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YANG-MILLS CONNECTIONS IN THE ORTHONORMAL FRAME BUNEDLES OVER EINSTENIN NORMAL HOMOGENEOUS MANIFOLDS

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Abstract: In this paper, we derived a necessary and sufficient condition for the connection in the orthonormal frame bundle defined by the Levi-Civita connection of an Einstein normal homogeneous manifold (M, g) to be a Yang-Mills connection, and then give some examples.

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1. Introduction and Statement of Results

Yang-Mills connections in a principal fibre bundle $P(M, G)$ are the extrema of the Yang-Mills functional of the space of all the connections in $P(M, G)$ into $\mathbb{R}$

$$\mathcal{YM}(\omega) := \frac{1}{2}\int_M |F(\omega)|^2 dv,$$

whrere $F(\omega)$ is the curvature form of a connection ω in $P(M, G)$. The purpose of this paper is to find Yang-Mills connections in a principal fibre bundle. In

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case of $\dim(M) \not> 4$ in $P(M, G)$, the theories of Yang-Mills connections have been investigated (M. Itoh and H. Nakajima [5]). But, the studies of Yang-Mills connections in a principal fibre bundle $P(M, G)$ with $\dim M > 4$ such that the base Riemannian manifold (M, g) is neither symmetric nor Kaehlerian are rarely ever seen. In this paper, we find Yang-Mills connections in $P(M, G)$, whose base manifold (M, g) is naturally reductive, but neither symmetric nor Kaehlerian. The main purpose is to show whether or not the connection form in the orthonormal frame bundle defined by the Levi-Civita connection of a naturally reductive homogeneous Riemannian manifold (M, g) is a Yang-Mills connection.

In Section 2, we summarize well known facts concerning homogeneous geometric structures on Riemannian compact connected homogeneous spaces (M, g), and get a necessary and sufficient condition for the connection form in the orthonormal frame bundle $O(G/H,\ g)$ defined by the Levi-Civita connection of a compact naturally reductive homogeneous space $(G/H,\ g)$ to be a Yang-Mills connection.

In Section 3, we obtain a complete condition for the connection form in $O(M, G)$ defined by the Levi-Civita connection of Einstein naturally reductive homogeneous spaces (M, g) to be a Yang-Mills connection. As a by-product of the complete condition above, we get thefollowing corollary.

Corollary 3.2. *Let G be a compact connected semisimple Lie group with the canonical metric g. Then the connection form in the orthonormal frame bundle $O(G, g)$ defined by the Levi-Civita connection of (G, g) is a Yang-Mills connection.*

Finally, we introduce a Wang-Ziller's complete condition (M. Wang and W. Ziller [12], (1.5) Corollary) for G/H (with G compact connected semisimple, and H a torus of G) to be Einstein. Using this criterion, we get following results.

Theorem 3.4. *Let G be a compact connected semisimple Lie group, and T a maximal torus. Let $\mathbf{g}$ and $\mathbf{t}$ be the corresponding Lie algebras and B the killing form of $\mathbf{g}^c$. Assume all nonzero roots of $\mathbf{g}^c$ relative to $\mathbf{t}^c$ have the same length with respect to B. Then the connection form in the orthonormal frame bundle $O(G/T,\ g)$ defined by the Levi-Civita connection of the normal homogeneous space $(G/T,\ g)$ is a Yang-Mills connection.*

Corollary 3.5. *Let T be a maximal torus of $SU(n)$ and $(SU(n)/T,\ g)$ a normal homogeneous space. Then the connection form in $O(SU(n)/T,\ g)$ defined by the Levi-Civita connection of $(SU(n)/T,\ g)$ is a Yang-Mills connection.*

2. A Yang-Mills Equation in the Orthonormal Frame Bundle over a Compact Naturally Reductive Homogeneous Space

Let G be a connected Lie group and H a closed subgroup. We denote by $\mathbf{g}$ and $\mathbf{h}$ the corresponding Lie algebras. A homogeneous space G/H is called reductive if there exists a vector space decomposition $\mathbf{g} = \mathbf{h} \oplus \mathbf{m}$ such that $Ad(h)\mathbf{m} \subset \mathbf{m}$ for any $h \in H$, where $Ad(h)$ denotes the adjoint representation of H in $\mathbf{g}$. In this case, $\mathbf{h} \oplus \mathbf{m}$ is called a reductive decomposition of $\mathbf{g}$. If G is a compact connected Lie group, we may regard G/H as a reductive homogeneous space. In this paper, we denote by p_o the point represented by the coset H.

For the calculus, we take a neighbourhood U of the identity element e in G and a subset N of G in such a way that:

(a) $N := U \cap \exp(\mathbf{m})$,

(b) the projection $\pi : G \longrightarrow G/H$ is a diffeomorphism of N onto a neighbourhood $\pi(N)$ of the origin p_o in G/H.

Now for an element $X \in \mathbf{m}$, we define a vector field X^* on the neighbourhood $\pi(N)$ of p_o in G/H by

$$X^*_{xH} := (\tau_x)_* X \in T_{\pi(x)}(\pi(N)), \qquad x \in N, \tag{2.1}$$

where τ_x denotes the transformation of G/H, which is induced by $x \in G$, and $(\tau_x)_*$ is the differential of τ_x at p_o. Let $\langle , \rangle$ be an inner product, which is invariant with respect to $Ad(H)$ on $\mathbf{m}$. From $\mathbf{m} \equiv T_{p_o}(G/H)$, we see that this inner product $\langle , \rangle$ determines an invariant Riemannian metric $g_{\langle , \rangle}$ on G/H, which is reductive. Then the connection funtion α on $\mathbf{m} \times \mathbf{m}$ corresponding to the invariant Riemannian connection of $(G/H, g_{\langle , \rangle})$ is given as follows (K. Nomizu [8, p. 15]):

$$\alpha(X,Y) = 2^{-1}[X,Y]_{\mathbf{m}} + U(X,Y), \qquad (X, Y \in \mathbf{m}), \tag{2.2}$$

where $U(X,Y)$ is determined by

$$2\langle U(X,Y), Z\rangle = \langle [Z,X]_{\mathbf{m}}, Y\rangle + \langle X, [Z,Y]_{\mathbf{m}}\rangle, \qquad (X, Y, Z \in \mathbf{m}), \tag{2.3}$$

and $X_{\mathbf{m}}$ denotes the $\mathbf{m}$-component of an element $X \in \mathbf{g} = \mathbf{h} \oplus \mathbf{m}$. The expression for the value at p_o of the curvature tensor field R is as follows:

$$\begin{aligned} R(X,Y)Z =& \alpha(X,\ \alpha(Y,Z)) \ - \ \alpha(Y,\ \alpha(X,Z)) \\ & - \alpha([X,Y]_{\mathbf{m}}, Z) - [[X,Y]_{\mathbf{h}}, Z], \qquad (X, Y, Z \in \mathbf{m}). \end{aligned} \tag{2.4}$$

Let $\{X_i\}_{i=1}^n$ be an orthonormal basis of $(\mathbf{m}, \langle,\rangle)$ and $\{H_b\}_{b=n+1}^m$ a basis of $\mathbf{h}$, where m=dim$G$. Then, $\{X_i^*\}_i$ is a locally defined orthonormal frame field. Let $\{\theta_*^i\}_i$ be the dual 1-forms to the orthonormal frame field $\{X_i^*\}_i$ on $\pi(N)$ and $\{\theta^i\}_i$ the dual basis of $\{X_i\}_i$.

Throughout this paper, indices h, i, j, k, l, s (resp. b, c, d) run over $\{1, 2, \cdots, n\}$ (resp. $\{n+1, n+2, \cdots, m\}$). For the Levi-Civita connection ∇ of $(G/H, g_{\langle,\rangle})$, we put on $\pi(N)$

$$\nabla_{X_i^*} X_j^* = \sum_k \Gamma_{ij}^{\ k} X_k^*, \qquad \nabla_{X_i^*} \theta_*^j = -\sum_l \Gamma_{il}^{\ j} \theta_*^l, \tag{2.5}$$

where $\Gamma_{ij}^{\ k}$ are C^∞-functions on $\pi(N)$. Then the connection form ω and the curvature form Ω in the orthonormal frame bundle $O(G/H,\ g_{\langle,\rangle})$ defined by the Levi-Civita connection of $(G/H,\ g_{\langle,\rangle})$ are defined as follows:

$$\omega_j^i = \sum_k \Gamma_{kj}^{\ i} \theta_*^k, \tag{2.6}$$

$$\begin{aligned}\Omega_j^i = 2^{-1} \sum_{k,l} \theta^i (\alpha(X_k,\ \alpha(X_l, X_j)) - \alpha(X_l,\ \alpha(X_k, X_j)) \\ - \alpha([X_k, X_l]_{\mathbf{m}},\ X_j) - [\ [X_k, X_l]_{\mathbf{h}},\ X_j]\ )\ \theta_*^k \wedge \theta_*^l.\end{aligned} \tag{2.7}$$

We denote $(\nabla_{X_k^*}\Omega)(X_j^*, X_i^*)$, $\omega(X_j^*)$ and $\Omega(X_j^*, X_i^*)$ by $\nabla_k \Omega_{ji}$, ω_j and Ω_{ji}, respectively. The connection ω in (2.6) is a Yang-Mills connection (I. Mogi and M. Itoh [7, p. 107]) if and only if

$$(\delta_\omega \Omega)(X_i^*) = -\sum_j (\nabla_j \Omega_{ji} + [\omega_j,\ \Omega_{ji}]) = 0 \qquad \text{for each } i. \tag{2.8}$$

Since each $\tau_x, (x \in G)$, is an isometry, in order for the connection in (2.8) to be a Yang-Mills connection, it is sufficient to show that the equation (2.8) is satisfied at the identity coset $p_o \in G/H$.

From now on, as a matter of convenience we put

$$\begin{cases} [X_j, X_k] =: \sum_l D_{jk}^{\ l} X_l + \sum_b D_{jk}^{\ b} H_b, \quad [X_j, H_b] =: \sum_l D_{jb}^{\ l} X_l, \\ \Gamma_{jk}^{\ i}(p_o) =: \Gamma_{jk}^{\ i} =: 2^{-1} D_{jk}^{\ i} + U_{jk}^{\ i}, \\ E_{(k)}{}_j^i := \sum_l \{\ (\nabla_l \Omega_j^i)(X_l, X_k) + [\omega_l, \Omega_{lk}]_j^i\ \}, \\ \qquad\qquad\qquad - (\delta_\omega \Omega)(X_k) =: (E_{(k)}{}_j^i). \end{cases} \tag{2.9}$$

Then we have from (2.3), (2.6), (2.7) and (2.9)

$$\begin{cases} D_{jk}^{\ l} = -D_{kj}^{\ l}, \quad D_{jk}^{\ b} = -D_{kj}^{\ b}, \quad D_{jb}^{\ l} = -D_{bj}^{\ l}, \quad U_{jk}^{\ i} = U_{kj}^{\ i}, \\ \Gamma_{jk}^{\ i} = -\Gamma_{ji}^{\ k}, \quad \omega_j^i = -\omega_i^j, \quad \Omega_j^i = -\Omega_i^j, \quad E_{(k)}{}_j^i = -E_{(k)}{}_i^j. \end{cases} \tag{2.10}$$

Using (2.2), (2.5) – (2.10), we get

$$\begin{aligned} \sum_l (\nabla_l\ \Omega_j^i)(X_l, X_k) &= \sum_{s,l,h} \{\ \Gamma_{ll}^{\ h}(D_{hk}^{\ s}\Gamma_{sj}^{\ i} - \Gamma_{hs}^{\ i}\Gamma_{kj}^{\ s} + \Gamma_{hj}^{\ s}\Gamma_{ks}^{\ i}) \\ &\quad + \Gamma_{lk}^{\ h}(\Gamma_{hs}^{\ i}\Gamma_{lj}^{\ s} - D_{hl}^{\ s}\Gamma_{sj}^{\ i} - \Gamma_{hj}^{\ s}\Gamma_{ls}^{\ i})\ \} \\ &\quad + \sum_{l,h}\sum_b (\ \Gamma_{ll}^{\ h} D_{hk}^{\ b} D_{bj}^{\ i} - \Gamma_{lk}^{\ h} D_{hl}^{\ b} D_{bj}^{\ i}\ ), \\ \sum_l\ [\omega_l, \Omega_{lk}]_j^i &= \sum_{s,l,h} \{\ \Gamma_{lh}^{\ i}(\Gamma_{ls}^{\ h}\Gamma_{kj}^{\ s} - \Gamma_{sj}^{\ h} D_{lk}^{\ s} - 2\Gamma_{ks}^{\ h}\Gamma_{lj}^{\ s}) \\ &\quad + \Gamma_{lj}^{\ h}(D_{lk}^{\ s}\Gamma_{sh}^{\ i} + \Gamma_{ks}^{\ i}\Gamma_{lh}^{\ s})\ \} \\ &\quad + \sum_{l,h}\sum_b (\Gamma_{lh}^{\ i} D_{kl}^{\ b} D_{bj}^{\ h} - \Gamma_{lj}^{\ h} D_{kl}^{\ b} D_{bh}^{\ i}). \end{aligned} \tag{2.11}$$

A Riemannian reductive homogeneous space $(G/H, g_{\langle,\rangle})$ with a fixed redutive decomposition $\mathbf{g} = \mathbf{h} \oplus \mathbf{m}$ is called *naturally reductive* iff U in (2.3) is identically zero. Now we assume that $(G/H, g)$ is a compact Riemannian naturally reductive homogeneous space. Then we get from $U = 0$, (2.9) and (2.10),

$$2\ \Gamma_{jk}^{\ i} = D_{jk}^{\ i} = -D_{ik}^{\ j} = -D_{ji}^{\ k}, \quad \Gamma_{jk}^{\ i} = -\Gamma_{kj}^{\ i}. \tag{2.12}$$

Thus, we obtain from (2.11) and (2.12) the following result.

Proposition 2.1. *If $(G/H, g)$ is a compact naturally reductive homogeneous space with a fixed reductive decomposition $\mathbf{g} = \mathbf{h} \oplus \mathbf{m}$, then a necessary and sufficient condition for the connection form in the orthonomal frame bundle $O(G/H,\ g)$ defined by the Levi-Civita connection of $(G/H, g)$ to be a Yang-Mills connection is that*

$$\begin{aligned} E_{(k)}{}_j^i = \sum_{s,l,h} &\{D_{lk}^{\ h}(D_{hs}^{\ i} D_{lj}^{\ s} - 4^{-1} D_{hl}^{\ s} D_{sj}^{\ i}) \\ &+ 8^{-1} D_{lh}^{\ s}(D_{ks}^{\ i} D_{lj}^{\ h} - D_{kj}^{\ s} D_{lh}^{\ i})\ \} \\ &+ 2^{-1}\sum_{l,h}\sum_b \{D_{kl}^{\ b}(D_{bj}^{\ h} D_{lh}^{\ i} - D_{bh}^{\ i} D_{lj}^{\ h}) - D_{hl}^{\ b} D_{bj}^{\ i} D_{lk}^{\ h}\ \} = 0. \end{aligned}$$

3. Yang-Mills Connections in the Orthonomal Frame Bundles over Compact Einstein Naturally Redutive Homogeneous Space

We retain the notations as in Section 1. In this section, let $(G/H,\ g)$ be a compact Einstein naturally reductive homogeneous space. Then from (2.4), (2.7), (2.10) and (2.12), we see immediately that the expression of the Ricci tensor $\mathcal{R}$ of $(G/H,\ g)$ is given as follows:

$$\mathcal{R}_{jl} := \mathcal{R}(X_j, X_l) = c\ \delta_{jl} = -4^{-1}\sum_{k,s} D_{jk}^{\ s} D_{ls}^{\ k} - \sum_k \sum_b D_{jb}^{\ k} D_{lk}^{\ b}\,, \tag{3.1}$$

for some constant c. Using the facts $\langle [X_j, H_b], X_k\rangle + \langle X_j, [X_k, H_b]\rangle = 0$, $\mathcal{R}_{jl} = \mathcal{R}_{lj}$ and (2.12), we have

$$\sum_k \sum_b D_{jk}^{\ b} D_{lb}^{\ k} = \sum_k \sum_b D_{lk}^{\ b} D_{jb}^{\ k}, \quad D_{jb}^{\ k} = D_{bk}^{\ j}. \tag{3.2}$$

Using the Jacobi identity $[X_h, [X_j, X_l]] + [X_j, [X_l, X_h]] + [X_l, [X_h, X_j]] = 0$, we obtain

$$\sum_s \left(D_{jl}^{\ s} D_{hs}^{\ i} + D_{lh}^{\ s} D_{js}^{\ i} + D_{hj}^{\ s} D_{ls}^{\ i}\right) = -\sum_b \left(D_{jl}^{\ b} D_{hb}^{\ i} + D_{lh}^{\ b} D_{jb}^{\ i} + D_{hj}^{\ b} D_{lb}^{\ i}\right), \tag{3.3}$$

$$\sum_s \left(D_{jl}^{\ s} D_{hs}^{\ b} + D_{lh}^{\ s} D_{js}^{\ b} + D_{hj}^{\ s} D_{ls}^{\ b}\right) = 0. \tag{3.4}$$

Transverting (3.3) with $D_{hk}^{\ l}$ and using (2.12), we obtain

$$\sum_{h,l,s} D_{jl}^{\ s} D_{hs}^{\ i} D_{hk}^{\ l} = -\sum_{h,l} \sum_b \left(D_{jl}^{\ b} D_{hb}^{\ i} + 2^{-1} D_{lh}^{\ b} D_{jb}^{\ i}\right) D_{hk}^{\ l} + 2^{-1} \sum_{h,l,s} D_{hl}^{\ s} D_{js}^{\ i} D_{lh}^{\ k}. \tag{3.5}$$

Substituting (3.1) into (3.5), and using (2.12), we get

$$\sum_{s,l,h} D_{lj}^{\ s} D_{hs}^{\ i} D_{lk}^{\ h} = \sum_{s,l} \sum_b \left(2 D_{kb}^{\ l} D_{ls}^{\ b} D_{js}^{\ i} - D_{jl}^{\ b} D_{sb}^{\ i} D_{sk}^{\ l} + 2^{-1} D_{sl}^{\ b} D_{jb}^{\ i} D_{sk}^{\ l}\right) + 2c D_{kj}^{\ i}. \tag{3.6}$$

Thus we obtain from Proposition 2.1, (3.1), (3.2) and (3.6)

Proposition 3.1. *If $(G/H,\ g)$ is a compact Einstein naturally reductive homogeneous space, then a necessary and sufficient condition for the connection form in the orthonormal frame bundle $O(G/H,\ g)$ defined by the Levi-Civita connection of $(G/H,\ g)$ to be a Yang-Mills connection is that*

$$\begin{aligned} E_{(k)}{}^i_j = \sum_{l,h}\sum_{b}\{\ & 2^{-1}D_{kl}^{\ b}(D_{lh}^{\ i}D_{bj}^{\ h} - D_{lj}^{\ h}D_{bh}^{\ i}) \\ &+ D_{lh}^{\ b}(2^{-1}D_{bj}^{\ h}D_{kl}^{\ i} - D_{bk}^{\ h}D_{lj}^{\ i}) \\ &+ D_{bi}^{\ h}(2^{-1}D_{lh}^{\ b}D_{kj}^{\ l} - D_{jl}^{\ b}D_{hk}^{\ l})\ \} = 0. \end{aligned} \tag{3.7}$$

By virtue of this proposition, we get the following result.

Corollary 3.2. *Let G be a compact connected semisimple Lie group with the canonical metric g. Then the connection form in the orthonormal frame bundle $O(G,g)$ defined by the Levi-Civita connection of (G,g) is a Yang-Mills connection.*

Proof. Since the metric g is minus the Killing form of $\mathbf{g}$, (G,g) is naturally reductive. Moreover, it is well known that the expression of the Ricci tensor $\mathcal{R}$ of (G,g) is given as follows (W.A. Poor [9, p. 195]):

$$\mathcal{R}(X_i\ ,X_j) = 4^{-1}\ \delta_{ij}, \tag{3.8}$$

that is, (G,g) is an Einstein manifold of Ricci curvature 4^{-1}. Hence by help of the proposition above, the proof of this corollary is completed.

A Riemannian homogeneous space $(G/H,g)$ is called *normal homogeneous* if g is canonically induced from a biinvariant metric on G. A normal homogeneous space is naturally reductive.

From now on, let G be a compact, connected, semisimple Lie group and T a maximal torus of G. Let $\mathbf{t}$ be the Lie algebra of T and $\mathbf{t}^c$ (resp. $\mathbf{g}^c$) the complexification of $\mathbf{t}$ (resp. $\mathbf{g}$). We denote by Δ the set of all nonzero roots with respect to $\mathbf{t}^c$, and by Δ^+ the set of all positive roots with respect to a fixed linear order in the dual space of $\{H \in \mathbf{g}^c \mid \alpha(H) \in R \text{ for any } \alpha \in \Delta\}_R$. Let B be the Killing form of $\mathbf{g}^c$. We define an inner product $(\ ,\ )$ on $\mathbf{m} = T_{p_o}(G/T)$ as follows: For $\alpha \in \Delta$, let E_α be a root vector such that $B(E_\alpha, E_{-\alpha}) = -1$ and $N_{\alpha,\beta} = N_{-\alpha,-\beta}$ for $\alpha, \beta \in \Delta \quad (\alpha+\beta \neq 0)$, where $N_{\alpha,\beta}$ are real numbers defined by

$$\begin{cases} [E_\alpha, E_\beta] = N_{\alpha,\beta}E_{\alpha+\beta} & \text{if } \ \alpha,\beta,\alpha+\beta \in \Delta, \quad \text{and} \\ N_{\alpha,\beta} = 0 & \text{if } \ 0 \neq \alpha+\beta \notin \Delta. \end{cases} \tag{3.9}$$

Hence, $[E_\alpha, E_{-\alpha}] = -H_\alpha$, H_α being determined by $B(H, H_\alpha) = \alpha(H)$ for any $H \in \mathbf{t}$. For $\alpha \in \Delta$, put $U_\alpha = E_\alpha + E_{-\alpha}$, $V_\alpha = \sqrt{-1}(E_\alpha - E_{-\alpha})$ which belong to $\mathbf{m}(\subset \mathbf{g})$. Then we take an inner product (,) on $\mathbf{m} = T_{p_o}(G/T)$ such that

$$\{X_\alpha := (1/\sqrt{2})\ U_\alpha, \quad X_{\overline{\alpha}} := (1/\sqrt{2})\ V_\alpha \mid \alpha \in \Delta^+\} \tag{3.10}$$

is an orthonormal basis of $\mathbf{m}$ with respect to (,).

Then (,) determines an invariant Riemannian metric g on G/T, and $(G/T,\ g)$ is normal homogeneous.

Let $\{H_b\}_{b=n+1}^{n+s}$ be a base of $\mathbf{t}$, where $s = \dim T$ and $m = (n+s) = \dim G$. From (3.9) and (3.10), we obtain the following equations:

$$\begin{cases} X_\alpha = X_{-\alpha}, \quad X_{\overline{\alpha}} = -X_{\overline{-\alpha}}, \quad [X_\alpha, X_{\overline{\alpha}}] = \sqrt{-1}\ H_\alpha, \\ [H_b, X_\alpha] = -\sqrt{-1}\ \alpha(H_b)\ X_{\overline{\alpha}}, \\ [H_b, X_{\overline{\alpha}}] = \sqrt{-1}\ \alpha(H_b)\ X_\alpha, \\ [X_\beta, X_\alpha] = (1/\sqrt{2})\ (N_{\beta,\alpha}\ X_{\beta+\alpha} + N_{\beta,-\alpha}\ X_{\beta-\alpha}), \\ [X_\beta, X_{\overline{\alpha}}] = (1/\sqrt{2})(N_{\beta,\alpha}\ X_{\overline{\beta+\alpha}} - N_{\beta,-\alpha}\ X_{\overline{\beta-\alpha}}), \\ [X_{\overline{\beta}}, X_{\overline{\alpha}}] = (1/\sqrt{2})(N_{\beta,-\alpha}\ X_{\beta-\alpha} - N_{\beta,\alpha}\ X_{\beta+\alpha}), \end{cases} \tag{3.11}$$

for each $\alpha, \beta \in \Delta$ $(\alpha \neq \beta)$. It is well known (S. Helgason [2, p. 171]) that for $\alpha, \beta, \gamma \in \Delta$ with $\alpha + \beta + \gamma = 0$

$$N_{\alpha,\beta} = N_{\beta,\gamma} = N_{\gamma,\alpha}. \tag{3.12}$$

By (3.11) and (3.12), $E_{(\gamma)}{}^\alpha_\beta$ in (3.7) is

$$\begin{aligned} E_{(\gamma)}{}^\alpha_\beta = 2^{-1}\sqrt{-1} \sum_{\mu\in\Delta^+} \{\ &([H_\mu, X_\beta],\ X_{\overline{\mu}})\ ([X_\gamma, X_\mu],\ X_\alpha) \\ &- 2\ ([H_\mu, X_\gamma],\ X_{\overline{\mu}})\ ([X_\mu, X_\beta],\ X_\alpha) \\ &- ([H_\mu, X_\alpha],\ X_{\overline{\mu}})\ ([X_\gamma, X_\beta],\ X_\mu)\ \} \\ +2^{-1}\sqrt{-1}\ \{\ &([H_\gamma, X_\beta],\ [X_\alpha, X_{\overline{\gamma}}]) + ([H_\gamma, X_\alpha],\ [X_{\overline{\gamma}}, X_\beta]) \\ &- 2([H_\beta, X_\alpha],\ [X_\gamma, X_{\overline{\beta}}])\ \}. \end{aligned} \tag{3.13}$$

Moreover, we get from (3.11) – (3.13)

$$\begin{aligned} E_{(\gamma)}{}^\alpha_\beta\ &= (2\sqrt{2})^{-1}\ N_{\gamma,\alpha}\ \delta_{\beta,\gamma+\alpha}\ \{2\alpha(H_\beta) - \beta(H_\gamma) + \alpha(H_\gamma)\} \\ &\quad - (2\sqrt{2})^{-1}\ N_{\gamma,-\alpha}\ \delta_{\beta,\gamma-\alpha}\ \{2\alpha(H_\beta) + \beta(H_\gamma) + \alpha(H_\gamma)\}. \end{aligned} \tag{3.14}$$

Similarly using (3.11) and (3.12), we have from (3.7)

$$\begin{cases} E_{(\gamma)}{}^{\alpha}_{\overline{\beta}} \;=\; E_{(\overline{\gamma})}{}^{\alpha}_{\beta} \;=\; E_{(\overline{\gamma})}{}^{\overline{\alpha}}_{\overline{\beta}} \;=\; 0, \\ E_{(\gamma)}{}^{\overline{\alpha}}_{\overline{\beta}} \;=\; (2\sqrt{2})^{-1}\; N_{\gamma,\alpha}\; \delta_{\beta,\gamma+\alpha}\; \{2\alpha(H_\beta) - \beta(H_\gamma) + \alpha(H_\gamma)\} \\ \qquad\qquad + (2\sqrt{2})^{-1}\; N_{\gamma,-\alpha}\; \delta_{\beta,\gamma-\alpha}\; \{2\alpha(H_\beta) + \beta(H_\gamma) + \alpha(H_\gamma)\}, \\ E_{(\overline{\gamma})}{}^{\alpha}_{\overline{\beta}} \;=\; (2\sqrt{2})^{-1}\; N_{\gamma,\alpha}\; \delta_{\beta,\gamma+\alpha}\; \{2\alpha(H_\beta) - \beta(H_\gamma) + \alpha(H_\gamma)\} \\ \qquad\qquad - (2\sqrt{2})^{-1}\; N_{\gamma,-\alpha}\delta_{\beta,\gamma-\alpha}\; \{2\alpha(H_\beta) + \beta(H_\gamma) + \alpha(H_\gamma)\}. \end{cases} \tag{3.15}$$

On the other hand, Wang and Ziller (M. Wang and W. Ziller [12, p. 568]) got an important result as follows:

Lemma 3.3. *Let G be a compact connected semisimple Lie group and H a torus in G. Then the normal homogeneous space $(G/H,\ g_B)$ is Einstein iff the torus H is maximal and all nonezero roots of $\mathbf{g}^c$ have the same length with respect to the Killing form B. Hence, G is locally a product of $SU(n), SO(2n), E_6, E_7$ or E_8.*

Thus, by virtue of (3.12), (3.14), (3.15), the last identity of (2.10) and Lemma 3.3, we obtain the following theorem.

Theorem 3.4. *Let G be a compact connected simisimple Lie group, and T a maximal torus. Let $\mathbf{g}$ and $\mathbf{t}$ be the corresponding Lie algebras and B the Killing form of $\mathbf{g}^c$.Assume all nonzero roots of $\mathbf{g}^c$ relative to $\mathbf{t}^c$ have the same length with respect to B. Then the connection form in the orthonomal frame bundle $O(G/T,\ g)$ defined by the Livi-Civita connection of the normal homogeneous space $(G/T,\ g)$ is a Yang-Mills connection.*

As examples of normal homogeous spaces $(G/T,\ g)$ which satisfy all the conditions in the Theorem 3.4, we can take flag manifolds

$$(SU(n)/T,\ g)\quad (n \geq 2) \qquad \text{and} \qquad (SO(2n)/T,\ g) \qquad (n \geq 2),$$

which are normal homogeneous. Thus, we get the following corollary.

Corollary 3.5. *Let T be a maximal torus of $SU(n)$ and $(SU(n)/T,\ g)$ a normal homogeneous space. Then the connection form in $O(SU(n)/T,\ g)$ defined by the Levi-Civita connection of $(SU(n)/T,\ g)$ is a Yang-Mills connecion.*

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