \mathfrak{R})$ is called a quasitriangular Hopf algebra if

- $\Delta'(a) = \sigma \circ \Delta = \mathfrak{R}\Delta(a)\mathfrak{R}^{-1}$.
- $(\Delta \otimes 1)\mathfrak{R} = \mathfrak{R}_{13}\mathfrak{R}_{23}$.
- $(1 \otimes \Delta)\mathfrak{R} = \mathfrak{R}_{13}\mathfrak{R}_{12}$.

Some algebraic manipulation leads then to (cf. [8, 13]) (♣) $\mathfrak{R}_{12}\mathfrak{R}_{13}\mathfrak{R}_{23} = \mathfrak{R}_{23}\mathfrak{R}_{13}\mathfrak{R}_{12}$ (quantum Yang-Baxter equation - QYBE). This is a crucial concept associated with the idea of a quantum group and leads to braiding, knot theory, etc., canonical forms exist for simple Lie groups (cf. [8]). For $\mathfrak{U}(\mathfrak{s}\ell_2)$ one gets

$$\mathfrak{R} = q^{-(1/2)H \otimes H} \exp_{q^{-2}}\{-(q - q^{-1})\lambda X^+ \otimes \tilde{X}^-\}, \tag{5.8}$$

$$\exp_q x = e_q^x = \sum_0^\infty \frac{x^n}{[n,q]!},\ [x,q] = \frac{q^x - 1}{1 - 1}\,,$$

and $\tilde{X}^+ \sim q^H X^+$ with $X^- \sim X^- q^{-H}$. Next, we note that in a representation ρ of $A = \mathfrak{U}_q(\mathfrak{g})$. Definition 9.6 leads to (**A49**) $R(\rho \otimes \rho)\Delta'(a) = (\rho \otimes \rho)\Delta(a)R$, where $\Delta' = \sigma \circ \Delta$ and $R = (\rho \otimes \rho)\mathfrak{R}$. Using **(A48)** we get (**A50**) $(\rho \otimes \rho)\Delta = (1 \otimes T)(T \otimes 1) = T_2T_1$, $(\rho \otimes \rho)\Delta' = (T \otimes 1)(1 \otimes T) = T_1T_2$ and via **(A49)** one obtains then (**A51**) $RT_1T_2 = T_2T_1R$. Now in general, one can pack a basis for $\mathfrak{U}_q(\mathfrak{g})$ into L^+ and L^-, which for $SL_q(2)$ takes the form

$$L^+ = \begin{pmatrix} q^{-H/2} & \lambda X^+ \\ 0 & q^{H/2} \end{pmatrix},\ L^- = \begin{pmatrix} q^{H/2} & 0 \\ -\lambda X^- & -q^{-H/2} \end{pmatrix}, \tag{5.9}$$

where $\lambda = q - q^{-1}$. One can think here now of the $Fun_q(G) - \mathfrak{U}_q(\mathfrak{g})$ duality pairing in the form (**A52**) $< L_0^+, T_1T_2 >= \tilde{R}_{10}\tilde{R}_{20}$ and $< L_0^-, T_1T_2 >= \tilde{R}_{01}^{-1}\tilde{R}_{02}^{-1}$ (action in $\mathbf{C}^{\otimes 3}$), where $\tilde{R} = q^{1/2}R$ is a normalization. Thus, the $T_1 \in \mathfrak{U}_q^*(\mathfrak{g})$ in the sense indicated. In any case (**A52**) makes the duality between $Fun_q(G)$ and $\mathfrak{U}_q(\mathfrak{g})$ explicit (cf. also [15]). Strictly speaking the algebra generated by the entries $(L^{\pm})^i_j$ becomes isomorphic to $\mathfrak{U}_q(\mathfrak{g})$ after a completion (which is omitted here - cf. [21]).

Now the action of $\mathfrak{U}_q(\mathfrak{g})$ on $Fun_q(G)$ is analogous to vector field action on functions over a group and is defined via (**A53**) $\ell(T) =< \ell \otimes id, \Delta(T) >$ for $\ell \in \mathfrak{U}_q(\mathfrak{g})$ and $T \in Fun_q(G)$, from which the relation (**A54**) $L_1^+T_2 = T_2\tilde{R}_{21}L_1^+$ and $L_1^-T_2 = T_2\tilde{R}_{12}^{-1}L_1^-$ follows. Now define (**A55**) $Y = L^+(L^-)^{-1}$ and **(A54)** takes the form (**A56**) $Y_1T_2 = T_2\tilde{R}_{21}Y_1\tilde{R}_{12}$, so (4.20) can be rewritten as (**A57**) $R_{21}Y_1R_{12}Y_2 = Y_2R_{21}Y_1R_{12}$. Next, one can define an exterior differential as (**A58**) $d = Tr(\mathfrak{D}^{-1}\Omega X)$, where Ω is a matrix of differential 1-forms to be defined and $\mathfrak{D}$ is a complex matrix which provides the invariance properties of the trace. Thus (**A59**) $Tr(\mathfrak{D}^{-1}M') = Tr(\mathfrak{D}^{-1}M)$, $(M')^i_j = S(T^i_k)T^\ell_j \otimes M^k_\ell$. Here, the usual trace (**A60**) $Tr(M') = S(T^i_\ell)T^k_i \otimes M^\ell_k$ is not invariant and $\mathfrak{D}$ is inserted to correct this. The q-trace is then defined as (**A61**) $Tr_qM = Tr(\mathfrak{D}^{-1}M)$. Finally, for the entries of Ω one uses $d^2 = 0$ and has forms

$$\omega^+\omega^- + \omega^-\omega^+ = 0,\ \omega^2\omega^- + q^{-2}\omega^-\omega^2 = -q^{-1}\lambda\omega^-\omega^1, \tag{5.10}$$

$$\omega^1\omega^+ + \omega^+\omega^1 = 0,\ \omega^1\omega^2 + \omega^2\omega^1 = -q^{-1}\lambda\omega^+\omega^-,\ \omega^1\omega^- + \omega^-\omega^1 = 0,$$
$$(\omega^1)^2 = (\omega^+)^2 = (\omega^-)^2 = 0,\ \omega^2\omega^+ + q^2\omega^+\omega^2 = q\lambda\omega^+\omega^1,$$
$$(\omega^2)^2 = q\lambda\omega^+\omega^-\,.$$

Here the matrix Ω and the quantum version of Maurer-Cartan equations are

$$\Omega = \begin{pmatrix} \omega^1 & \omega^- \\ \omega^+ & \omega^2 \end{pmatrix},\ d\omega^1 = -q^{-3}\omega^+\omega^-, \tag{5.11}$$

$$d\omega^+ = q^{-1}\omega^+(\omega^1 - \omega^2),\ d\omega^2 = q^{-1}\omega^+\omega^-,\ d\omega^- = q^{-1}(\omega^1 - \omega^2)\omega^-\,.$$

Remark 5.1. We note that for $\omega^1 - \omega^2 = \omega_1,\ \omega^+ = \omega_0$, and $\omega^- = \omega_2$ the equations (5.11) give

$$d\omega_1 = -(q^{-3} + q^{-1})\omega_0\omega_2,\ d\omega_0 = q^{-1}\omega_0\omega_1,\ d\omega_2 = q^{-1}\omega_1\omega_2\,. \tag{5.12}$$

This can be compared to the Maurer-Cartan equations from [5] (based on [34])

$$d\omega_0 = (q^2 + q^4)\omega_0 \wedge \omega_1,\ d\omega_1 = -\omega_0 \wedge \omega_2,\ d\omega_2 = (q^2 + q^4)\,,\omega_1 \wedge \omega_2\,, \quad (5.13)$$

which agree up to factors of q. This can be adjusted via $\omega_1 = (q^{-3} + q^{-1})\tilde{\omega}_1$ in (5.12) with $q \to q^{-1}$ (see Theorem 5.1 below for the corresponding quantum KdV type equation).

Example 5.2. Now go to [8, 34, 36] and recall that the algebraic properties of $SL(2,\mathbf{C})$ are stored in the coordinate Hopf algebra $\mathfrak{O}(SL(2)) = \mathbf{C}[u_1^1, u_2^1, u_1^2, u_2^2]/\{u_1^1 u_2^2 - u_2^1 u_1^2 = 1\}$ (recall $u_j^i((g_{k\ell})) = g_{ij}$). The Hopf algebras $\mathfrak{O}(SL_q(2))$ will be one parameter deformations of $\mathfrak{O}(SL(2))$. One defines first $\mathfrak{O}(M_q(2))$ to be the complex (associative) algebra with generators a, b, c, d satisfying (**A62**) $ab = qba,\ ac = qca,\ bd = qdb,\ cd = qdc,\ bc = cb,\ ad - da = (q - q^{-1})bc$, where $u_1^1 = a,\ u_2^1 = b,\ u_1^2 = c$, and $u_2^2 = d$. We omit Hopf algebra structure momentarily. One defines now (**A63**) $D_q = ad - qc = da - q^{-1}bc$ (quantum determinant) which belongs to the center of $\mathfrak{O}(M_q(2))$. Then, define $\mathfrak{O}(SL_q(2)) = \mathfrak{O}(M_q(2))/(D_q - 1)$ as the coordinate algebra of $SL_q(2))$. There is a compact real form $\mathfrak{O}(SU_q(2))$ for $q \in \mathbf{R}$ with involution $a^* = d,\ b^* = -qc,\ c^* = -q^{-1}b$, and $d^* = a$ so that $\mathfrak{O}(SU_q(2))$ is the star algebra generated by two elements a, c with relations (**A64**) $ac = qca,\ ac^* = qc^*a,\ cc^* = c^*c,\ a^*a + c^*c = 1,\ aa^* + q^2c^*c = 1$. For $|q| = 1$ there is also a real form $\mathfrak{O}(SL_q(2,\mathbf{R}))$ of $\mathfrak{O}(SL_q(2))$ with $a^* = a,\ b^* = b,\ c^* = c$, and $d^* = d$. Note that $SL_q(2)$ is known through its coordinate algebra but one can also envision matrices

$$M_m^n = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ (M_m^n)^\dagger = (M_m^n)^{-1}\,, \quad (5.14)$$

for suitable a, b, c, d (cf. [36]). Next, we recall the following universal enveloping algebras. In the terminology of [36] $\mathfrak{s}\ell_2$ has generators $X_\pm$, H with (**A65**) $[H, X_\pm] = \pm 2X_\pm$ and $[X_+, X_-] = H$. $\mathfrak{U}_q(\mathfrak{s}\ell_2)$ is then defined via generators 1, $X_\pm, q^{H/2}$, and $q^{-H/2}$ with

$$q^{H/2} X_\pm q^{-H/2} = q^{\pm 1} X_\pm,\ [X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}} \quad (5.15)$$

(cf. (5.6) where the Hopf algebra structure also appears). Another variation appears in [34]. Thus, for $q \neq 0$ and $q^2 \neq 1$ $\mathfrak{U}_q(\mathfrak{s}\ell_2)$ is the associative algebra with unit generated by E, F, K, K^{-1} satisfying

$$KK^{-1} = K^{-1}K = 1,\ KEK^{-1} = q^2 e, \quad (5.16)$$

$$KFK^{-1} = q^{-2}F: \ [E,F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

Then the sets $\{F^\ell K^m E^n,\ m \in \mathbf{Z},\ \ell, n \in \mathbf{N}_0\}$ or $\{E^n K^m F^\ell,\ m \in \mathbf{Z},\ \ell, n \in \mathbf{N}_0\}$ are vector space bases of $\mathfrak{U}_q(\mathfrak{s}\ell_2)$ (we omit temporarily the Hopf algebra structure).

Now one knows that for $a \in \mathfrak{U}(\mathfrak{g})$ and $f \in \mathfrak{O}(G)$ there is a dual pairing determined by (**A66**) $< a, f >= (\tilde{a}f)(e)$ where for $a = X_1 \cdots X_n$

$$(\tilde{a}f)(g) = \left.\frac{\partial^n}{\partial t_1 \cdots \partial t_n}\right|_{t=0} f(g \exp(t_1 X_1) \cdots \exp(t_n X_n)). \tag{5.17}$$

Then for $q^4 \neq 0, 1$ let $\breve{\mathfrak{U}}_q(\mathfrak{s}\ell_2)$ be the algebra over $\mathbf{C}$ with generators $E,\ F,\ K,\ K^{-1}$ satisfying

$$KK^{-1} = K^{-1}K = 1,\ KEK^{-1} = qE,\ KFK^{-1} = q^{-1}F, \tag{5.18}$$

$$[E,F] = \frac{K^2 - K^{-2}}{q - q^{-1}}.$$

One can show that there is an injective Hopf algebra homomorphism $\phi : \mathfrak{U}_q(\mathfrak{s}\ell_2) \to \breve{\mathfrak{U}}_q(\mathfrak{s}\ell_2)$ determined via $\phi(E) = EK$, $\phi(F) = K^{-1}F$, and $\phi(K) = K^2$. Under this injection $\mathfrak{U}_q(\mathfrak{s}\ell_2)$ is a Hopf subalgebra of $\breve{\mathfrak{U}}_q(\mathfrak{s}\ell_2)$. Now for $\mathfrak{g} = \mathfrak{s}\ell(2,\mathbf{C})$ and $G = SL(2,\mathbf{C})$ with generators H, E, F for $\mathfrak{s}\ell(2,\mathbf{C})$ satisfying $[H,E] = 2E$, $[H,F] = -2F$, and $[E,F] = H$ the dual pairing of (5.17) is expressed via (**A67**) $- < H, a >=< H, d >=< E, c >=< F, b >= 1$ and zero otherwise. There is a corresponding result with bracket $< , >\breve{}$ for $\breve{\mathfrak{U}}_q(\mathfrak{s}\ell_2)$ and $\mathfrak{O}(SL_q(2))$ such that (**A68**) $< K, a >\breve{} = q^{-1/2}$, $< K, d >\breve{} = q^{1/2}$, $< E, c >\breve{} =< F, b >\breve{} = 1$ and all other brackets are zero. Similarly there is a unique dual pairing $< , >$ for $\mathfrak{U}_q(\mathfrak{s}\ell_2)$ and $\mathfrak{O}(SL_q(2))$ such that the only nonvanishing brackets are (**A69**) $< K, a >= q^{-1}$, $< K, d >= q$, $< E, c >=< F, b >= 1$ and if q is not a root of unity both pairings are nondegenerate.

We consider now the quantum tangent space for $A = \mathfrak{O}(SL_q(2))$ when $q^2 \neq 1$. Let $a \sim u_1^1$, $b \sim u_2^1$, $c \sim u_1^2$, and $d \sim u_2^2$ with E, F, K generators of $\breve{\mathfrak{U}}(\mathfrak{s}\ell_2)$. We use the dual pairing **(A68)** and define three linear functionals on A via (**A70**) $X_0 = q^{-1/2}FK$, $X_2 = q^{1/2}EK$, and $X_1 = (1 - q^{-2})^{-1}(\epsilon - K^4)$, where the Hopf algebra structure is given by

$$\Delta(K) = K \otimes K,\ \Delta(E) = E \otimes K + K^{-1} \otimes E, \tag{5.19}$$

$$\Delta(F) = F \otimes K + K^{-1} \otimes F,$$

$$S(K) = K^{-1},\ S(E) = -qE,\ S(F) = -q^{-1}F,\ \epsilon(K) = 1,$$

$$\epsilon(E) = \epsilon(F) = 0\,.$$

This illustrates the advantage of $\breve{\mathfrak{U}}(\mathfrak{s}\ell_2)$ over $\mathfrak{U}_q(\mathfrak{s}\ell_2)$ in that $\Delta(E)$ and $\Delta(F)$ have the same type of formula. Now one gets (**A71**) $\Delta(X_j) = \epsilon\otimes X_j + X_j\otimes K^2$ $(j=0,2)$, $\Delta(X_1) = \epsilon\otimes X_1 + X_1\otimes K^4$. It follows that $T = Lin\{X_0, X_1, X_2\}$ is the quantum tangent space of a (unique up to isomorphism) left covariant FODC Γ over A (3-D calculus of Woronowicz). We recall here that an FODC Γ over A is left covariant if and only if there is a linear map (♠) $\Delta_L : \Gamma \to A\otimes\Gamma$ such that $\Delta_L(adb) = \Delta(a)(id\otimes d)\Delta(b)$. For such Γ the quantum tangent space is (**A72**) $T_\Gamma = \{X \in A',\, X(1) = 0,\, X(a) = 0$ for $a \in R_\Gamma = (ker(\epsilon))^2 = \{f \in A,\, f(e) = (df)(e) = 0\}\}$. One writes ${}_{inv}\Gamma = \{\rho\in\Gamma,\, \Delta_L(\rho) = 1\otimes\rho\}$ (left invariant elements). For $P_L : \Gamma \to {}_{inv}\Gamma$ defined via $P_L(a\rho) = \epsilon(a)P_L(\rho)$ with $P_L(\rho) = \sum S(\rho_{(-1)})\rho_0$, where $\Delta_L\rho = \sum\rho_{(-1)}\otimes\rho_0$ one defines (**A73**) $\omega_\Gamma : A \to {}_{inv}\Gamma$ by $\omega_\Gamma(a) = P_L(da)$ so $\omega(A) = {}_{inv}\Gamma$. In particular $R_\Gamma = \{a \in ker(\epsilon),\, \omega_\Gamma(a) = 0\}$ and one takes X_i as a basis of T_Γ with ω_j the dual basis of ${}_{inv}\Gamma$ (i.e. $(X_i, \omega_j) = \delta_{ij}$ where the nondegenerate pairing $(\,,\,) : T\times\Gamma \to \mathbf{C}$ is defined via (**A74**) $(X, a\cdot db) = \epsilon(a)X(b)$. Now let $\{\omega_0, \omega_1, \omega_2\}$ be a basis of ${}_{inv}\Gamma$, which is dual to the basis $\{X_0, X_1, X_2\}$ of T_Γ. Then since $X_0(b) = X_2(c) = X_1(a) = 1$ and $X_0(a) = X_0(c) = X_2(a) = X_2(b) = X_1(b) = X_2(c) = 0$ (due to **(A68)**) one obtains

$$\omega_0 = \omega(b),\ \omega_2 = \omega(c),\ \omega_1 = \omega(a) = -q^{-2}\omega(d),\ da = b\omega_2 + a\omega_1, \qquad (5.20)$$
$$db = a\omega_0 - q^2 b\omega_1,\ dc = c\omega_1 + d\omega_2,\ dd = -q^2 d\omega_1 + c\omega_0\,.$$

Some calculation then shows that (**A75**) $q^2X_1X_0 - q^{-2}X_0X_1 = (1+q^2)X_0,\ q^2X_2X_1 - q^{-2}X_1X_2 = (1+q^2)X_2$, and $qX_2X_0 - q^{-1}X_0X_2 = -q^{-1}X_1$ and these in fact correspond to MC equations. A next step could be to determine the associated bicovariant FODC and develop corresponding q-Lie algebraic ideas (i.e. T_Γ can be made into a Lie algebra - cf. [34]). First, however, one notes that if $\Gamma^\wedge = \oplus_0^\infty \Gamma^{\wedge n}$ is a DC over A based on $\Gamma^{\wedge 1} = \Gamma$ then (**A76**) $d\omega(a) = -\sum\omega(a_1)\otimes\omega(a_2)$ is called the Maurer-Cartan formula. This can be written via $\omega(a) = \sum X_i(a)\omega_i$ in the form (**A77**) $\sum X_i(a)d\omega_i = -\sum X_iX_j(a)\omega_i\wedge\omega_j$. Now one can determine a universal differential calculus (DC) $\Gamma^\wedge$ for any left covariant FODC Γ (see [34]) and for the $3-D$ calculi on $SL_q(2)$ with $A = \mathfrak{O}(SL_q(2))$ one has $(q^2 \neq -1)$ (**A78**) $\omega_j\wedge\omega_j = 0$ $(j = 0,1,2)$, $\omega_2\wedge\omega_0 = -q^2\omega_0\wedge\omega_2$, $\omega_1\wedge\omega_0 = -q^4\omega_0\wedge\omega_1$, and $\omega_2\wedge\omega_1 = -q^4\omega_1\wedge\omega_2$ in $\Gamma^{\wedge 2}$. The 3-form $\omega_0\wedge\omega_1\wedge\omega_2$ is a free left A module basis of $\Gamma^{\wedge 3}$ and $\Gamma^{\wedge n} = 0$ for $n \geq 4$ (note the 3-D calculus here is not bicovariant).

Now look at the Maurer-Cartan (MC) equations for the 3-D calculus in the form **(A76)** - **(A77)**. We know formulas for ω_i via (5.20) and X_i via **(A75)** and one would like to put variables x, t in the ω_i or the X_i to give equations as in Example 3.5. We write now A, B, C, D for a, b, c, d in order to avoid confusion involving dd and refer to **(A62)** for some relations. Some calculation then yields the MC equations in the form (**A79**) $\omega_1 = DdA - q^{-1}BdC,\ \omega_2 = qCdA - AdC$, and $\omega_0 = DdB - q^{-1}BdD$. Further calculation shows also that (cf. [34]) (**A80**) $d\omega(a) = \sum \omega(a_1) \wedge \omega(a_2)$. In summary one has

Proposition 5.1. Properties of the 3-D calculus include:

1. $AB = qBA,\ AC = qCA,\ BD = qDB,\ CD = qDC,\ BC = CB,\ AD - DA = (q - q^{-1})BC$.
2. $X_0 = q^{-1/2}FK,\ X_2 = q^{1/2}EK,\ X_1 = (1 - q^{-2})^{-1}(\epsilon - K^4)$.
3. $\Delta(K) = K \otimes K,\ \Delta(E) = E \otimes K + K^{-1} \otimes E,\ \Delta(F) = F \otimes K + K^{-1} \otimes F,\ S(K) = K^{-1},\ S(E) = -qE,\ S(F) = -q^{-1}F,\ \epsilon(K) = 1,\ \epsilon(E) = \epsilon(F) = 0$.
4. $\Delta(X_j) = \epsilon \otimes X_j + X_j \otimes K^2\ (j = 0, 2),\ \Delta(X_1) = \epsilon \otimes X_1 + X_1 \otimes K^4$.
5. $\omega_0 = \omega(B),\ \omega_2 = \omega(C),\ \omega_1 = \omega(A) - q^{-2}\omega(D),\ dA = B\omega_2 + A\omega_1,\ dB = A\omega_0 - q^2 B\omega_1,\ dC = C\omega_1 + D\omega_2,\ dD = -q^2 D\omega_1 + C\omega_0$.
6. $\omega_j A = q^{-1}A\omega_j,\ \omega_j B = qB\omega_j,\ \omega_j C = q^{-1}C\omega_j\ (j = 0, 2),\ \omega_1 A = q^{-2}A\omega_1,\ \omega_1 B = q^2 B\omega_2,\ \omega_1 C = q^{-2}C\omega_1,\ \omega_1 D = q^2 D\omega_1$.
7. $q^2 X_1 X_0 - q^{-2} X_0 X_1 = (1+q^2)X_0,\ q^2 X_2 X_1 - q^{-2} X_1 X_2 = (1+q^2)X_2,\ qX_2X_0 - q^{-1}X_0X_2 = -q^{-1}X_1$.
8. $\omega_j \wedge \omega_j = 0\ (j = 0, 1, 2),\ \omega_2 \wedge \omega_0 = -q^2 \omega_0 \wedge \omega_2,\ \omega_1 \wedge \omega_0 = -q^4 \omega_0 \wedge \omega_1,\ \omega_2 \wedge \omega_1 = -q^4 \omega_1 \wedge \omega_2$.
9. $\omega_1 = DdA - q^{-1}BdC,\ \omega_2 = qCdA - AdC,\ \omega_0 = DdB - q^{-1}BdD$.

In principle one can now adapt the method of Example 3.6 as in [18] (where terms $A_\mu = g^{-1}\partial_\mu g$ are employed) by using (7) in Proposition 5.1 and then from [34], using (5), (8), and **(A80)**, there results (**A81**) $d\omega_0 = (q^2 + q^4)\omega_0 \wedge \omega_1,\ d\omega_1 = -\omega_0 \wedge \omega_2,\ d\omega_2 = (q^2 + q^4)\omega_1 \wedge \omega_2$. Now a map $f : U \to SL_q(2)$ (as in Example 3.6) means simply expressing A, B, C, D as functions of (x, t) and the pullback of forms from Γ built over $\mathfrak{O}(SL_q(2))$ should be modelable on the standard procedure from differential geometry. Thus, for a manifold map $f : M \to N$ one has maps $f_* : TM \to TN$ and $f^* : T^*N \to T^*M$ defined via $f_* v(g) = v(g \circ f)$ $(v \in T_p(M))$ and $f^*(dg) = d(g \circ f)$ $(dg \in T^*_{f(p)}(N))$.

The analogue here would work from formal power series g in terms of $A^sC^rB^n$ or $D^sC^rB^n$ and formal power series $f(x,t)$ in $x^{\alpha}t^{\beta}$ (i.e. $f \in \mathbf{C}[[x,t]]$). The $X_a \in T_{\Gamma}$ correspond to elements in TN and $\Gamma \sim T^*N$ so in some sense e.g. (**A82**) $f^*(da) = d(a(x,t))$, etc. (we proceed in a somewhat ad hoc manner). Thus, one could use **A79)** to express the ω_i in terms of dA, dB, dC, dD and **(A81)** for the MC equations. First, however, let us simply write following Example 3.6 (**A83**) $\omega_1^1 \sim \omega_1 = \eta dx + \mu dt$, $\omega_2^1 \sim w_0 = \alpha dx + \nu dt$, and $\omega_1^2 \sim \omega_2 = \gamma dx + \beta dt$ (so $\eta \sim \eta$, $A \sim \mu$, $r \sim \alpha$, $B \sim \beta$, $u \sim \gamma$, $C \sim \nu$). Then (**A84**) $d\omega_0 = (q^2+q^4)\omega_0 \wedge \omega_1$, $d\omega_1 = -\omega_0 \wedge \omega_2$, and $d\omega_2 = (q^2+q^4)\omega_1 \wedge \omega_2$. Pick again η constant, $\gamma = u(x,t)$, and $\alpha = 1$ so (♠) $\omega_1 = \eta dx + \mu dt$, $\omega_0 = dx + \nu dt$, $\omega_2 = udx + \beta dt$. Assume first $dh = \partial_x h dx + h_t dt$ for ∂_x possibly a q-derivative. Then **A84)** for $d\omega_1$ implies (•) $\partial_x \mu - \nu u + \beta = 0$ which corresponds to (1) in Example 3.6 and for $d\omega_0$ we get $\partial_x \nu = (q^2+q^4)(\mu - \nu\eta)$ analogous to (3) in Example 3.6. Finally, from $d\omega_2$ there arises (♦) $\partial_x \beta - u_t = (q^2+q^4)(\eta\beta - \mu u)$ analogous to (2) in Example 3.6. Summarizing (with $q^2 + q^4 = \mathfrak{Q}$)

1. $\partial_x \mu - \nu u + \beta = 0$.
2. $-u_t + \partial_x \beta = \mathfrak{Q}(\eta\beta - \mu u)$.
3. $\partial_x \nu = \mathfrak{Q}(\mu - \nu\eta)$.

To eliminate as in Example 3.6 one has from (1) and (3) (♣) $\partial_x \nu = \mathfrak{Q}(\mu - \nu\eta)$, $\partial_x \mu = u\nu - \beta$ Hence (**A85**) $\beta = \nu u - \partial_x(\nu\eta + (1/\mathfrak{Q})\partial_x \nu) = \nu u - \eta \partial_x \nu - (1/\mathfrak{Q})\partial_x^2 \nu$ as in (3). Now put this in (2) to get

$$u_t = \partial_x \beta - \mathfrak{Q}(\eta\beta - \mu u) = (\partial_x \nu)u + \nu \partial_x u - \eta \partial_x^2 \nu - (1/\mathfrak{Q})\partial_x^3 \nu - \tag{5.21}$$
$$-\mathfrak{Q}\eta(\nu u - \eta\partial_x \nu - (1/\mathfrak{Q})\partial_x^2 \nu) + \mathfrak{Q}u((1/\mathfrak{Q})\partial_x \nu + \nu\eta),$$

and modeled Example 3.6 one would try $\nu = \eta^2 - Pu$ and $P = 1/\mathfrak{Q}$ to get (**A86**) $u_t = \frac{1}{\mathfrak{Q}^2}\partial_x^3 u - \frac{3}{2}u\partial_x u$, which actually can be rescaled to become a standard KdV equation with variables depending on q. This assumes ∂_x is a standard derivative. If e.g. $\partial_x \sim \partial_q^x$ (in what follows $\partial_q^x \sim D_q^x$) then we need only check $\partial_t \partial_q^x = \partial_q^x \partial_t$ and note the assumption $dxdt = -dtdx$. However, for $\partial_x \sim \partial_q^x$ the term $\partial_q^x(\nu u)$ becomes (**A87**) $\partial_q^x(\nu u) = \partial_q^x \nu D_x u + \nu \partial_q^x u$ where $D_x u = u(qx,t)$. This changes some terms in (5.21) and leads to the following theorem.

Theorem 5.1. *Assuming* **(A83)** *with $dxdt = -dtdx$ then a quantum KdV type equation arising from MC equations can be written as* (**A88**) $u_t = \frac{1}{\mathfrak{Q}^2}\partial_q^3 u - \frac{2}{\mathfrak{Q}}u\partial_q u - D_x u \partial_q u$, *where $\partial_x = \partial_q^x$.*

Example 5.3. We note now that the search for Burger's equation or qKP type equations on a quantum plane is somewhat distorted since only D_q^x is involved and other variables t_i play a traditional role. Thus, we go to [12, 23, 24] for some background information since ad hoc calculations are not clearly meaningful. Thus, a DC over A is another associative algebra $\Omega^*(A)$ with a differential d, which plays the role of the deRham DC and must tend to this in the commutative limit. The DC is what gives structure to the set of "points" and determines the dimension. It would determine the nearest neighbors in a lattice. Over a given A there are many possible DC and the choice depends on what limit manifold is in mind. One can start with derivations as a set of linear maps of the algebra into itself satisfying a Leibnitz rule and use them as a basis for constructing differential forms or one can start with a set of differential forms and construct a set of possibly twisted derivations, which are dual to the forms (twisted in the sense of obeying a modified Leibnitz rule). One can introduce derivations $e_a f = [\lambda_a, f]$ and suppose the algebra generated by x^i with $1 \leq i \leq n$. Defining $df(e_q) = e_a f$ one finds in general that $dx^i(e_a) \neq \delta_a^i$. Alternatively, one constructs a new basis θ^a dual to the basis or derivations ∂_i obeying a modified Leibnitz rule such that $dx^i(\partial_j) = \delta_j^i$. In general both approaches are equivalent. By construction the θ^a commute with elements of A and define the structure of the 1-forms as a bimodule over A. It is also appropriate to add an extra generator Λ called the dilatator and its inverse Λ^{-1} chosen so that (**A89**) $x^i\Lambda = q\Lambda x^i$ with Λ unitary, since r and Λ do not commute the center of the corresponding extension is trivial. For $n = 1$ now there are two generators x and Λ with $x\Lambda = q\Lambda x$ and one chooses x Hermitian and $q \in \mathbf{R}^+$ with $q > 1$. This is a modified version of the Weyl algebra with q real instead of having unit modulus.

We go then to a quantum line $\mathbf{R}_q^1$ coupled with a time variable (e.g. $A_q = C(\mathbf{R}) \otimes \mathbf{R}_q^1$). One possibility involves

$$x\Lambda = q\Lambda x,\ xdx = qdxx,\ dx\Lambda = q\Lambda dx,\ xd\Lambda = qd\Lambda x,\ e_1 x = q\Lambda x, \tag{5.22}$$

$$e_1\Lambda = 0,\ df(e_1) = e_1 f,\ e_2\Lambda = q\Lambda x,\ e_2 x = 0,\ df(e_2) = e_2 f$$

We mimic **(A77)** and its heuristic derivation in taking (**A90**) $\omega_1^1 \sim \omega_1 = \eta dx + \mu dt + \phi d\Lambda,\ \omega_2^1 \sim \omega_0 = \alpha dx + \nu dt + \psi d\Lambda$, and $\omega_1^2 \sim \omega_2 = \gamma dx + \beta dt + \chi d\Lambda$. The MC equations **(A75)** are (**A91**) $d\omega_0 = \mathfrak{Q}(\omega_0 \wedge \omega_1),\ d\omega_1 = -\omega_0 \wedge \omega_2,\ d\omega_2 = \mathfrak{Q}(\omega_1 \wedge \omega_2)$ and we must find expressions df and $d\omega$ for functions and 1-forms. First (recalling e.g. Example 3.5) we have from **(A90)**, (**A92**) $dx^2 = xdx + dxx = xdx + q^{-1}xdx = (1+q^{-1})xdx,\ \cdots,\ dx^n = [n]_{q^{-1}} x^{n-1} dx$. Since $\Lambda d\Lambda$ does not arise in (5.22) we go back to Example 3.4 of the generalized q-plane which has relations ($y \sim \Lambda$) (**A93**) $xdx = qdxx,\ dx\Lambda = q\Lambda dx,\ xd\Lambda = qd\Lambda x, d\Lambda\Lambda = q\Lambda d\Lambda,\ dx^2 = d\Lambda^2 = 0$, and $dxd\Lambda + qd\Lambda dx = 0$. Modeled on this for consistency we add the additional hypothesis (**A94**) $d\Lambda\Lambda = q\Lambda d\Lambda$ and then from Example 3.4 (**A95**) $d\Lambda^m = [m]_q\Lambda^{m-1},\ df = D_\Lambda\partial^x_{q^{-1}}fdx + \partial^\Lambda_q fd\Lambda$. We are after some form of qKdV now and not qKP - the variable Λ is purely an artifice to give "quantum" meaning to **R**. Note here (**A96**) $dx\Lambda^m = q^m\Lambda^m dx$ plays a role in **(A95)**. Now from **(A91)** we write

$$d\omega_0 = (D_\Lambda\partial^x_{q^{-1}}\alpha dx + \partial^\Lambda_q\alpha d\Lambda + \alpha_t dt)dx + (D_\Lambda\partial^x_{q^{-1}}\nu dx + \partial^\Lambda_q\nu d\Lambda + \nu_t dt)dt$$
$$+ (D_\Lambda\partial_{q^{-1}}\psi dx + \partial^\Lambda_q\psi d\Lambda + \psi_t dt)d\Lambda$$
$$= \mathfrak{Q}(\omega_0 \wedge \omega_1) = \mathfrak{Q}(\alpha dx + \nu dt + \psi d\Lambda) \wedge (\eta dx + \mu dt + \phi d\Lambda)\,. \quad (5.23)$$

This leads to

$$\partial_q^\Lambda \alpha d\Lambda dx + \alpha_t dtdx + D_\Lambda \partial_{q^{-1}}^x \nu dxdt + \partial_q^\Lambda \nu d\Lambda dt + D_\Lambda \partial_{q^{-1}}^x \psi dxd\Lambda + \psi_t dtd\Lambda = \mathfrak{Q}\{\alpha dx\eta dx + \alpha dx\mu dt + \alpha dx\phi d\Lambda + \nu dt\eta dx + \nu dt\mu dt + \nu dt\phi d\Lambda + \psi d\Lambda\eta dx + \psi d\Lambda\mu dt + \psi d\Lambda\phi d\Lambda\}. \quad (5.24)$$

Now e.g. $dx\eta = dx\sum \eta_{nmk}x^n\Lambda^m t^k = \sum \eta_{nmk}q^{-n}x^n q^m x^n \Lambda^m t^k dx = D_x^{-1}D_\Lambda \eta dx$ (since $dxx^n = q^{-n}x^n dx$ and **(A96)** holds). Similarly $d\Lambda x = q^{-1}xd\Lambda \Rightarrow d\Lambda x^n = q^{-n}x^n d\Lambda$ and $d\Lambda\Lambda = q\Lambda d\Lambda \Rightarrow d\Lambda\Lambda^m = q^m\Lambda^m d\Lambda$ so $d\Lambda\eta = D_x^{-1}D_\Lambda\eta d\Lambda$ and one has (assuming $dxdt + dtdx = 0$ and $d\Lambda dt + dtd\Lambda = 0$ and omitting some calculations)

$$D_\Lambda \partial_{q^{-1}}^x \psi - q^{-1}\partial_q^\Lambda \alpha = \mathfrak{Q}(\alpha D_x^{-1}D_\Lambda\phi - q^{-1}\phi D_x^{-1}D_\Lambda\eta), \quad (5.25)$$

$$D_\Lambda \partial_{q^{-1}}^x \nu - \alpha_t = \mathfrak{Q}[\alpha D_x^{-1}D_\Lambda\mu - \nu\eta],\ \partial_q^\Lambda\nu - \psi_t = \mathfrak{Q}[\psi D_x^{-1}D_\Lambda\mu - \nu\phi],$$

with the $dxdt$ term in the middle. Similarly, computing for $d\omega_1$ leads to

$$D_\Lambda \partial_{q^{-1}}^x \phi - q^{-1}\partial_q^\Lambda \eta = q^{-1}D_x^{-1}D_\Lambda\chi - \psi D_x^{-1}D_\Lambda\psi, \quad (5.26)$$

$$D_\Lambda \partial_{q^{-1}}^x \mu - \eta_t = \nu\gamma - \alpha D_x^{-1}D_\Lambda\beta,\ \partial_q^\Lambda\mu - \phi_t = \nu\chi - \psi D_x^{-1}D_\Lambda\beta$$

Finally, from $d\omega_2$

$$D_\Lambda \partial_{q^{-1}}^x \chi - q^{-1}\partial_q^\Lambda \gamma = \mathfrak{Q}(\eta D_x^{-1}D_\Lambda\chi - q^{-1}\phi D_x^{-1}D_\Lambda\gamma), \quad (5.27)$$

$$D_\Lambda \partial_{q^{-1}}^x \beta - \gamma_t = \mathfrak{Q}(\eta D_x^{-1}D_\Lambda\beta - \mu\gamma),$$

$$\partial_q^\Lambda\beta - \chi_t = \mathfrak{Q}(\phi D_x^{-1}D_\Lambda\beta - \chi\mu).$$

We look first at the $dxdt$ equations in (5.25), (5.26), and (5.27) to get

$$D_\Lambda \partial_{q^{-1}}^x \nu - \alpha_t = \mathfrak{Q}[\alpha D_x^{-1}D_\Lambda\mu - \nu\eta], \quad (5.28)$$

$$D_\Lambda \partial_{q^{-1}}^x \mu - \eta_t = \nu\gamma - \alpha D_x^{-1}D_\Lambda\beta,\ D_\Lambda \partial_{q^{-1}}^x \beta - \gamma_t = \mathfrak{Q}(\eta D_x^{-1}D_\Lambda\beta - \mu\gamma)$$

Compare this with the enumeration before (5.21), namely (**A97**) $\partial_x\nu = \mathfrak{Q}(\mu - \nu\eta)$, $-u_t + \partial_x\beta = \mathfrak{Q}(\eta\beta - \mu u)$, $\partial_x\mu - \nu\mu + \beta = 0$, where $u \sim \gamma$ and $r \sim \alpha$. If we take $\eta =$ constant and $\alpha = 1$ again in (5.28) there results

$$D_\Lambda \partial_{q^{-1}}^x \nu = \mathfrak{Q}(D_x^{-1}D_\Lambda\mu - \eta\nu), \quad (5.29)$$

$$D_\Lambda \partial_{q^{-1}}^x \beta - u_t = \mathfrak{Q}(\eta D_x^{-1}D_\Lambda\beta - \mu u),\ D_\Lambda \partial_{q^{-1}}^x \mu = \nu u - D_x^{-1}D_\Lambda\beta$$

This is quite parallel, modulo shifts D_x and D_Λ. Thus,

1. $\partial_x\mu - \nu u + \beta = 0 \sim D_\Lambda \partial^x_{q^{-1}}\mu = \nu u - D_x^{-1}D_\Lambda\beta$.

2. $\partial_x\beta - u_t = \mathfrak{Q}(\eta\beta - \mu u) \sim D_\Lambda\partial^x_{q^{-1}}\beta - u_t = \mathfrak{Q}(\eta D_x^{-1}D_\Lambda\beta - \mu u)$,.

3. $\partial_x\nu = \mathfrak{Q}(\mu - \nu\eta) \sim D_\Lambda\partial^x_{q^{-1}}\nu = \mathfrak{Q}(D_x^{-1}D_\Lambda\mu - \eta\nu)$.

It is interesting that there are no ∂_q^Λ terms here, just shifts D_Λ. Moreover only $D_\Lambda\beta$ appears but pairs $(D_\Lambda\mu,\, \mu)$ and $(D_\Lambda\nu,\, \nu)$ both appear. We reduce matters as before. Thus, set $D_\Lambda\beta = \tilde{\beta}$ and then from (1) and (3) we have $(\hat{\partial}_x \sim \partial^x_{q^{-1}})$ (**A98**) $\hat{\partial}_x D_\Lambda\nu = \mathfrak{Q}(D_x^{-1}D_\Lambda\mu - \eta\nu)$ and $\hat{\partial}_x D_\Lambda\mu = \nu u - D_x^{-1}\tilde{\beta}$. This means $D_\Lambda\mu = D_x[(1/\mathfrak{Q})\hat{\partial}_x D_\Lambda\nu + \eta\nu]$ and $D_x^{-1}\tilde{\beta} = \nu u - \hat{\partial}_x D_x\left(\frac{1}{\mathfrak{Q}}\hat{\partial}_x D_\Lambda\nu + \eta\nu\right)$. Note now (**A99**) $D_x\hat{\partial}_x f = D_x\partial^x_{q^{-1}} f = \partial^x_q f$ so

$$D_x^{-1}\tilde{\beta} = \nu u - \hat{\partial}_x\left[\frac{1}{\mathfrak{Q}}\partial^x_q D_\Lambda\nu + \eta D_x\nu\right]$$
$$\Rightarrow \tilde{\beta} = D_x(\nu u) - \partial^x_q\left[\frac{1}{\mathfrak{Q}}\partial^x_q D_\Lambda\nu + \eta D_x\nu\right]. \quad (5.30)$$

Note also $\partial^x_q D_x f = qD_x\partial^x_q f$ so $\tilde{\beta} = D_x(\nu u) - \frac{D_\Lambda}{\mathfrak{Q}}\left(\partial^x_q\right)^2\nu - \eta q D_x\partial^x_q\nu$ (since $D_\Lambda\partial^x_q = \partial^x_q D_\Lambda$). Putting this in (2) now involves computing

$$D_\Lambda\partial^x_{q^{-1}}\tilde{\beta} = D_\Lambda q\partial^x_q(\nu u) - \frac{1}{\mathfrak{Q}}D_\Lambda^2\partial^x_{q^{-1}}(\partial^x_q)^2\nu - \eta q^2(\partial^x_q)^2\nu \quad (5.31)$$

(note $\partial^x_{q^{-1}}D_x f = q\partial^x_q f$). However,

$$\begin{aligned}\partial^x_{q^{-1}}\partial^x_q f &= \frac{1}{(q^{-1}-1)x}\left[\partial^x_{q^{-1}}f - \partial^x_q f\right],\\ \partial^x_q\partial^x_{q^{-1}} f &= \frac{1}{(q-1)x}\left[\partial^x_q f - \partial^x_{q^{-1}}f\right].\end{aligned} \quad (5.32)$$

Then after some calculation, using (2), $\partial^x_q(\nu u) = \partial^x_q\nu D_x u + \nu\partial^x_q u$, and (**A100**) $\mathfrak{Q}D_x^{-1}D_\Lambda\mu = \partial^x_{q^{-1}}D_\Lambda\nu + \eta\nu\mathfrak{Q} \Rightarrow D_\Lambda\mu = D_x(\eta\nu) +$ $(1/\mathfrak{Q}D_x\partial^x_{q^{-1}}D_\Lambda\nu = (1/\mathfrak{Q})\partial^x_q D_\Lambda\nu + D_x\eta\nu$, one gets (puting $\nu = \eta^2 - Pu$)

$$\begin{aligned}u_t = \frac{1}{\mathfrak{Q}}&D_\Lambda^2\partial^x_{q^{-1}}(\partial^x_q)^2 Pu + \eta q^2(\partial^x_q)^2 Pu - \eta D_x^{-1}D_\Lambda(\partial^x_q)^2 Pu - u\partial^x_q Pu\\ &- D_\Lambda q(\partial^x_q Pu)D_x u - D_\Lambda qPu\partial^x_q u + D_\Lambda q\eta^2\partial^x_q u - \mathfrak{Q}\eta^2 q\partial^x_q Pu - \mathfrak{Q}\eta^3 u\\ &+ \mathfrak{Q}\eta u Pu + \mathfrak{Q}uD_\Lambda^{-1}D_x\eta^3 - \mathfrak{Q}uD_\Lambda^{-1}D_x\eta Pu\end{aligned} \quad (5.33)$$

The terms in u, u^2, $\partial_x u$ which cancelled before now have the form

$$\eta^2 q(D_\Lambda \partial_q^x u - \mathfrak{Q}\partial_q^x Pu),\ -\mathfrak{Q}\eta^3 u + \mathfrak{Q}uD_\Lambda^{-1}D_x\eta^3, \tag{5.34}$$

$$\mathfrak{Q}\eta u(1 - D_\Lambda^{-1}D_x)Pu\,.$$

Then $P = 1/\mathfrak{Q}$ gives terms (●●) $\eta^2 q(D_\Lambda\partial_q^x u - \partial_q^x u)$, $\eta u(1 - D_\Lambda^{-1}D_x)u$, neither of which vanish. However, from [12] one can assume $\Lambda \to 1$ as $q \to 1$ so both terms in (●●) vanish in the limit as desired. Moreover, (5.33) tends to ($\mathfrak{Q} \to 2$ and $P \to 1/2$) $u_t = \frac{1}{4}\partial_q^x u - \frac{3}{2}u\partial_x u$ as desired.

Theorem 5.2. *If $u = u(x,t)$ does not depend on Λ one obtains a qKdV type equation based on the quantum line in the form (note $\partial_{q^{-1}}^x D_x f = q\partial_q^x f \sim (\partial_{q^{-1}}^x D_x)^2 f = q^2(\partial_q^x)^2 f$)*

$$u_t = \frac{1}{\mathfrak{Q}^2}\partial_{q^{-1}}^x(\partial_q^x)^2 u + \frac{\eta}{\mathfrak{Q}}\left((\partial_{q^{-1}}^x D_x)^2 - D_x^{-1}(\partial_q^x)^2\right)u$$
$$- \frac{1}{\mathfrak{Q}}[q(\partial_q^x)D_x u + (1+q)u\partial_q^x u] + \eta u(1 - D_x)u\,. \tag{5.35}$$

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