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A GENERALIZATION OF PELEG'S THEOREM WITH APPLICATIONS TO A SYSTEM OF VARIATIONAL INEQUALITIES

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Abstract: The main object of this paper is to introduce the general Peleg KKM Theorem on pseudo H-spaces. As applications, we derive some new results for fixed point theorems and the system of variational inequalities.

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1. Introduction

In very recent years, Marchi and Martínez-Legaz [7] extended the Peleg Theorem [8] to H-spaces and obtained some generalizations of Fan-Browder Fixed Point Theorem. They also discussed the Ky Fan type inequalities and the intersection theorem for sets with convex sections. Park and Kim [10] gave

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a Peleg Theorem on G-convex spaces and applications to a whole intersection theorem. They also derived existence theorems of equilibrium points in qualitative games. In this paper, we shall introduce the general Peleg KKM Theorem on pseudo H-spaces. As an application, we shall derive some new results for fixed point theorem and the system of variational inequalities.

Let X be a non-empty set. We denote by 2^X the family of all subsets of X, by $|X|$ the cardinality of X. If X is a subset of a vector space, $co(X)$ denotes the convex hull of X. Let Δ^n denote the standard n-simplex $co\{e_1, \ldots, e_{n+1}\}$, where e_i is the ith unit vector in $\mathbf{R}^{n+1}$ for $i = 1, 2, \ldots, n+1$.

We first recall the definition of the pseudo H-space and q-map as follows.

Definition 1.1. [6] Let X be a topological space, D be a nonempty set. The triple (X, D, q) is said to be a pseudo H-space if for each nonempty finite subset A of D, the mapping $q : \Delta^{|A|-1} \to 2^X$ is upper semi-continuous with nonempty compact values. If $D = X$, the triple (X, D, q) can be written by (X, q).

If the map q is single-valued and we set $\Gamma(A) = q(\Delta^{|A|-1})$ for each nonempty finite subset A of X, then (X, D, Γ) forms a G-convex space.

Example 1.2. For any given G-convex space (X, D, Γ) with D is a nonempty finite set. Let Y be a topological space and $F : X \to 2^Y$ be upper semi-continuous with nonempty compact values. Then there is a function $f : \Delta^{|D|-1} \to \Gamma_D \subset X$ and for any nonempty finite subset A of D, the function $f : \Delta^{|A|-1} \to \Gamma_A$. Define $q \doteq F \circ f : \Delta^{|A|-1} \to 2^Y$. Then q is upper semi-continuous with nonempty compact values. Therefore, (Y, D, q) forms a pseudo H-space.

If A, B are two nonempty finite subsets with $A \subset B$ and $|A|+1 = |B|$. Then $\Delta^{|A|-1}$ is a face of $\Delta^{|B|-1}$ corresponding to A. The set $\Delta^{|A|-1}$ is homeomorphic to the set $\Delta^{|A|-1} \times \{0\} \subset \Delta^{|B|-1}$. In this case, we shall replace the notation "$q(\Delta^{|A|-1} \times \{0\}) \subset q(\Delta^{|B|-1})$" by "$q(\Delta^{|A|-1}) \subset q(\Delta^{|B|-1})$". The other cases (i.e., $|A| + 1 < |B|$) are similar to the above discussion.

Let P and Q be two non-empty sets in a pseudo H-space (X, D, q). We say that P is *q-convex relative to Q* if for each nonempty finite subset A of D with $A \subset Q$, we have $q(\Delta^{|A|-1}) \subset P$. We note that if Q is non-empty and P is G-convex relative to Q, then P is automatically non-empty. If $P = Q$, we say P is a *q-convex set* of X.

The *lower inverse* of a set-valued map $F : X \to 2^Y$ is the set-valued map $F^- : Y \to 2^X$ defined by $F^-(B) = \{x \in X : B \cap F(x) \neq \emptyset\}$ for $B \in 2^Y$. A subset W of Y is called *compactly closed* (*compactly open*, resp.) if, for any compact set K in Y, $W \cap K$ is closed (open, resp.) in K.

Let Ω_1 and Ω_2 be nonempty sets. Let $F : \Omega_1 \to 2^{\Omega_2}$ and $\{A_k\}_{k=1}^n$ be a family of sets in Ω_2. We say that the family $\{A_k\}_{k=1}^n$ has *intersection property* with respect to F if the following property hold:

$$F^-(\bigcap_{k=1}^n A_k) = \bigcap_{k=1}^n F^-(A_k).$$

We say that the family $\{A_k\}_{k=1}^n$ has *intersection property* with respect to F on nonempty set $K \subset \Omega_2$ if the family $\{A_k \cap K\}_{k=1}^n$ has *intersection property* with respect to F, that is, the following property hold:

$$F^-(\bigcap_{k=1}^n A_k \cap K) = \bigcap_{k=1}^n F^-(A_k \cap K).$$

We note that if F is a single valued function, then it satisfies the above two intersection properties.

Through out this paper, we denote $I = \{1, 2, \ldots, m\}$ and $X = \prod_{k\in I} X_k$.

2. Main Results

We first need the generalized KKM theorem due to Peleg [8].

Peleg Theorem. *For $k \in I$, let N_k be a nonempty finite set and $\Delta^{|N_k|-1}$ be $(|N_k| - 1)$-simplex in $\mathbf{R}^{|N_k|}$. If C_i^k, $i \in \{1, 2, \ldots, |N_k|\}$ and $k \in I$, are closed subsets of $\Delta^{|N_1|-1} \times \Delta^{|N_2|-1} \times \cdots \times \Delta^{|N_m|-1}$ such that for each $Q_k \subset N_k$, $k \in I$,*

$$\Delta^{|N_1|-1} \times \Delta^{|N_2|-1} \times \cdots \times \Delta^{|Q_k|-1} \times \cdots \times \Delta^{|N_m|-1} \subset \bigcup_{j=1}^{|Q_k|} C_j^k,$$

where $\Delta^{|Q_k|-1}$ denotes the face of $\Delta^{|N_k|-1}$ corresponding to Q_k. Then we have

$$\bigcap_{k=1}^m \bigcap_{i=1}^{|N_k|} C_i^k \neq \emptyset.$$

Let (X_k, D_k, q_k) be a pseudo H-space, $k \in I$ and N_k be any nonempty finite subset of D_k for $k \in I$. Let $q : \Delta^{|N_1|-1} \times \cdots \times \Delta^{|N_m|-1} \to 2^X$ be the mapping, which is defined by

$$q(\alpha^1, \alpha^2, \ldots, \alpha^m) = (q_1(\alpha^1), q_2(\alpha^2), \ldots, q_m(\alpha^m)),$$

where $\alpha^k \in \Delta^{|N_k|-1}, k \in I$.

Now we shall use the Peleg Theorem to prove our main result.

Theorem 2.1. *Let (X_k, D_k, q_k) be a pseudo H-space, $k \in I$ and N_k be any nonempty finite subset of D_k for $k \in I$. If C_x^k, $x \in N_k$ and $k \in I$, are compactly closed subsets of $X = \prod_{k \in I} X_k$, which have intersection property with respect to q on any nonempty compact set in X such that for each $A_k \in 2^{N_k} \setminus \{\emptyset\}$, $k \in I$,*

$$q_1(\Delta^{|A_1|-1}) \times q_2(\Delta^{|A_2|-1}) \times \cdots \times q_m(\Delta^{|A_m|-1}) \subset \bigcap_{k=1}^{m} \bigcup_{x \in A_k} C_x^k,$$

then

$$\bigcap_{k=1}^{m} \bigcap_{x \in N_k} C_x^k \neq \emptyset.$$

Furthermore, if the product space X is compact or there are $k \in I$ and $x \in N_k$ such that C_x^k is compact, then

$$\bigcap_{k=1}^{m} \bigcap_{x \in D_k} C_x^k \neq \emptyset.$$

Proof. Let $q : \Delta^{|N_1|-1} \times \cdots \times \Delta^{|N_m|-1} \to 2^X$ be the mapping defined as above. Then

$$q(\Delta^{|A_1|-1} \times \cdots \times \Delta^{|A_m|-1}) \subset q_1(\Delta^{|A_1|-1}) \times \cdots \times q_m(\Delta^{|A_m|-1}) \subset \bigcap_{k=1}^{m} \bigcup_{x \in A_k} C_x^k.$$

Hence,

$$q(\Delta^{|A_1|-1} \times \cdots \times \Delta^{|A_m|-1}) \subset \bigcap_{k=1}^{m} \bigcup_{x \in A_k} (C_x^k \cap q(\Delta^{|N_1|-1} \times \cdots \times \Delta^{|N_m|-1})).$$

Since $q(\Delta^{|N_1|-1} \times \cdots \times \Delta^{|N_m|-1})$ is compact and C_x^k is compactly closed, $E_x^k \doteq C_x^k \cap q(\Delta^{|N_1|-1} \times \cdots \times \Delta^{|N_m|-1})$ is closed in $q(\Delta^{|N_1|-1} \times \cdots \times \Delta^{|N_m|-1})$. Since I is finite and q_k is upper semi-continuous with nonempty compact values for each $k \in I$ and $x \in A_k$, $q^-(E_x^k)$ is closed. Moreover,

$$\Delta^{|A_1|-1} \times \cdots \times \Delta^{|A_m|-1} \subset q^-(\bigcap_{k=1}^m \bigcup_{x \in A_k} E_x^k) \subset \bigcap_{k=1}^m \bigcup_{x \in A_k} q^-(E_x^k).$$

By Peleg Theorem and the intersection perproty,

$$q^-(\bigcap_{k=1}^m \bigcap_{x \in N_k} E_x^k) = \bigcap_{k=1}^m \bigcap_{x \in N_k} q^-(E_x^k) \neq \emptyset.$$

Therefore,

$$\bigcap_{k=1}^m \bigcap_{x \in N_k} C_x^k \supset \bigcap_{k=1}^m \bigcap_{x \in N_k} E_x^k \neq \emptyset.$$

Furthermore, if the product space X is compact or there are $k \in I$ and $x \in N_k$ such that C_x^k is compact, then it is easy to see that

$$\bigcap_{k=1}^m \bigcap_{x \in D_k} C_x^k \neq \emptyset.$$

□

Now we have the following corollary which was derived by Marchi and Martínez-Legaz ([7], Lemma 3).

Corollary 2.2. *For $k \in I$, let X_k be a topological space, N_k a nonempty finite set and $\{\Gamma_A^k\}_{\emptyset \neq A \subset N_k}$ a family of nonempty contractible subsets of X_k such that $A \subset B$ implies $\Gamma_A^k \subset \Gamma_B^k$. If C_x^k, $x \in N_k$, $k \in I$, are closed subsets of $X = \prod_{k=1}^m X_k$ such that for each $k \in I$ and $A_k \in 2^{N_k} \setminus \{\emptyset\}$,*

$$\Gamma_{A_1}^1 \times \cdots \times \Gamma_{A_m}^m \subset \bigcap_{k=1}^m \bigcup_{x \in A_k} C_x^k,$$

then

$$\bigcap_{k=1}^m \bigcap_{x \in N_k} C_x^k \neq \emptyset.$$

Furthermore, if the product space X is compact or there are $k \in I$ and $x \in N_k$ such that C_x^k is compact, then

$$\bigcap_{k=1}^m \bigcap_{x \in D_k} C_x^k \neq \emptyset.$$

Proof. For each $k \in I$ and for each nonempty set $A_k \subset N_k$, by Horvath theorem ([5], Theorem 1), there is a continuous function $f_k : \Delta^{|A_k|-1} \to \Gamma^k_{A_k}$. We can choose $q_k(\alpha^k) = \{f_k(\alpha^k)\}$ for all $\alpha^k \in \Delta^{|A_k|-1}$. Then, for each $k \in I$, (X_k, D_k, q_k) forms a pseudo H-space, and

$$q_1(\Delta^{|A_1|-1}) \times \cdots \times q_m(\Delta^{|A_m|-1}) \subset \Gamma^1_{A_1} \times \cdots \times \Gamma^m_{A_m} \subset \bigcap_{k=1}^m \bigcup_{x \in A_k} C^k_x.$$

By Theorem 2.1, we know that

$$\bigcap_{k=1}^m \bigcap_{x \in N_k} C^k_x \neq \emptyset.$$

Furthermore, if the product space X is compact or there are $k \in I$ and $x \in N_k$ such that C^k_x is compact, then

$$\bigcap_{k=1}^m \bigcap_{x \in D_k} C^k_x \neq \emptyset.$$

□

The proof of the following result is similar to that of Corollary 2.2 by taking into account of Example 2.1 and hence will be omitted.

Corollary 2.3. *For $k \in I$, let (X_k, D_k, Γ^k) be a G-convex space. If C^k_x, $x \in D_k$, $k \in I$, are compactly closed subsets of $X = \prod_{k=1}^m X_k$ such that for each $k \in I$ and $A_k \in 2^{D_k} \setminus \{\emptyset\}$,*

$$\Gamma^1_{A_1} \times \cdots \times \Gamma^m_{A_m} \subset \bigcap_{k=1}^m \bigcup_{x \in A_k} C^k_x,$$

then

$$\bigcap_{k=1}^m \bigcap_{x \in N_k} C^k_x \neq \emptyset.$$

Furthermore, if the product space X is compact or there are $k \in I$ and $x \in N_k$ such that C^k_x is compact, then

$$\bigcap_{k=1}^m \bigcap_{x \in D_k} C^k_x \neq \emptyset.$$

3. Applications

In this section, we shall give some applications of Theorem 2.1. First, we establish a Fan-Browder type fixed point theorem on pseudo H-space.

Theorem 3.1. *For each $k \in I$, let (X_k, q_k) be a pseudo H-space and $S_k, T_k : X \to 2^{X_k}$ be set-valued mappings such that*

(1) *for each $k \in I$ and $x_k \in X_k$, the set $S_k^{-1}(x_k)$ is compactly open,*

(2) *for each $k \in I$ and $x \in X$, if $S_k(x)$ is nonempty, then $T_k(x)$ is q-convex relative to $S_k(x)$,*

(3) *for each $x \in X$, there exists $k = k(x) \in I$ such that $S_k(x) \neq \emptyset$,*

(4) *there are $k \in I$ and $x_k \in X_k$ such that $X \setminus S_k^{-1}(x_k)$ is compact,*

(5) *the family $\{X \setminus S_k^{-1}(x_k) : k \in I,\ x_k \in X_k\}$ has intersection property with respect to q on any nonempty compact set in X.*

Then there exist $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m) \in X$ and $\bar{k} \in I$ such that $\bar{x}_{\bar{k}} \in T_{\bar{k}}(\bar{x})$.

Proof. For each $k \in I$, let $F_k,\ G_k : X_k \to 2^X$ be the mappings defined by $G_k(x_k) = X \setminus T_k^{-1}(x_k)$ and $F_k(x_k) = X \setminus S_k^{-1}(x_k)$ for $x_k \in X_k$. Then the mapping F_k has compactly closed values for each $k \in I$. From (3), we have $\bigcup_{k=1}^m \bigcup_{x_k \in X_k} S_k^{-1}(x_k) = X$. This implies that

$$\bigcap_{k=1}^{m} \bigcap_{x_k \in X_k} F_k(x_k) = X \setminus \bigcup_{k=1}^{m} \bigcup_{x_k \in X_k} S_k^{-1}(x_k) = \emptyset.$$

Hence, by Theorem 2.1 with $D_k = X_k$ for each $k \in I$, there exist nonempty finite subsets A_k of X_k, $k \in I$, such that

$$q_1(\Delta^{|A_1|-1}) \times q_2(\Delta^{|A_2|-1}) \times \cdots \times q_m(\Delta^{|A_m|-1}) \not\subset \bigcap_{k=1}^{m} \bigcup_{x_k \in A_k} F_k(x_k).$$

Therefore, there exists $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m)$ such that

$$\bar{x} \in q_1(\Delta^{|A_1|-1}) \times q_2(\Delta^{|A_2|-1}) \times \cdots \times q_m(\Delta^{|A_m|-1}) \setminus \bigcap_{k=1}^{m} \bigcup_{x_k \in A_k} F_k(x_k).$$

Then $\bar{x} \in X$ and for some $\bar{k} \in I$, $\bar{x} \notin F_{\bar{k}}(x_{\bar{k}})$ for all $x_{\bar{k}} \in A_{\bar{k}}$. That is, $A_{\bar{k}} \subset S_{\bar{k}}(\bar{x})$ and it follows from (2) that, $q_{\bar{k}}(\Delta^{|A_{\bar{k}}|-1}) \subset T_{\bar{k}}(\bar{x})$. This implies that, for this $\bar{k}$, $\bar{x}_{\bar{k}} \in T_{\bar{k}}(\bar{x})$. □

The condition (5) of Theorem 3.1 is fulfilled if each mapping q_k, $k \in I$, is single-valued.

By using the result of Theorem 3.1, we can derive the following generalization of Corollary 8 in [7], which generalized the Fan result ([4], Lemma 4).

Theorem 3.2. *For each $k \in I$, let (X_k, q_k) be a pseudo H-space and $X = \prod_{k \in I} X_k$. Let $A_k,\ B_k \subset X_k \times X$, for $k \in I$. Suppose that*

(1) *for each $k \in I$ and $y_k \in X_k$, the set $\{x \in X : (y_k, x) \in B_k\}$ is compactly closed in X, and the family $\{\{x \in X : (y_k, x) \in B_k\} : k \in I,\ y_k \in X_k\}$ has intersection property with respect to q on any nonempty compact set in X,*

(2) *for any $x = (x_1, x_2, \ldots, x_m) \in X$ and for each $k \in I$, $(x_k, x) \in A_k$,*

(3) *for any $x \in X$, if the sets $\{y_k \in X_k : (y_k, x) \notin B_k\}$, $k \in I$, are nonempty, then the set $\{y_k \in X_k : (y_k, x) \notin A_k\}$ is q-convex relative to the set $\{y_k \in X_k : (y_k, x) \notin B_k\}$,*

(4) *there are $k \in I$ and $y_k \in X_k$ such that the set $\{x \in X : (y_k, x) \in B_k\}$ is compact.*

Then there is an $\bar{x} \in X$ such that $X_k \times \{\bar{x}\} \subset B_k$ for each $k \in I$.

Proof. For each $k \in I$, we define $S_k,\ T_k : X \to 2^{X_k}$ by

$$S_k(x) = \{y_k \in X_k : (y_k, x) \notin B_k\},$$

and

$$T_k(x) = \{y_k \in X_k : (y_k, x) \notin A_k\}.$$

By (1), $S_k^{-1}(y_k)$ is compactly open in X_k for each $k \in I$ and $y_k \in X_k$. For each $k \in I$, by (3), if $S_k(x)$ is nonempty, then for each nonempty finite subset C_k of $S_k(x)$, we have $q_k(\Delta^{|C_k|-1}) \subset T_k(x)$. Also by (2), we know that $x_k \notin T_k(x)$ for each $x \in X$ and $k \in I$. Applying Theorem 3.1, there is an $\bar{x} \in X$ such that $S_k(\bar{x}) = \emptyset$ for all $k \in I$. This implies

$$(x_k, \bar{x}) \in B_k \text{ for all } k \in I \text{ and } x_k \in X_k,$$

and hence, there is an $\bar{x} \in X$ such that $X_k \times \{\bar{x}\} \subset B_k$ for all $k \in I$. □

Next, we consider application of Theorem 2.1 to the system of variational inequalities. Given a family of functions $\{f_k : X_k \times X \to \mathbf{R}\}_{k \in I}$, we can consider the system of variational inequalities (in short, SVI) which is to find $\bar{x} \in X$ such that for each $k \in I$, $f_k(x_k, \bar{x}) \geq 0$, for all $x_k \in X_k$.

Theorem 3.3. *For each $k \in I$, let (X_k, q_k) be a pseudo H-space and let f_k and g_k be two real-valued functions defined on $X_k \times X$. Suppose that*

(1) *for each $k \in I$ and $y_k \in X_k$, the mapping $x \to f_k(y_k, x)$ is upper semi-continuous on any compact subsets of X,*

(2) *for any $x = (x_1, x_2, \ldots, x_m) \in X$ and each $k \in I$, $g_k(x_k, x) \geq 0$,*

(3) *for any $x \in X$ and each $k \in I$, if the the set $\{y_k \in X_k : f_k(y_k, x) < 0\}$ is nonempty, then the set $\{y_k \in X_k : g_k(y_k, x) < 0\}$ is q-convex relative to the set $\{y_k \in X_k : f_k(y_k, x) < 0\}$,*

(4) *there are $k \in K$ and $y_k \in X_k$ such that the set $\{x \in X : f_k(y_k, x) \geq 0\}$ is compact,*

(5) *the family $\{\{x \in X : f_k(x_k, x) \geq 0\} : k \in I,\ x_k \in X_k\}$ has intersection property with respect to q on any nonempty compact set in X.*

Then there is an $\bar{x} \in X$ such that for all $x_k \in X_k$, $f_k(x_k, \bar{x}) \geq 0$ for each $k \in I$.

Proof. For each $k \in I$, let $A_k = \{(x_k, x) \in X_k \times X : g_k(x_k, x) \geq 0\}$ and $B_k = \{(x_k, x) \in X_k \times X : f_k(x_k, x) \geq 0\}$. Then all the conditions of Theorem 3.2 hold, and we can deduce the conclusion that there is an $\bar{x} \in X$ such that $X_k \times \{\bar{x}\} \subset B_k$ for each $k \in I$. That is, there is an $\bar{x} \in X$ such that for each $k \in I$, $f_k(x_k, \bar{x}) \geq 0$, for all $x_k \in X_k$. □

Theorem 3.4. *For each $k \in I$, let (X_k, q_k) be a pseudo H-space, and let f_k and g_k be two real-valued functions defined on $X \times X_k$. Suppose that*

(1) *for each $k \in I$ and $y_k \in X_k$, the mapping $x \to f_k(y_k, x)$ is upper semi-continuous on any nonempty compact set of X, and the family $\{\{x \in X : f_k(x_k, x) \geq 0\} : k \in I,\ x_k \in X_k\}$ has intersection property with respect to q on any nonempty compact set in X,*

(2) *for any $x = (x_1, x_2, \ldots, x_m) \in X$ and each $k \in I$, $g_k(x_k, x) \geq 0$,*

(3) *for each $k \in I$, $f_k(y_k, x) < 0$ implies $g_k(y_k, x) < 0$ for all $(y_k, x) \in X_k \times X$,*

(4) *for each $k \in I$ and $x \in X$, either* (a) *if the the set $\{y_k \in X_k : f_k(y_k, x) < 0\}$ is nonempty, then the set $\{y_k \in X_k : f_k(y_k, x) < 0\}$ is q-convex set, or* (b) *if the the set $\{y_k \in X_k : g_k(y_k, x) < 0\}$ is nonempty, then the set $\{y_k \in X_k : g_k(y_k, x) < 0\}$ is q-convex set,*

(5) *there are $k \in K$ and $y_k \in X_k$ such that the set $\{x \in X : f_k(y_k, x) \geq 0\}$ is compact.*

Then there is an $\bar{x} \in X$ such that for each $k \in I$, $f_k(x_k, \bar{x}) \geq 0$, for all $x_k \in X_k$.

Proof. We will show that: if the the set $\{y_k \in X_k : f_k(y_k, x) < 0\}$ is nonempty, then for any nonempty finite subset C_k of $\{y_k \in X_k : f_k(y_k, x) < 0\}$, we have $q_k(\Delta^{|C_k|-1}) \subset \{y_k \in X_k : g_k(y_k, x) < 0\}$. Then the result will follow from Theorem 3.3.

To this end, for each $x \in X$, let C_k be any nonempty finite subset of $\{y_k \in X_k : f_k(y_k, x) < 0\}$. First, we assume that the condition (a) of (4) holds. Then we have $q_k(\Delta^{|C_k|-1}) \subset \{y_k \in X_k : f_k(y_k, x) < 0\}$. Then for each $y \in q_k(\Delta^{|C_k|-1})$, $f_k(y, x) < 0$. From (3), $g_k(y, x) < 0$, and hence $q_k(\Delta^{|C_k|-1}) \subset \{y_k \in X_k : g_k(y_k, x) < 0\}$. Secondly, we assume that the condition (b) of (4) holds. For each $y \in C_k$, $f_k(y, x) < 0$, and then we have $g_k(y, x) < 0$ by (3). Hence C_k is a finite subset of $\{y_k \in X_k : g_k(y_k, x) < 0\}$ and we have

$$q_k(\Delta^{|C_k|-1}) \subset \{y_k \in X_k : g_k(y_k, x) < 0\}.$$

□

Finally, let E_k be a Hausdorff topological vector space with its topological dual E_k^*, X_k be a nonempty convex subset of E_k and A_k be an operator defined from X into E_k^* for $k \in I$. The case $f_k(y_k, x) = \langle A_k(x), y_k - x_k \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the pairing between E_k^* and E_k for each $k \in I$ was discussed by Pang [9] with applications in equilibrium problems in finite-dimensional spaces. For existence results, see, e.g., Ansari and Yao [1], Bianchi [2], Cohen and Chaplais [3], Zhu and Marcotte [11].

Corollary 3.5. *For each $k \in I$, let A_k be an operator from X into E_k^*. Suppose that*

(1) *for each $k \in I$, A_k is upper semi-continuous on any compact subsets of X,*

(2) *there is a family $\{g_k : k \in I\}$ of real-valued functions defined on $X_k \times X$ such that*

  (a) *for each $k \in I$, $\langle A_k(x), y_k - x_k \rangle < 0$ implies $g_k(y_k, x) < 0$ for all $(y_k, x) \in X_k \times X$,*

  (b) *for any $x = (x_1, x_2, \ldots, x_m) \in X$, $x_k \in X_k$ and each $k \in I$, $g_k(x_k, x) \geq 0$,*

(3) *there are* $k \in I$ *and* $y_k \in X_k$ *such that the set* $\{x \in X : \langle A_k(x), y_k - x_k \rangle \geq 0\}$ *is compact.*

Then there is an $\bar{x} \in X$ *such that for each* $k \in I$, $\langle A_k(\bar{x}), y_k - \bar{x}_k \rangle \geq 0$ *for all* $y_k \in X_k$.

Proof. For each $k \in I$, we define $f_k(y_k, x) = \langle A_k(x), y_k - x_k \rangle$ for all $(y_k, x) \in X_k \times X$. Then, by Theorem 3.4, there is an $\bar{x} \in X$ such that for each $k \in I$, $\langle A_k(\bar{x}), y_k - \bar{x}_k \rangle \geq 0$ for all $y_k \in X_k$. □

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